Martingale Property and Pricing for Time-homogeneous Diffusion Models in Finance

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Zhenyu Cui

Abstract

The thesis studies the martingale properties, probabilistic methods and efficient unbiased Monte Carlo simulation methods for various time-homogeneous diffusion models commonly used in mathematical finance. Some of the popular stochastic volatility models such as the Heston model, the Hull-White model and the 3/2 model are special cases. The thesis consists of the following three parts:

Part I of the thesis studies martingale properties of stock prices in stochastic volatility models driven by time-homogeneous diffusions. We find necessary and sufficient conditions for the martingale properties. The conditions are based on the local integrability of certain deterministic test functions.

Part II of the thesis studies probabilistic methods for determining the Laplace transform of the first hitting time of an integral functional of a time-homogeneous diffusion, and pricing an arithmetic Asian option when the stock price is modeled by a time-homogeneous diffusion. We also consider the pricing of discrete variance swaps and discrete gamma swaps in stochastic volatility models based on time-homogeneous diffusions.

Part III of the thesis studies the unbiased Monte Carlo simulation of option prices when the characteristic function of the stock price is known but its density function is unknown or complicated.

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To my parents

My father, Jianjun Cui

My mother, Ping Fan

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Chapter 1

Outline of the Thesis

The following is the outline of the thesis with descriptions of each individual chapter.

Part I: Martingale properties in time-homogeneous diffusion models

Chapter 2: Martingale properties in correlated stochastic volatility models

This chapter generalizes the results in Mijatović and Urusov (2012c) to the arbitrary correlation case and proposes easy-to-check necessary and sufficient conditions for the martingale properties of stock prices in correlated stochastic volatility models, where the stochastic variance is modeled by a time-homogeneous diffusion. Our contribution to this literature is first to unify and generalize the results on convergence or divergence of integral functionals of time-homogeneous diffusions, and also to provide unified new proofs to the main results in Mijatović and Urusov (2012c) without the use of the concept *separating times* introduced by Cherny and Urusov (2004). Results in this chapter are applied to verifying martingale properties in four popular correlated stochastic volatility models, are consistent with and complement the literature.

Part II: Probabilistic pricing methods

Chapter 3: First hitting times of integrated time-homogeneous diffusions

This chapter studies properties of the first hitting time of the integral functional of a time-homogeneous diffusion to a fixed level. We provide a unified probabilistic approach with an alternative proof to the main results in Metzler (2013). The links between the first hitting times and integral functionals of diffusions are established, and the relevant literature is connected. In the last part of the chapter, we show the link between the pricing of an arithmetic Asian option and the first hitting time of the integral functional of a time-homogeneous diffusion, and we give an analytical formula for the price of an arithmetic Asian option in the Black-Scholes setting. We also provide financial motivations behind the study of this first hitting time.

Chapter 4: Prices and asymptotics of some discrete volatility derivatives

This chapter is based on the publication Bernard and Cui (2013) forthcoming in the *Applied Mathematical Finance*. It presents explicit expressions for fair strikes of discretely sampled and continuously sampled variance swaps in the Heston, the Hull-White, the Schöbel-Zhu, and the mixed exponential jump diffusion models. They are consistent with the literature, more explicit (as there are no sums involved in the discrete fair strikes), and easier to use. Asymptotic expansions are new and consistent with theoretical results in the recent literature. In the Heston model, we also derive a new closed-form formula for a special type of discrete gamma swap, and obtain the asymptotics of its fair strike with respect to key parameters.

Part III: Efficient unbiased Monte Carlo simulation methods and applications

Chapter 5: Nearly exact option price simulation using characteristic functions

This chapter is based on the publication Bernard, Cui and McLeish (2012) in the *International Journal of Theoretical and Applied Finance*. We propose a new approach to perform a nearly unbiased simulation using inversion of the characteristic function. As an application, we are able to give unbiased estimates of the prices of forward starting options in the Heston model and of continuously monitored Parisian options in the Black-Scholes framework. This method of simulation can be applied to a problem for which the characteristic function is known but the corresponding probability density function is complicated.

The contribution here is that we can unbiasedly simulate directly from the characteristic function of (for example) the log stock price. In continuous time models used in finance, it is usually the case that the characteristic function of the log stock price is given. Examples are affine processes (Duffie, Pan and Singleton (2000)), and time-changed Lévy process (Carr et al (2003), Carr and Wu (2004)). Applications of the results can be in the simulation of exotic option prices when the stock prices are modeled as time-changed Lévy processes.

Part I

Martingale properties in time-homogeneous diffusion models

Chapter 2

Martingale properties in correlated stochastic volatility models

2.1 Introduction

There are several recent papers proposing sufficient conditions (Lions and Musiela (2007)) or necessary and sufficient conditions (Blei and Engelbert (2009), Mijatović and Urusov (2012c), Mijatović, Novak and Urusov (2012)) to verify when the stochastic exponential of a continuous local martingale is a true martingale or a uniformly integrable martingale. A relevant application in finance is to check if the discounted stock price is a true martingale in a general time-homogeneous stochastic volatility model with arbitrary correlation.

This problem has been extensively studied and dates back from Girsanov (1960), who posed the problem of deciding whether a stochastic exponential is a true martingale or not. Gikhman and Skorohod (1972), Liptser and Shiryaev (1972), Novikov (1972) and Kazamaki (1977) provided sufficient conditions for the martingale property of a stochastic exponential. Novikov's criterion is easy to apply in practical situations, but for concrete models in mathematical finance it may not always be verified. In the setting of Brownian motions, refer to Kramkov and Shiryaev (1998), Cherny and Shiryaev (2001) and Ruf (2013b) for improvements of criteria of Novikov (1972) and Kazamaki (1977). For affine processes, similar questions have been considered in Kallsen and Shiryaev (2002), Kallsen and Muhle-Karbe (2010), and Mayerhofer, Muhle-Karbe, and Smirnov (2011). In Kotani (2006) and Hulley and Platen (2011), they obtain necessary and sufficient conditions for a one-dimensional regular strong Markov continuous local martingale to be a true martingale. In the strand of stochastic exponentials based on time-homogeneous diffusions, Engelbert and Schmidt (1984) provided analytic conditions for the martingale property, and Stummer (1993) provided further analytic conditions when the diffusion coefficient is the identity. In the context of stochastic volatility models, Sin (1998), Andersen and Piterbarg (2007), and Lions and Musiela (2007) provided easily verifiable conditions. Blanchet and Ruf (2012) describe a method to decide on the martingale property of a non-negative local martingale based on weak convergence considerations. A recent paper by Karatzas and Ruf (2013) provides the precise relationship between explosions of one-dimensional stochastic differential equations and the martingale properties of related stochastic exponentials. For an overview of stochastic exponentials and related problem of martingale properties, refer to Rheinländer (2010) and the references therein.

This chapter makes two contributions to the current literature. First, we provide a complete classification of the convergence or divergence properties of integral functionals of time-homogeneous diffusions based on the local integrability of certain deterministic test functions. Theorem 2.3.1 unifies and generalizes the work of Salminen and Yor (2006) and Khoshnevisan, Salminen, and Yor (2006) under weaker assumptions. Second, we extend some results in Mijatović and Urusov (2012b, 2012c) from the case $\rho = 1$ to the case of arbitrary correlation (see Proposition 2.4.1 and Proposition 2.4.2). In our proofs, we do not make use of the concept of *separating times* introduced by Cherny and Urusov (2004).

In this chapter, the new results, which contribute to the current literature, are as follows: Corollary 2.2.1, Proposition 2.2.3, Lemma 2.2.4, Proposition 2.2.4, Proposition 2.2.5, Lemma 2.3.1, Lemma 2.3.2, Theorem 2.3.1, Corollary 2.3.1, Theorem 2.3.2, Corollary 2.3.2, Proposition 2.4.1, Proposition 2.4.2, Proposition 2.4.3, Proposition 2.4.4, Proposition 2.4.5, Proposition 2.5.1, Proposition 2.5.2, Proposition 2.5.3, Lemma 2.5.1, Proposition 2.5.4, Proposition 2.5.5, Proposition 2.5.6 Proposition 2.5.7, Proposition 2.5.8, Proposition 2.5.9, Proposition 2.5.10, Proposition 2.5.11, Proposition 2.5.12, and Theorem 2.6.1.

The chapter is organized as follows. Section 2.2 presents some technical tools from Ruf (2013b) using our notation. Section 2.3 presents the main result of the chapter, which is a complete classification of the convergence or divergence properties of integral functionals of time-homogeneous diffusions. Section 2.4 shows the application of our main result to generalizing some results in Mijatović and Urusov (2012b, 2012c) to the arbitrary correlation case with new proofs. Section 2.5 studies in detail the martingale properties in four popular stochastic volatility models. Section 2.6

illustrates some key results from stochastic time-change. Section 2.7 provides an alternative proof to the Engelbert-Schmidt type zero-one law for a time-homogeneous diffusion. Section 2.8 recalls the statement and proof of a result from Karatzas and Shreve (1991). Section 2.9 concludes the chapter.

2.2 Necessary and sufficient conditions for the martingale property

2.2.1 Probabilistic setup

Denote the state space of the variance process $Y = (Y_t)_{t\in[0,\infty)}$ as $J = (\ell, r), -\infty \leqslant \ell < r \leqslant \infty$, and set $\bar{J} = [\ell, r]$. Assume that Y satisfies the following SDE

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \quad Y_0 = x_0, \tag{2.1}$$

where $\mu, \sigma : J \to \mathbb{R}$ are Borel functions, $x_0 \in \mathbb{R}$, and assume that μ, σ satisfy the Engelbert-Schmidt conditions

$$\forall x \in J, \quad \sigma(x) \neq 0, \quad \text{and} \quad \frac{1}{\sigma^2(\cdot)}, \quad \frac{\mu(\cdot)}{\sigma^2(\cdot)} \in L^1_{loc}(J). \tag{2.2}$$

$L^1_{loc}(J)$ denotes the class of locally integrable functions, i.e. the functions $J \to \mathbb{R}$ that are integrable on compact subsets of J.

Now we rephrase Definition 5.1, p329, Karatzas and Shreve (1991) (accommodating the possibility of exploding solutions) using our notation.

Definition 2.2.1. *A weak solution up to an explosion time of equation* (2.1) *is a triple* $(Y, W), (\Omega, \mathcal{G}, Q), \{\mathcal{G}_t\}$ *where*

(i) $(\Omega, \mathcal{G}, Q)$ *is a probability space, and* $\{\mathcal{G}_t\}$ *is a filtration of sub-*σ*-fields of* $\mathcal{G}$ *satisfying the usual conditions;*

(ii) $Y = \{Y_t, \mathcal{G}_t; 0 \leqslant t < \infty\}$ *is a continuous, adapted,* $\mathbb{R} \cup \{\pm\infty\}$-

valued process with $|Y_0| < \infty$ *a.s., and* $\{W_t, \mathcal{G}_t; 0 \leqslant t < \infty\}$ *is a standard one-dimensional Brownian motion;*

(iii) with $\zeta_n = \inf\{t \in [0, \infty) : |Y_t| \geqslant n\}$*, we have*

$$Q\left(\int_0^{t\wedge\zeta_n} (|\mu(Y_s)| + \sigma^2(Y_s))ds < \infty\right) = 1; \quad \forall 0 \leqslant t < \infty \tag{2.3}$$

and

(iv)

$$Q\left(Y_{t\wedge\zeta_n} = Y_0 + \int_0^t \mu(Y_s)\mathbb{1}_{s\leqslant\zeta_n}ds + \int_0^t \sigma(Y_s)\mathbb{1}_{s\leqslant\zeta_n}dW_s; \forall 0 \leqslant t < \infty\right) = 1 \tag{2.4}$$

valid for every $n \geqslant 1$*.*

We refer to $\zeta = \lim_{n\to\infty} \zeta_n$ *as the explosion time for* Y*.*

The Engelbert-Schmidt condition (2.2) guarantees that the SDE (2.1) has a unique in law weak solution as described in Definition 2.2.1 that possibly exits its state space J (see Theorem 5.15, p341, Karatzas and Shreve (1991)). From Definition 2.2.1, it is equivalent to say that there exists a triple $(Y, W), (\Omega, \mathcal{G}, P), \mathcal{G}_t$ such that Y solves the SDE

$$dY_t = \mu(Y_t)\mathbb{1}_{t<\zeta_n}dt + \sigma(Y_t)\mathbb{1}_{t<\zeta_n}dW_t, \quad Y_0 = x_0, \tag{2.5}$$

for all ζ_n defined in Definition 2.2.1.

We can similarly define a weak solution for one stochastic differential equation with arbitrary state space (l, r). Denote the possible exit time[1] of Y from its state space by ζ (as in Definition 2.2.1), i.e. $\zeta = \inf\{u > 0, Y_u \notin J\}$, P-a.s., which means that on $\{\zeta = \infty\}$ the trajectories of Y do not exit J, and on $\{\zeta < \infty\}$, $\lim_{t\to\zeta} Y_t = r$ or $\lim_{t\to\zeta} Y_t = \ell$, P-a.s. Y is defined such that it stays at its exit point, which means that ℓ and r are absorbing boundaries. The following terminology will be used: "Y may exit the state

[1] Refer to Karatzas and Ruf (2013) for a detailed study of the distribution of this exit time in a one-dimensional time-homogeneous diffusion setting.

space J at r" means $P(\zeta < \infty, \lim_{t\to\zeta} Y_t = r) > 0$.

Then we may enlarge the space and filtration to introduce a Brownian motion $W^{(2)}$ independent of W. Let $Z = (Z_t)_{t\in[0,\infty)}$ denote the (discounted) stock price, and define

$$Z_t = \exp\left\{\rho \int_0^{t\wedge\zeta} b(Y_u)dW_u + \sqrt{1-\rho^2}\int_0^{t\wedge\zeta} b(Y_u)dW_u^{(2)} - \frac{1}{2}\int_0^{t\wedge\zeta} b^2(Y_u)du\right\}, \quad t \in [0,\infty), \tag{2.6}$$

where $b : J \to \mathbb{R}$ is a Borel function, and the constant correlation satisfies $-1 \leqslant \rho \leqslant 1$. Denote $W^{(1)} = \rho W + \sqrt{1-\rho^2}W^{(2)}$, we have

$$Z_t = \exp\left\{\int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)} - \frac{1}{2}\int_0^{t\wedge\zeta} b^2(Y_u)du\right\}, \quad t \in [0,\infty), \tag{2.7}$$

and it is easy to verify that Z and Y satisfy the following system of SDEs

$$\begin{aligned} dZ_t &= Z_t b(Y_t)dW_t^{(1)}, \quad Z_0 = 1, \\ dY_t &= \mu(Y_t)dt + \sigma(Y_t)dW_t, \quad Y_0 = x_0, \end{aligned} \tag{2.8}$$

Now we define the space accommodating all four processes $(Y, Z, W, W^{(1)})$.

Let $\Omega_1 := \overline{C}((0,\infty), \bar{J})$ be the space of continuous functions $\omega_1 : (0,\infty) \to \bar{J}$ that start inside J and can exit, i.e. there exists $\zeta(\omega_1) \in (0,\infty]$ such that $\omega_1(t) \in J$ for $t < \zeta(\omega_1)$ and in the case $\zeta(\omega_1) < \infty$ we have either $\omega_1(t) = r$ for $t \geqslant \zeta(\omega_1)$ (hence also $\lim_{t\to\zeta(\omega_1)} \omega_1(t) = r$) or $\omega_1(t) = \ell$ for $t \geqslant \zeta(\omega)$ (hence also $\lim_{t\to\zeta(\omega_1)} \omega_1(t) = \ell$).

Let $\Omega_2 := \overline{C}((0,\infty), [0,\infty])$ be the space of continuous functions $\omega_2 : (0,\infty) \to [0,\infty]$ with $\omega_2(0) = 1$ that satisfy $\omega_2(t) = \omega_2(t \wedge T_0(\omega_2) \wedge T_\infty(\omega_2))$ for all $t \geqslant 0$, where $T_0(\omega_2)$ and $T_\infty(\omega_2)$ denote the first hitting times of 0 and ∞ by ω_2.

Let $\Omega_3 = \overline{C}([0,\infty), (-\infty,\infty))$ be the space of continuous functions $\omega_3 : [0,\infty) \to (-\infty,\infty)$ with $\omega_3(0) = 0$.

Let $\Omega_4 = \overline{C}([0,\infty), (-\infty,\infty))$ be the space of continuous functions

$\omega_4 : [0,\infty) \to (-\infty,\infty)$ with $\omega_4(0) = 0$.

Define the canonical process

$$(Y_t(\omega_1), Z_t(\omega_2), W_t(\omega_3), W_t^{(1)}(\omega_4)) := (\omega_1(t), \omega_2(t), \omega_3(t), \omega_4(t))$$

for all $t \geqslant 0$, and let $\mathcal{F}_t$ denote the usual right continuous filtration generated by the canonical process. The σ-field is $\mathcal{F} = \bigvee_{t\in[0,\infty)} \mathcal{F}_t$. Note that T_0 and T_∞ are stopping times adapted to $\mathcal{F}$ and either or both can take the value ∞. Also ω_2 is continuous on $[0, T_\infty(\omega_2))$.

From now on, processes are defined in this filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)})$. Let P be the probability measure induced by the canonical process on the space $(\Omega, \mathcal{F})$.

Define the Borel set $\mathcal{B}(\mathbb{R})$ as the smallest σ-algebra that contains the open intervals of $\mathbb{R}$. In what follows, $\lambda(.)$ denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. In the following, assume that[2] $\lambda(x \in (\ell, r) : b^2(x) > 0) > 0$, and assume the following local integrability condition

$$\forall x \in J, \quad \sigma(x) \neq 0, \quad \text{and} \quad \frac{b^2(\cdot)}{\sigma^2(\cdot)} \in L^1_{loc}(J). \tag{2.9}$$

Remark 2.2.1. *In the literature (e.g. Andersen and Piterbarg (2007)), there is a more general class of stochastic volatility models where the (discounted) stock price has non-linear diffusion coefficients in Z. For example, a general model is as follows*

$$\begin{aligned} dZ_t &= Z_t^{\alpha} b(Y_t) \mathbb{1}_{t\in[0,\zeta)} dW_t^{(1)}, \quad Z_0 = 1, \\ dY_t &= \mu(Y_t)\mathbb{1}_{t\in[0,\zeta)} dt + \sigma(Y_t)\mathbb{1}_{t\in[0,\zeta)} dW_t, \quad Y_0 = x_0, \end{aligned}$$

where $W_t^{(1)}$ and W_t are standard $\mathcal{F}_t$-Brownian motions, with $E[dW_t^{(1)} dW_t] = \rho dt$. ρ is the constant correlation coefficient and $-1 \leqslant \rho \leqslant 1$. Here $1 \leqslant \alpha \leqslant 2$. The difficulty of dealing with this model lies mainly in obtaining an explicit representation of Z in terms of functionals of only Y. Thus

[2] Note that this is in the same condition as in Mijatović and Urusov (2012b, 2012c), and Cherny and Urusov (2006).

in this chapter we only focus on model in (2.8)*.*

Lemma 2.2.1. *(Mijatović and Urusov (2012c))*

If the condition (2.9) *holds, then*

$$\int_0^t b^2(Y_u)du < \infty \ P\text{-}a.s. \ on \ \{t < \zeta\}, \quad t \in [0, \infty)$$

Proof. For the proof, refer to p5 of Mijatović and Urusov (2012c). ☐

Fix an arbitrary constant $c \in J$ and introduce the scale function $s(.)$ of the SDE (2.1) under P

$$s(x) := \int_c^x \exp\left\{-\int_c^y \frac{2\mu}{\sigma^2}(u)du\right\} dy, \quad x \in \bar{J}. \tag{2.10}$$

Recall the following result from Cherny and Urusov (2006) using our notation.

Lemma 2.2.2. *(Lemma* 5.7*, p149 of Cherny and Urusov (2006))*

Assume the conditions (2.2)*,* (2.9) *hold for the SDE* (2.1)*, and* $s(\ell) = -\infty$*,* $s(r) = \infty$*. Then* $\int_0^\infty b^2(Y_u)du = \infty$*,* P*-a.s.*

Proof. For the proof, refer to Lemma 5.7, p149 of Cherny and Urusov (2006). ☐

2.2.2 Properties of non-negative continuous local martingales

We now recall some results from Ruf (2013b) concerning non-negative continuous local martingales, and we apply them in the setting of time-homogeneous diffusions as in (2.7). Ruf (2013b) does not specify the form of the continuous local martingale L_t, and in our setting $L_t = \int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)}$. Thus we modify his proofs where appropriate. To cast the setting of Ruf

(2013b) into the current notation, the process in (2.7) under P can be rewritten as $Z_t = \mathcal{E}(L_t) = \exp\left(L_t - \langle L\rangle_t/2\right)$ where $L_t = \int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)}$ is a continuous local martingale under P.

Lemma 2.2.3. *(Lemma 1, Ruf (2013b), case of time-homogeneous diffusions)*

Assume the conditions (2.2) *and* (2.9) *for the SDE* (2.1). *Under P, consider a continuous local martingale $L_t = \int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)}$, and its quadratic variation is $\langle L\rangle_t = \int_0^{t\wedge\zeta} b^2(Y_u)du$. For a predictable positive stopping time $\tau > 0$, define $Z_t = \mathcal{E}(L_t), t \in [0,\tau)$. Then the random variable $Z_\tau := \lim_{t\uparrow\tau} Z_t$ exists, is non-negative and satisfies*

$$\left\{\int_0^{\tau\wedge\zeta} b^2(Y_u)du < \infty\right\} = \{Z_\tau > 0\}, \quad P\text{-}a.s.$$

Proof. Consider a sequence of stopping times $\tau_n \to \tau$ such that Z_{τ_n} is a martingale. Then by the submartingale convergence theorem (Theorem 1.3.15, p17, Karatzas and Shreve (1991)), $Z_{\tau_n} = \exp(L_{\tau_n} - \frac{1}{2}\langle L\rangle_{\tau_n})$ converges almost surely to a non-negative random variable $Z_\tau = \exp(L_\tau - \frac{1}{2}\langle L\rangle_\tau)$.

On the set $\{\langle L\rangle_\tau < \infty\}$, since

$$\ln(Z_\tau) = L_\tau - \frac{1}{2}\langle L\rangle_\tau = \langle L\rangle_\tau\left(\frac{L_\tau}{\langle L\rangle_\tau} - \frac{1}{2}\right),$$

it follows from the Dambis-Dubins-Schwartz theorem (Ch.V, Theorem 1.6, Revuz and Yor (1999)), $\frac{L_t}{\langle L\rangle_t} = \frac{B_{\langle L\rangle_t}}{\langle L\rangle_t}$, $t \in [0,\tau]$ for some Brownian motion B on an extended probability space and it is therefore finite, P-a.s.

On the set $\{\langle L\rangle_\tau = \infty\}$, since

$$\ln(Z_{\tau_n}) = L_{\tau_n} - \frac{1}{2}\langle L\rangle_{\tau_n} = \langle L\rangle_{\tau_n}\left(\frac{L_{\tau_n}}{\langle L\rangle_{\tau_n}} - \frac{1}{2}\right),$$

it follows again from the Dambis-Dubins-Schwartz theorem, $\frac{L_{\tau_n}}{\langle L\rangle_{\tau_n}} = \frac{B'_{\langle L\rangle_{\tau_n}}}{\langle L\rangle_{\tau_n}}$, for some Brownian motion B' on an extended probability space, and $\frac{B'_{\langle L\rangle_{\tau_n}}}{\langle L\rangle_{\tau_n}} \to$

0, P-a.s. as $n \to \infty$ from the law of iterated logarithm (Theorem 2.9.23, p112, Karatzas and Shreve (1991)). Then $\ln(Z_{\tau_n}) = \langle L \rangle_{\tau_n} \left(\frac{L_{\tau_n}}{\langle L \rangle_{\tau_n}} - \frac{1}{2} \right) \to -\infty$, P-a.s. as $n \to \infty$, so that $Z_\tau = 0$, P-a.s.

Therefore $Z_\tau = 0$, P-a.s. on the set $\{\langle L \rangle_\tau = \infty\}$, and $Z_\tau > 0$, P-a.s. on the set $\{\langle L \rangle_\tau < \infty\}$. This completes the proof. $\square$

As an application of Lemma 2.2.3, we have the following result.

Corollary 2.2.1. *Assume*[3] *conditions* (2.2) *and* (2.9) *for the SDE* (2.1). *Under* P*, with the process* Z *defined in* (2.7)*, for* $t \in [0, \infty)$

$$\{Z_t = 0\} = \left\{ \zeta \leqslant t < \infty, \int_0^\zeta b^2(Y_u) du = \infty \right\}, \quad P\text{-a.s.}$$

Proof. From Lemma 2.2.3,

$$\{Z_t = 0\} = \left\{ \int_0^{t \wedge \zeta} b^2(Y_u) du = \infty \right\}, \quad P\text{-a.s.}$$

From Lemma 2.2.1, $\int_0^t b^2(Y_u) du < \infty$ P-a.s. on the set $\{t < \zeta, t \in [0, \infty)\}$. Therefore

$$\{Z_t = 0\} = \left\{ \zeta \leqslant t < \infty, \int_0^\zeta b^2(Y_u) du = \infty \right\}, \quad P\text{-a.s.}$$

This completes the proof. $\square$

Note that similar results as Lemma 2.2.3 and Corollary 2.2.1 also hold under $\widetilde{P}$ with a suitable stochastic exponential $\mathcal{E}(.)$.

Definition 2.2.2. *We say that a stopping time* τ *is bounded, if there exists some* $0 \leqslant t_0 < \infty$*, such that* $\tau \leqslant t_0$*.*

Proposition 2.2.1. *(Theorem 2, Ruf (2013b), case of time-homogeneous diffusions*[4]*)*

[3]This is stated after equation (7) on p4, Mijatović and Urusov (2012c), and after equation (2.4) on p228, Mijatović and Urusov (2012b). Here we provide a proof.

[4]Ruf (2013b)'s result applies to *continuous* non-negative local martingales, and thus

On the space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,\infty)})$, with the process Z defined in (2.7), assume the conditions (2.2) and (2.9). Then we have

(1) There exists a unique probability measure $\widetilde{P}$ on the same space such that, for any bounded stopping time ν and for all non-negative $\mathcal{F}_\nu$-measurable random variables S

$$\mathbb{E}^{\widetilde{P}}\left[\frac{1}{Z_\nu} S \mathbb{1}_{\infty > Z_\nu > 0}\right] = \mathbb{E}^{P}\left[S \mathbb{1}_{Z_\nu > 0}\right], \tag{2.11}$$

where we define $\frac{1}{Z_\nu}\mathbb{1}_{\infty>Z_\nu>0} = 0$ on $\{Z_\nu = 0\}$ from the usual convention.

(2) Under P, for $t \in [0, T_0)$, define $L_t := \int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)}$, and it is a continuous P-local martingale. Then under $\widetilde{P}$, for $t \in [0, T_\infty)$, we have that $\widetilde{L}_t^* := L_t - \langle L\rangle_t = \int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)} - \int_0^{t\wedge\zeta} b^2(Y_u)du$ is a continuous $\widetilde{P}$-local martingale. Here T_0 and T_∞ are defined in Section 2.2.1 as the first hitting times to 0 and ∞ by Z.

(3) Under $\widetilde{P}$, for $t \in [0, T_\infty)$

$$1/Z_t = \mathcal{E}(-\widetilde{L}_t^*) = \exp\left\{-\int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)} + \frac{1}{2}\int_0^{t\wedge\zeta} b^2(Y_u)du\right\}.$$

Proof. For statement (1), for details of the proof, refer to the proofs of Theorem 2 in Ruf (2013b), and Theorem 2.1 in Carr, Fisher and Ruf (2013).

For statement (2), we need to show that $\widetilde{L}_t^* = \int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)} - \int_0^{t\wedge\zeta} b^2(Y_u)du$ is a $\widetilde{P}$-local-martingale on $[0, T_\infty)$. Denote R_n as the first hitting time of Z to the level n, and set $\tau_n = R_n \wedge n$ for all $n \in \mathbb{N}$. Let U_n denote the first hitting time of $\frac{1}{n}$ by Z. This is done by showing that, with $\tau_n \wedge U_n$ the first passage time to either n or $\frac{1}{n}$, $\widetilde{L}^*_{t\wedge\tau_n\wedge U_n} = \int_0^{t\wedge\zeta\wedge\tau_n\wedge U_n} b(Y_u)dW_u^{(1)} - \int_0^{t\wedge\zeta\wedge\tau_n\wedge U_n} b^2(Y_u)du$ is a $\widetilde{P}$-local martingale. This follows from $\widetilde{P}(\lim_{n\to\infty} \tau_n \wedge U_n = T_\infty) = 1$, the Girsanov theorem (Ch.VIII, Theorem 1.4 in Revuz and Yor (1999)) and the equivalence of P and $\widetilde{P}$ on

to our setting of stochastic volatility models based on diffusions. For non-negative local martingales, there is a more general result in Theorem 2.1, p6 of Carr, Fisher and Ruf (2013).

$\mathcal{F}_{\tau_n \wedge U_n}$.

For statement (3), under $\widetilde{P}$, for $t < T_\infty$

$$\begin{aligned}\frac{1}{Z_t} &= \exp\left\{-\int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)} + \frac{1}{2}\int_0^{t\wedge\zeta} b^2(Y_u)du\right\}\\ &= \exp\left\{-\int_0^{t\wedge\zeta} b(Y_u)dW_u^{(1)} + \int_0^{t\wedge\zeta} b^2(Y_u)du - \frac{1}{2}\int_0^{t\wedge\zeta} b^2(Y_u)du\right\}\\ &= \mathcal{E}(-\widetilde{L}_t^*).\end{aligned}$$

This completes the proof. □

Proposition 2.2.2. *(Corollary 2, Ruf (2013b), case of time-homogeneous diffusions)*

Assume conditions (2.2) *and* (2.9)*, for* $T \in [0, \infty)$*,* Z_t *is a* P*-martingale for* $t \in [0, T]$*, i.e.* $\mathbb{E}^P[Z_T] = 1$*, if and only if* $\widetilde{P}\left(\frac{1}{Z_T} > 0\right) = 1$.

Proof. Denote R_n as the first hitting time of Z to the level n. Define $\tau_n = R_n \wedge n$ for all $n \in \mathbb{N}$. Since $\widetilde{P}$ is absolutely continuous with respect to P on $\mathcal{F}_{\tau_n}$, we have $\tau_n \uparrow T_\infty$ both P-a.s. and $\widetilde{P}$-a.s., as $n \to \infty$. For $T \in [0, \infty)$, substitute $\nu = T \wedge \tau_n$ (note that $\nu < T_\infty$ both P-a.s. and $\widetilde{P}$-a.s.) and $S = Z_{T\wedge\tau_n} \geqslant 0$ for $n \in \mathbb{N}$ into the equation (2.11) of Proposition 2.2.1

$$\mathbb{E}^{\widetilde{P}}\left[\frac{1}{Z_{T\wedge\tau_n}} Z_{T\wedge\tau_n} \mathbb{1}_{\infty > Z_{T\wedge\tau_n} > 0}\right] = \mathbb{E}^P\left[Z_{T\wedge\tau_n} \mathbb{1}_{Z_{T\wedge\tau_n} > 0}\right].$$

Equivalently

$$\widetilde{P}\left(\frac{1}{Z_{T\wedge\tau_n}} > 0\right) = \mathbb{E}^P[Z_{T\wedge\tau_n} \mathbb{1}_{Z_{T\wedge\tau_n} > 0}]. \tag{2.12}$$

Since τ_n is non-negative and non-decreasing, by the monotone convergence theorem taking limits on both sides of (2.12) as $n \to \infty$

$$
\begin{aligned}
\widetilde{P}\left(\frac{1}{Z_{T\wedge T_\infty}} > 0\right) &= \mathbb{E}^P[Z_{T\wedge T_\infty}\mathbb{1}_{Z_{T\wedge T_\infty}>0}] \\
&= \mathbb{E}^P\left[Z_T\mathbb{1}_{Z_T>0}\right], \quad \text{since } P(T_\infty = \infty) = 1, \\
&= \mathbb{E}^P[Z_T], \quad \text{since } \mathbb{E}^P\left[Z_T\mathbb{1}_{Z_T=0}\right] = 0.
\end{aligned}
$$

Note that ∞ is an absorbing boundary for Z, then $Z_{T\wedge T_\infty} = Z_T$ both P-a.s. and $\widetilde{P}$-a.s. for $T \in [0,\infty)$. Therefore $\mathbb{E}^P[Z_T] = 1$ if and only if $\widetilde{P}\left(\frac{1}{Z_T} > 0\right) = 1$. This completes the proof. □

Now we seek to determine the SDE satisfied by Y under $\widetilde{P}$.

Proposition 2.2.3. *Assume the conditions* (2.2) *and* (2.9) *for the SDE* (2.1). *Under $\widetilde{P}$, for $-1 \leqslant \rho \leqslant 1$, the diffusion Y satisfies the following SDE up to ζ*

$$
dY_t = (\mu(Y_t) + \rho b(Y_t)\sigma(Y_t))\mathbb{1}_{t\in[0,\zeta)}dt + \sigma(Y_t)\mathbb{1}_{t\in[0,\zeta)}d\widetilde{W}_t, \quad Y_0 = x_0. \tag{2.13}
$$

Proof. Consider the system of SDEs in (2.8), from the Cholesky decomposition, $dW_t^{(1)} = \rho dW_t + \sqrt{1-\rho^2}dW_t^{(2)}$, where W and $W^{(2)}$ are standard independent Brownian motions under P.

Define for $t \in [0,\infty)$

$$
\widetilde{W}_t := \begin{cases} W_t - \rho\int_0^t b(Y_u)du, & \text{if } t < \zeta, \\ W_\zeta - \rho\int_0^\zeta b(Y_u)du + \widetilde{\beta}_{t-\zeta}, & \text{if } t \geqslant \zeta, \end{cases} \tag{2.14}
$$

where $\widetilde{\beta}$ is a standard $\widetilde{P}$-Brownian motion independent of W with $\widetilde{\beta}_0 = 0$.

Define $\xi_n = \zeta \wedge \tau_n$, and consider the process $\widetilde{W}$ up to ξ_n. Since $\mathcal{F}_{\xi_n} \subset \mathcal{F}_{\tau_n}$, it follows from Proposition 2.2.1 that $\widetilde{P}$ restricted to $\mathcal{F}_{\xi_n}$ is absolutely continuous with respect to P restricted to $\mathcal{F}_{\xi_n}$ for $n \in \mathbb{N}$. Then from the

Girsanov Theorem (Ch.VIII, Theorem 1.12, p331 of Revuz and Yor (1999))

$$
\begin{aligned}
&W_t - \langle W_t, \int_0^t b(Y_u) dW_u^{(1)} \rangle \\
&= W_t - \langle W_t, \rho \int_0^t b(Y_u) dW_u \rangle - \langle W_t, \sqrt{1-\rho^2} \int_0^t b(Y_u) dW_u^{(2)} \rangle \\
&= W_t - \rho \int_0^t b(Y_u) du. \\
&:= \widetilde{W}_t
\end{aligned}
$$

is a $\widetilde{P}$-Brownian motion for $t \in [0, \xi_n)$ and $n \in \mathbb{N}$.

We first prove a lemma concerning the relative magnitude of ζ and T_∞.

Lemma 2.2.4. *Assume the conditions* (2.2) *and* (2.9) *for the SDE* (2.1), *then we have* $\zeta \leqslant T_0 \wedge T_\infty$, *$P$-a.s. and $\widetilde{P}$-a.s.*

Proof. We aim to prove the following four statements:

(i) $P(\zeta \leqslant T_\infty) = 1$,

(ii) $\widetilde{P}(\zeta \leqslant T_\infty) = 1$,

(iii) $P(\zeta \leqslant T_0) = 1$,

and (iv) $\widetilde{P}(\zeta \leqslant T_0) = 1$.

Since $P(T_\infty = \infty) = 1$, statement (i) follows trivially.

For statement (ii), under $\widetilde{P}$, clearly $\widetilde{P}(\zeta \leqslant T_\infty, T_\infty = \infty) = 1$ follows. On the set $\{T_\infty < \infty\}$, we want to apply Lemma 2.2.1, the proof of which is on p5 of Mijatović and Urusov (2012c). Note that their proof requires that Y is continuous on the stochastic interval $[0, \zeta)$ (which is satisfied in our setting), and also needs $\langle Y, Y \rangle_t = \int_0^t \sigma^2(Y_u) du$ to hold. Note that the change of measure from P to $\widetilde{P}$ does not modify the quadratic variation of Y, and thus $\langle Y, Y \rangle_t = \int_0^t \sigma^2(Y_u) du$ holds both P-a.s. and $\widetilde{P}$-a.s. Given that the condition (2.9) is satisfied, Lemma 2.2.1 also works for $\widetilde{P}$. Then

$$
\widetilde{P}\left(\int_0^{T_\infty \wedge \zeta} b^2(Y_u) du < \infty, T_\infty < \zeta \right) = \widetilde{P}(T_\infty < \zeta).
$$

Note that by definition $\widetilde{P}(Z_{T_\infty} = \infty) = 1$, then

$$\begin{aligned}
\widetilde{P}(T_\infty < \zeta) &= \widetilde{P}\left(\int_0^{T_\infty \wedge \zeta} b^2(Y_u)du < \infty, T_\infty < \zeta\right) \\
&= \widetilde{P}\left(\int_0^{T_\infty} b^2(Y_u)du < \infty, T_\infty < \zeta, Z_{T_\infty} = \infty\right) \\
&= \widetilde{P}\left(\langle L\rangle_{T_\infty} < \infty, T_\infty < \zeta, \ln(Z_{T_\infty}) = \infty\right) \\
&= \widetilde{P}\left(\langle L\rangle_{T_\infty} < \infty, T_\infty < \zeta, \ln(Z_{T_\infty}) = \langle L\rangle_{T_\infty}\left(\frac{L_{T_\infty}}{\langle L\rangle_{T_\infty}} - \frac{1}{2}\right) = \infty\right) \\
&= \widetilde{P}\left(\langle L\rangle_{T_\infty} < \infty, T_\infty < \zeta, \ln(Z_{T_\infty}) = \langle L\rangle_{T_\infty}\left(\frac{B_{\langle L\rangle_{T_\infty}}}{\langle L\rangle_{T_\infty}} - \frac{1}{2}\right) = \infty\right) \\
&= 0,
\end{aligned}$$

and here the second last equality is due to the Dambis-Dubins-Schwartz theorem (Ch.V, Theorem 1.6, Revuz and Yor (1999)) for some Brownian motion B on an extended probability space. The last equality holds because $\frac{B_{\langle L\rangle_{T_\infty}}}{\langle L\rangle_{T_\infty}}$ is finite $\widetilde{P}$-a.s. on the set $\{\langle L\rangle_{T_\infty} < \infty\}$.

For statement (iii), clearly $P(\zeta \leqslant T_0, T_0 = \infty) = 1$ holds. On the set $\{T_0 < \infty\}$, note that by definition we have $\{Z_t = 0\} = \{T_0 \leqslant t < \infty\}$ under P. From Corollary 2.2.1, under P, we have

$$\{T_0 \leqslant t < \infty\} = \{Z_t = 0\} = \left\{\zeta \leqslant t < \infty, \int_0^\zeta b^2(Y_u) = \infty\right\} \subset \{\zeta \leqslant t < \infty\},$$

then clearly $P(\zeta \leqslant T_0) = 1$.

For statement (iv), since $\widetilde{P}(T_0 = \infty) = 1$ holds as a consequence of the proof of Proposition 2.2.1, the result follows. This completes the proof of the lemma. □

From monotone convergence, $\widetilde{P}(\lim_{n\to\infty} \tau_n = T_\infty) = 1$ and $\widetilde{P}(\lim_{n\to\infty} \xi_n = \zeta \wedge T_\infty) = 1$ hold. From Lemma 2.2.4, $\widetilde{P}(\zeta \leqslant T_\infty) = 1$, thus $\widetilde{P}(\lim_{n\to\infty} \xi_n = \zeta) = 1$. Recall that $\widetilde{W}_t$ is a standard $\widetilde{P}$-Brownian motion for $t \in [0, \xi_n)$. Taking limits as $n \to \infty$, then $\widetilde{W}_t$ is a standard $\widetilde{P}$-Brownian motion for $t \in [0, \zeta)$. From the construction in (2.14), it follows that $\widetilde{W}_t$ is a standard

$\widetilde{P}$-Brownian motion for $t \in [0, \infty)$.

Thus Y is governed by the following SDE under $\widetilde{P}$ for $t \in [0, \zeta)$

$$\begin{aligned} dY_t &= \mu(Y_t)dt + \sigma(Y_t)\left(d\widetilde{W}_t + \rho b(Y_t)dt\right) \\ &= (\mu(Y_t) + \rho b(Y_t)\sigma(Y_t))dt + \sigma(Y_t)d\widetilde{W}_t, \quad Y_0 = x_0. \end{aligned} \tag{2.15}$$

This completes the proof. □

In order to verify $\mathbb{E}^P[Z_T] = 1$, the equivalent condition in Proposition 2.2.2 can be transformed into a condition related to integral functionals of Y under $\widetilde{P}$ as shown in the following proposition.

Proposition 2.2.4. *Assume*[5] *conditions* (2.2) *and* (2.9), *for* $T \in [0, \infty)$, Z_t *is a* P*-martingale for* $t \in [0, T]$, *i.e.* $\mathbb{E}^P[Z_T] = 1$, *if and only if* $\widetilde{P}\left(\int_0^{T\wedge\zeta} b^2(Y_u)du < \infty\right) = 1$.

Proof. Define $\tau_n = R_n \wedge n$ for all $n \in \mathbb{N}$ similarly as before. The left hand side of (2.12) can be rewritten as

$$\begin{aligned} \widetilde{P}\left(\frac{1}{Z_{T\wedge\tau_n}} > 0\right) &= \widetilde{P}\left(\frac{1}{Z_{T\wedge\tau_n}} = \mathcal{E}(-\widetilde{L}^*_{T\wedge\tau_n}) > 0\right) \\ &= \widetilde{P}\left(\langle -\widetilde{L}^*\rangle_{T\wedge\tau_n} < \infty\right) \\ &= \widetilde{P}\left(\int_0^{T\wedge\zeta\wedge\tau_n} b^2(Y_u)du < \infty\right), \end{aligned} \tag{2.16}$$

and the first equality is because of $\widetilde{P}(T \wedge \tau_n < T_\infty) = 1$ for $n \in \mathbb{N}$, and Proposition 2.2.1(3). The second equality is because of Lemma 2.2.3 applied to the stochastic exponential $\mathcal{E}(-\widetilde{L}^*_{T\wedge\tau_n})$. Since τ_n is non-negative and non-decreasing, by the monotone convergence theorem taking limits

[5] A similar result also appears in Theorem 1, p6 of Ruf (2013a).

on both sides of (2.16) as $n \to \infty$

$$\widetilde{P}\left(\frac{1}{Z_{T\wedge T_\infty}} > 0\right) = \widetilde{P}\left(\int_0^{T\wedge\zeta\wedge T_\infty} b^2(Y_u)du < \infty\right)$$
$$= \widetilde{P}\left(\int_0^{T\wedge\zeta} b^2(Y_u)du < \infty\right), \tag{2.17}$$

and the last equality is because $\widetilde{P}(\zeta \leqslant T_\infty) = 1$ from Lemma 2.2.4. From (2.17) combined with Proposition 2.2.2, and note that $Z_{T\wedge T_\infty} = Z_T$, $\widetilde{P}$-a.s., then for $T \in [0,\infty)$

$$\mathbb{E}^P[Z_T] = \widetilde{P}\left(\frac{1}{Z_T} > 0\right) = \widetilde{P}\left(\int_0^{T\wedge\zeta} b^2(Y_u)du < \infty\right). \tag{2.18}$$

This completes the proof. □

The following is the necessary and sufficient condition for the uniform integrable martingale.

Proposition 2.2.5. *Assume conditions* (2.2) *and* (2.9). *Then Z is a uniformly integrable P-martingale on $[0,\infty]$, i.e. $\mathbb{E}^P[Z_\infty] = 1$, if and only if*

$$\widetilde{P}\left(\int_0^{\zeta} b^2(Y_u)du < \infty\right) = 1.$$

Proof. Recall from Proposition 2.2.4

$$\mathbb{E}^P[Z_T] = \widetilde{P}\left(\int_0^{T\wedge\zeta} b^2(Y_u)du < \infty\right), \quad \text{for } T \in [0,\infty). \tag{2.19}$$

Similar as the proof of Lemma 2.2.3, consider a sequence of stopping times $\xi_n \to \infty$, $n \in \mathbb{N}$, such that Z_{ξ_n} is a martingale. Then by the submartingale convergence theorem (Theorem 1.3.15, p17, Karatzas and Shreve (1991)), $Z_\infty := \lim_{n\to\infty} Z_{\xi_n}$ exists and is a non-negative continuous local martingale, and thus a supermartingale due to Fatou's lemma. Then the left hand side of (2.19) has a well-defined limit $\mathbb{E}^P[Z_\infty]$ as $T \to \infty$. Since $\int_0^{T\wedge\zeta} b^2(Y_u)du < \infty$ is non-negative and non-decreasing, from the monotone convergence theorem, we have that the right hand side of (2.19) also has a well-defined limit $\widetilde{P}\left(\int_0^\zeta b^2(Y_u)du < \infty\right)$ as $T \to \infty$. From the above analysis, as $T \to \infty$ on both sides of (2.19)

$$\mathbb{E}^P[Z_\infty] = \widetilde{P}\left(\int_0^\zeta b^2(Y_u)du < \infty\right). \tag{2.20}$$

Thus $\mathbb{E}^P[Z_\infty] = 1$ if and only if $\widetilde{P}\left(\int_0^\zeta b^2(Y_u)du < \infty\right) = 1$. This completes the proof. $\square$

2.3 Classification of convergence properties of integral functionals of time-homogeneous diffusions

The Engelbert-Schmidt zero-one law was initially proved in the Brownian motion case (see Engelbert and Schmidt (1981) or Proposition 3.6.27, p216 of Karatzas and Shreve (1991)). Engelbert and Tittel (2002) obtain a generalized Engelbert-Schmidt type zero-one law for the integral functional $\int_0^t f(X_s)ds$, where f is a non-negative Borel function and X is a strong Markov continuous local martingale. In an expository paper, Mijatović and Urusov (2012a) consider the case of a one-dimensional time-homogeneous diffusion and their Theorem 2.11 gives the corresponding zero-one law results. They provide two proofs that circumvent the use of Jeulin's lemma. The first proof is based on William's theorem (Ch.VII, Corollary 4.6, p317, Revuz and Yor (1999)). The second proof is based on the first Ray-Knight theorem (Ch.XI, Theorem 2.2, p455, Revuz and Yor (1999)).

Recall the scale function $s(.)$ defined in (2.10), and introduce the following test functions for $x \in \bar{J}$, with a constant $c \in J$.

$$\begin{aligned} v(x) &\equiv \int_c^x (s(x) - s(y)) \frac{2}{s'(y)\sigma^2(y)} dy, \\ v_b(x) &\equiv \int_c^x (s(x) - s(y)) \frac{2b^2(y)}{s'(y)\sigma^2(y)} dy. \end{aligned} \tag{2.21}$$

Note that if $s(\infty) = \infty$, then $v(\infty) = \infty$ and $v_b(\infty) = \infty$ by the definition in (2.21). Define $\widetilde{s}(.)$, $\widetilde{v}(.)$ and $\widetilde{v}_b(.)$ similarly based on the SDE (2.13) under $\widetilde{P}$.

We have the following Engelbert-Schmidt type zero-one law for the SDE (2.1) under P, which is precisely the Theorem 2.11 of Mijatović and Urusov (2012a) with $f(.) = b^2(.)$ using our notation.

Proposition 2.3.1. *(Engelbert-Schmidt type zero-one law for a time-homogeneous diffusion, Theorem* 2.11 *of Mijatović and Urusov (2012a))*

Assume that the function $f = b^2 : J \to [0, \infty]$ *satisfies* $b^2/\sigma^2 \in L^1_{loc}(J)$, *and let* $s(r) < \infty$.

(i)If $v_b(r) < \infty$, *then* $\int_0^\zeta b^2(Y_u)du < \infty$, P*-a.s. on* $\left\{\lim_{t\to\zeta} Y_t = r\right\}$.

(ii)If $v_b(r) = \infty$, *then* $\int_0^\zeta b^2(Y_u)du = \infty$, P*-a.s. on* $\left\{\lim_{t\to\zeta} Y_t = r\right\}$.

The analogous results on the set $\{\lim_{t\to\zeta} Y_t = l\}$ *can be similarly stated.*

Clearly the above proposition has a counterpart for the SDE (2.13) under $\widetilde{P}$.

Proposition 2.3.2. *Assume that the function* $f = b^2 : J \to [0, \infty]$ *satisfies* $b^2/\sigma^2 \in L^1_{loc}(J)$, *and let* $\widetilde{s}(r) < \infty$.

(i)If $\widetilde{v}_b(r) < \infty$, *then* $\int_0^\zeta b^2(Y_u)du < \infty$, $\widetilde{P}$*-a.s. on* $\left\{\lim_{t\to\zeta} Y_t = r\right\}$.

(ii)If $\widetilde{v}_b(\ell) < \infty$, *then* $\int_0^\zeta b^2(Y_u)du = \infty$, $\widetilde{P}$*-a.s. on* $\left\{\lim_{t\to\zeta} Y_t = r\right\}$.

Analogous results on the set $\{\lim_{t\to\zeta} Y_t = l\}$ *can be similarly stated.*

The following result is Proposition 5.5.22 on p345 of Karatzas and Shreve (1991) using our notation. It classifies possible exit behaviors of the process Y at the boundaries of its state space J under P.

Proposition 2.3.3. *(Proposition* 5.5.22, *Karatzas and Shreve (1991))*

Assume the conditions (2.2) *and* (2.9), *let* Y *be a weak solution of* (2.1) *in* J *under* P, *with nonrandom initial condition* $Y_0 = x_0 \in J$. *Distinguish four cases:*

(a) If $s(\ell) = -\infty$ *and* $s(r) = \infty$*, then* $P(\zeta = \infty) = P(\sup_{0 \leqslant t < \infty} Y_t = r) = P(\inf_{0 \leqslant t < \infty} Y_t = \ell) = 1$.

(b) If $s(\ell) > -\infty$ *and* $s(r) = \infty$*, then* $P(\lim_{t \to \zeta} Y_t = \ell) = P(\sup_{0 \leqslant t < \zeta} Y_t < r) = 1$.

(c) If $s(\ell) = -\infty$ *and* $s(r) < \infty$*, then* $P(\lim_{t \to \zeta} Y_t = r) = P(\inf_{0 \leqslant t < \zeta} Y_t > l) = 1$.

(d) If $s(\ell) > -\infty$ *and* $s(r) < \infty$*, then* $P(\lim_{t \to \zeta} Y_t = \ell) = 1 - P(\lim_{t \to \zeta} Y_t = r) = \frac{s(r) - s(x_0)}{s(r) - s(\ell)}$. *Note that* $0 < \frac{s(r) - s(x_0)}{s(r) - s(\ell)} < 1$.

Analogous results also hold for the SDE (2.13) *under* $\widetilde{P}$.

Remark 2.3.1. *In the conditions* (*b*)*,* (*c*) *and* (*d*) *above, we make no claim concerning the finiteness of* ζ*. See Remark* 5.5.23 *on p*345 *of Karatzas and Shreve (1991). Note that conditions* (*b*) *and* (*c*) *are consequences of the expression in condition* (*d*) *by letting either* $s(r) = \infty$ *or* $s(\ell) = -\infty$*.*

Similar to the statements in Proposition 2.3.3, for the study of the convergence or divergence properties of integral functionals of time-homogeneous diffusions, we distinguish the following four exhaustive and disjoint cases under P:

Case (1): $s(\ell) = -\infty$, $s(r) = \infty$.

Case (2): $s(\ell) = -\infty$, $s(r) < \infty$.

Case (3): $s(\ell) > -\infty$, $s(r) = \infty$.

Case (4): $s(\ell) > -\infty$, $s(r) < \infty$.

Further divide each case above into the following subcases based on the finiteness of $v_b(r)$ and $v_b(\ell)$ as defined in (2.21):

Case (2) (i): $s(\ell) = -\infty$, $s(r) < \infty$, $v_b(r) = \infty$.

Case (2) (ii): $s(\ell) = -\infty$, $s(r) < \infty$, $v_b(r) < \infty$.

Case (3) (i): $s(\ell) > -\infty$, $s(r) = \infty$, $v_b(\ell) = \infty$.

Case (3) (ii): $s(\ell) > -\infty$, $s(r) = \infty$, $v_b(\ell) < \infty$.

Case (4) (i): $s(\ell) > -\infty$, $s(r) < \infty$, $v_b(r) = \infty$, $v_b(\ell) = \infty$.

Case (4) (ii): $s(\ell) > -\infty$, $s(r) < \infty$, $v_b(r) < \infty$, $v_b(\ell) = \infty$.

Case (4) (iii): $s(\ell) > -\infty$, $s(r) < \infty$, $v_b(r) = \infty$, $v_b(\ell) < \infty$.

Case (4) (iv): $s(\ell) > -\infty$, $s(r) < \infty$, $v_b(r) < \infty$, $v_b(\ell) < \infty$.

Remark 2.3.2. *Define* $\varphi_t := \int_0^t b^2(Y_u)du$, *for* $t \in [0, \zeta]$. *Recall that* $b^2(.)$ *is a non-negative Borel function, thus* φ_t *is a non-decreasing function for* $t \in [0, \zeta]$. φ_t *is in the form of a time integral, and it is clear that it is continuous for* $t \in [0, \zeta)$, *and is left continuous at* $t = \zeta$.

We now apply the Engelbert-Schmidt type zero-one law under P as in Proposition 2.3.1 to determine whether $P(\varphi_\zeta < \infty) = 1$ or $P(\varphi_\zeta = \infty) = 1$ in each of the cases above. We first prove two lemmas.

Lemma 2.3.1. *Assume the conditions* (2.2) *and* (2.9), *then "*$v_b(\ell) = \infty$ *and* $v_b(r) = \infty$*" are necessary and sufficient for* $P(\varphi_\zeta = \infty) = 1$.

Proof. For the sufficiency, divide into the following four distinct cases:

Case (1): $s(\ell) = -\infty$, $s(r) = \infty$. From Proposition 2.3.3 (d), $P(\zeta = \infty) = 1$. This combined with Lemma 2.2.1 implies $P(\varphi_\zeta = \infty) = 1$.

Case (2): $s(\ell) = -\infty$, $s(r) < \infty$. From Proposition 2.3.3 (c), $P(\lim_{t\to\zeta} Y_t = r) = 1$. If $v_b(r) = \infty$, then from Proposition 2.3.1 and Proposition 2.3.3, $P(\varphi_\zeta = \infty) = P(\varphi_\zeta = \infty, \lim_{t\to\zeta} Y_t = r) = P(\lim_{t\to\zeta} Y_t = r) = 1$.

Case (3): $s(\ell) > -\infty$, $s(r) = \infty$. The proof is similar to Case (2) above by switching the roles of ℓ and r, and applying Proposition 2.3.3 (b) and Proposition 2.3.1.

Case (4): $s(\ell) > -\infty$, $s(r) < \infty$. From Proposition 2.3.3 (d), $0 < p = P(\lim_{t\to\zeta} Y_t = r) < 1$. Since $v_b(r) = \infty$ and $v_b(\ell) = \infty$, from Proposition 2.3.1 for both cases of r and ℓ, $P(\varphi_\zeta = \infty) = P(\varphi_\zeta = \infty, \lim_{t\to\zeta} Y_t = r) + P(\varphi_\zeta = \infty, \lim_{t\to\zeta} Y_t = \ell) = P(\lim_{t\to\zeta} Y_t = r) + P(\lim_{t\to\zeta} Y_t = \ell) = 1$.

For the necessity, we only need to prove the contrapositive statement: "If at least one of $v_b(\ell)$ or $v_b(r)$ is finite, then $P(\varphi_\zeta = \infty) < 1$." Without loss of generality, assume that $v_b(\ell) < \infty$, because the case of $v_b(r) < \infty$ can be similarly proved. From Proposition 2.3.1, $P(\varphi_\zeta = \infty, \lim_{t\to\zeta} Y_t = \ell) = 0$.

Then $P(\varphi_\zeta = \infty) = P(\varphi_\zeta = \infty, \lim_{t\to\zeta} Y_t = \ell) + P(\varphi_\zeta = \infty, \lim_{t\to\zeta} Y_t = r) = P(\varphi_\zeta = \infty, \lim_{t\to\zeta} Y_t = r) \leqslant P(\lim_{t\to\zeta} Y_t = r)$. Divide into two cases:

Case (i): $s(\ell) > -\infty$, $s(r) = \infty$. From Proposition 2.3.3 (b), $P(\lim_{t\to\zeta} Y_t = r) = 0$.

Case (ii): $s(\ell) > -\infty$, $s(r) < \infty$. From Proposition 2.3.3 (d), $0 < p = P(\lim_{t\to\zeta} Y_t = r) < 1$.

In both cases $P(\lim_{t\to\zeta} Y_t = r) < 1$, thus $P(\varphi_\zeta = \infty) < 1$, and the necessity follows. This completes the proof. □

Lemma 2.3.2. *Assume the conditions* (2.2) *and* (2.9)*, and* $s(\ell) > -\infty$*,* $s(r) < \infty$*, then "*$v_b(\ell) < \infty$ *and* $v_b(r) < \infty$*" are necessary and sufficient for* $P(\varphi_\zeta < \infty) = 1$*.*

Proof. With $s(\ell) > -\infty$ and $s(r) < \infty$, denote $p = P(\lim_{t\to\zeta} Y_t = r) = 1 - P(\lim_{t\to\zeta} Y_t = \ell)$. From Proposition 2.3.3 (d), we have that $0 < p < 1$. For the sufficiency, assume that $v_b(\ell) < \infty$ and $v_b(r) < \infty$ hold, we aim to prove that $P(\varphi_\zeta < \infty) = 1$.

From Proposition 2.3.1, $P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = r) = P(\lim_{t\to\zeta} Y_t = r)$ and $P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = \ell) = P(\lim_{t\to\zeta} Y_t = \ell)$. Then

$$\begin{aligned} P(\varphi_\zeta < \infty) &= P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = r) + P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = \ell) \\ &= P(\lim_{t\to\zeta} Y_t = r) + P(\lim_{t\to\zeta} Y_t = \ell) = 1. \end{aligned}$$

For the necessity, we only need to prove the contrapositive argument: "If at least one of $v_b(\ell)$ and $v_b(r)$ is infinite, then $P(\varphi_\zeta < \infty) < 1$." Without loss of generality, assume that $v_b(\ell) = \infty$, because the case of $v_b(r) = \infty$ can be similarly proved. From Proposition 2.3.1, $P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = \ell) = 0$,

and

$$\begin{aligned} P(\varphi_\zeta < \infty) &= P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = r) + P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = \ell) \\ &= P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = r) \\ &\leqslant P(\lim_{t\to\zeta} Y_t = r) < 1. \end{aligned}$$

Thus the necessity follows. This completes the proof. □

We now give a detailed study of the function $\varphi_t, t \in [0, \zeta]$ under P using the Engelbert-Schmidt type zero-one law.

Theorem 2.3.1. *Assume the conditions* (2.2) *and* (2.9)*. Then we have the following properties*[6] *for the function* $\varphi_t, t \in [0, \zeta]$ *under* P.

(i) $\varphi_t < \infty$ *P-a.s. on* $\{0 \leqslant t < \zeta\}$.

(ii) $P(\varphi_\zeta < \infty) = 1$ *if and only if at least one of the following conditions is satisfied:*

(a) $v_b(r) < \infty$ *and* $s(\ell) = -\infty$,

(b) $v_b(\ell) < \infty$ *and* $s(r) = \infty$,

(c) $v_b(r) < \infty$ *and* $v_b(\ell) < \infty$.

(iii) $P(\varphi_\zeta = \infty) = 1$ *if and only if* $v_b(r) = \infty$ *and* $v_b(\ell) = \infty$.

Remark 2.3.3. *We summarize the results of Theorem 2.3.1 in Table 2.1.*

Proof. Statement (i) follows from Lemma 2.2.1.

For statement (ii), the detailed proof for each of the case in the table is as follows:

Case (1): from Lemma 2.2.2, $P(\varphi_\zeta = \infty) = 1$.

Case (2): from Proposition 2.3.3, $P(\lim_{t\to\zeta} Y_t = r) = 1$. For the two

[6]In Khoshnevisan, Salminen, and Yor (2006), they obtained deterministic criteria for the convergence or divergence of perpetual integral functionals of time-homogeneous diffusions. They also consider weak solutions to the SDE similar to (2.1). However, in Theorem 2, p110 of Khoshnevisan, Salminen, and Yor (2006), they assume the twice differentiability of the function $g(.)$ defined in their paper, while our assumptions here concern the local integrability of certain deterministic functions and are weaker.

Case	$s(\ell)$	$s(r)$	$v_b(\ell)$	$v_b(r)$	Conclusion
(1)	$-\infty$	∞	∞	∞	$P(\varphi_\zeta < \infty) = 0, \quad Z_\infty = 0$, P-a.s.
(2)(i)	$-\infty$	$< \infty$	∞	∞	$P(\varphi_\zeta < \infty) = 0, \quad Z_\infty = 0$, P-a.s.
(2)(ii)	$-\infty$	$< \infty$	∞	$< \infty$	$P(\varphi_\zeta < \infty) = 1, \quad Z_\infty > 0$, P-a.s.
(3)(i)	$> -\infty$	∞	∞	∞	$P(\varphi_\zeta < \infty) = 0, \quad Z_\infty = 0$, P-a.s.
(3)(ii)	$> -\infty$	∞	$< \infty$	∞	$P(\varphi_\zeta < \infty) = 1, \quad Z_\infty > 0$, P-a.s.
(4)(i)	$> -\infty$	$< \infty$	∞	∞	$P(\varphi_\zeta < \infty) = 0, \quad Z_\infty = 0$, P-a.s.
(4)(ii)	$> -\infty$	$< \infty$	∞	$< \infty$	$0 < P(\varphi_\zeta < \infty) < 1, \quad 0 < P(Z_\infty = 0) < 1$
(4)(iii)	$> -\infty$	$< \infty$	$< \infty$	∞	$0 < P(\varphi_\zeta < \infty) < 1, \quad 0 < P(Z_\infty = 0) < 1$
(4)(iv)	$> -\infty$	$< \infty$	$< \infty$	$< \infty$	$P(\varphi_\zeta < \infty) = 1, \quad Z_\infty > 0$, P-a.s.

Table 2.1: Classification table for the positivity of the stock price

subcases, from Proposition 2.3.1, Case (2) (i) is necessary and sufficient for $P(\varphi_\zeta = \infty) = 1$, and Case (2) (ii) is necessary and sufficient for $P(\varphi_\zeta < \infty) = 1$.

Case (3): from Proposition 2.3.3, $P(\lim_{t\to\zeta} Y_t = \ell) = 1$. For the two subcases, from Proposition 2.3.1, Case (3) (i) is necessary and sufficient for $P(\varphi_\zeta = \infty) = 1$, and Case (3) (ii) is necessary and sufficient for $P(\varphi_\zeta < \infty) = 1$.

Case (4): from Proposition 2.3.3, $1 > p = P(\lim_{t\to\zeta} Y_t = r) = 1 - P(\lim_{t\to\zeta} Y_t = \ell) > 0$. For individual subcases:

Case (4) (i) is necessary and sufficient for $P(\varphi_\zeta = \infty) = 1$ from Lemma 2.3.1.

Case (4) (ii): from Proposition 2.3.1, $v_b(\ell) = \infty$ implies that $P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = \ell) = 0$. Then $P(\varphi_\zeta < \infty) = P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = r) + P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = \ell) = P(\varphi_\zeta < \infty, \lim_{t\to\zeta} Y_t = r) \leqslant P(\lim_{t\to\zeta} Y_t = r) < 1$.

Case (4) (iii): the proof is similar to the proof of Case (4) (ii) by switching the roles of ℓ and r.

Case (4) (iv) is necessary and sufficient for $P(\varphi_\zeta < \infty) = 1$ from Lemma 2.3.2.

The classification above is exhaustive, and Table 2.1 follows. The "conclusion" column in Table 2.1 is based on the classification of whether $P(\varphi_\zeta < \infty) = 1$ or $P(\varphi_\zeta = \infty) = 1$ combined with Lemma 2.2.3.

Statement (iii) follows from Lemma 2.3.1. This completes the proof. $\square$

We have the following corollary of Theorem 2.3.1 under $\widetilde{P}$

Corollary 2.3.1. *Assume the conditions* (2.2) *and* (2.9), *then*

(i) $\varphi_t < \infty$ $\widetilde{P}$*-a.s. on* $\{0 \leqslant t < \zeta\}$.

(ii) $\widetilde{P}(\varphi_\zeta < \infty) = 1$ *if and only if at least one of the following conditions is satisfied:*

(a) $\widetilde{v}_b(r) < \infty$ *and* $\widetilde{s}(\ell) = -\infty$,

(b) $\widetilde{v}_b(\ell) < \infty$ *and* $\widetilde{s}(r) = \infty$,

(c) $\widetilde{v}_b(r) < \infty$ *and* $\widetilde{v}_b(\ell) < \infty$.

(iii) $\widetilde{P}(\varphi_\zeta = \infty) = 1$ *if and only if* $\widetilde{v}_b(r) = \infty$ *and* $\widetilde{v}_b(\ell) = \infty$.

Remark 2.3.4. *We summarize the results of Corollary 2.3.1 in Table 2.2.*

Case	$\widetilde{s}(\ell)$	$\widetilde{s}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$	Conclusion	U.I. Mart.
(1)	$-\infty$	∞	∞	∞	$\widetilde{P}(\varphi_\zeta < \infty) = 0, \quad \mathbb{E}^P[Z_\infty] < 1$	No
(2)(i)	$-\infty$	$< \infty$	∞	∞	$\widetilde{P}(\varphi_\zeta < \infty) = 0, \quad \mathbb{E}^P[Z_\infty] < 1$	No
(2)(ii)	$-\infty$	$< \infty$	∞	$< \infty$	$\widetilde{P}(\varphi_\zeta < \infty) = 1, \quad \mathbb{E}^P[Z_\infty] = 1$	Yes
(3)(i)	$> -\infty$	∞	∞	∞	$\widetilde{P}(\varphi_\zeta < \infty) = 0, \quad \mathbb{E}^P[Z_\infty] < 1$	No
(3)(ii)	$> -\infty$	∞	$< \infty$	∞	$\widetilde{P}(\varphi_\zeta < \infty) = 1, \quad \mathbb{E}^P[Z_\infty] = 1$	Yes
(4)(i)	$> -\infty$	$< \infty$	∞	∞	$\widetilde{P}(\varphi_\zeta < \infty) = 0, \quad \mathbb{E}^P[Z_\infty] < 1$	No
(4)(ii)	$> -\infty$	$< \infty$	∞	$< \infty$	$0 < \widetilde{P}(\varphi_\zeta < \infty) < 1, \quad \mathbb{E}^P[Z_\infty] < 1$	No
(4)(iii)	$> -\infty$	$< \infty$	$< \infty$	∞	$0 < \widetilde{P}(\varphi_\zeta < \infty) < 1, \quad \mathbb{E}^P[Z_\infty] < 1$	No
(4)(iv)	$> -\infty$	$< \infty$	$< \infty$	$< \infty$	$\widetilde{P}(\varphi_\zeta < \infty) = 1, \quad \mathbb{E}^P[Z_\infty] = 1$	Yes

Table 2.2: Classification table for the uniformly integrable martingale (U.I.Mart.)

Proof. The proof is similar to that of Theorem 2.3.1, and is thus omitted. We can construct Table 2.2. The "conclusion" column in Table 2.2 is based on the classification of whether $\widetilde{P}(\varphi_\zeta < \infty) = 1$ or $\widetilde{P}(\varphi_\zeta = \infty) = 1$

combined with Proposition 2.2.5. The "U.I.Mart." column in Table 2.2 is based on the classifications in the "conclusion" column in Table 2.2. This completes the proof. ☐

The following result provides necessary and sufficient conditions for $P(\varphi_{\zeta\wedge T} < \infty) = 1$, for $T \in [0, \infty)$.

Theorem 2.3.2. *Assume the conditions* (2.2) *and* (2.9)*, then for all* $T \in [0, \infty)$*,* $P(\varphi_{\zeta\wedge T} < \infty) = P(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty) = 1$ *if and only if at least one of the following conditions is satisfied:*

(a) $v(\ell) = v(r) = \infty$*,*

(b) $v_b(r) < \infty$ *and* $v(\ell) = \infty$*,*

(c) $v_b(\ell) < \infty$ *and* $v(r) = \infty$*,*

(d) $v_b(r) < \infty$ *and* $v_b(\ell) < \infty$*.*

Proof. First of all, from Feller's test of explosions, $v(\ell) < \infty$ if and only if $P(\zeta < \infty, \lim_{t\to\zeta} Y_t = l) > 0$. Similarly $v(r) < \infty$ if and only if $P(\zeta < \infty, \lim_{t\to\zeta} Y_t = r) > 0$.

On the set $\{\zeta = \infty\}$, from Lemma 2.2.1, $\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty$ P-a.s. Then $P(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta = \infty) = P(\zeta = \infty)$. We have the following decomposition

$$\begin{aligned}
&P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty\right) \\
&= P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta = \infty\right) + P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\quad + P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&= P(\zeta = \infty) + P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\quad + P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right). \qquad (2.22)
\end{aligned}$$

For the sufficiency, assuming that at least one of (a), (b), (c) and (d)

holds, we aim to prove $\varphi_{\zeta\wedge T} < \infty$ P-a.s.

Condition (a): from Feller's test of explosions, the condition (a) is equivalent to $P(\zeta = \infty) = 1$. Then from the decomposition (2.22), $P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty\right) = P(\zeta = \infty) = 1$.

Condition (b): $v(\ell) = \infty$ is equivalent to $P(\zeta < \infty, \lim_{t\to\zeta} Y_t = l) = 0$. Then $P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) = 0$, and from the decomposition (2.22)

$$
\begin{aligned}
&P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty\right) \\
&= P(\zeta = \infty) + P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&\geqslant P(\zeta = \infty) + P\left(\int_0^{\zeta} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right).
\end{aligned}
$$

If $v_b(r) < \infty$, then from Proposition 2.3.1, $P\left(\int_0^{\zeta} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) = P(\zeta < \infty, \lim_{t\to\zeta} Y_t = r)$. Then from the decomposition (2.22)

$$
\begin{aligned}
&P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty\right) \\
&\geqslant P(\zeta = \infty) + P(\zeta < \infty, \lim_{t\to\zeta} Y_t = r) \\
&= 1 - P(\zeta < \infty, \lim_{t\to\zeta} Y_t = \ell) = 1,
\end{aligned}
$$

and thus $P(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty) = 1$.

Condition (c): the proof is similar to the proof of condition (b) by switching the roles of ℓ and r.

Condition (d): if $v_b(r) < \infty$ and $v_b(\ell) < \infty$, then from Proposition 2.3.1, $P\left(\int_0^{\zeta} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) = P(\zeta < \infty, \lim_{t\to\zeta} Y_t = \ell)$, and $P\left(\int_0^{\zeta} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) = P(\zeta < \infty, \lim_{t\to\zeta} Y_t = r)$.

Then the decomposition (2.22) becomes

$$
\begin{aligned}
&P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty\right) \\
&= P(\zeta = \infty) + P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\qquad + P\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&= P(\zeta = \infty) + P(\zeta < \infty, \lim_{t\to\zeta} Y_t = \ell) + P(\zeta < \infty, \lim_{t\to\zeta} Y_t = r) \\
&= 1.
\end{aligned}
$$

This completes the proof of the sufficiency part.

For the necessity part, it is equivalent to proving its contra-positive statement: "If none of the conditions (a), (b), (c) or (d) holds, then there exits $T^* \in [0, \infty)$, such that $P(\varphi_{\zeta\wedge T^*} < \infty) < 1$".

We now seek to find the complement set to the conditions (a)-(d). The complement set of (a) is:

Case (1): $v(\ell) < \infty$, $v(r) = \infty$.

Case (2): $v(\ell) = \infty$, $v(r) < \infty$.

Case (3): $v(\ell) < \infty$, $v(r) < \infty$.

Further complement the above cases with the remaining conditions (b), (c), and (d), and we have the following subcases comprising the whole complement set:

Case (1)(i): $v(\ell) < \infty$, $v(r) = \infty$, $v_b(\ell) = \infty$, $v_b(r) < \infty$.

Case (1)(ii): $v(\ell) < \infty$, $v(r) = \infty$, $v_b(\ell) = \infty$, $v_b(r) = \infty$.

Case (2)(i): $v(\ell) = \infty$, $v(r) < \infty$, $v_b(\ell) = \infty$, $v_b(r) < \infty$.

Case (2)(ii): $v(\ell) = \infty$, $v(r) < \infty$, $v_b(\ell) = \infty$, $v_b(r) = \infty$.

Case (3)(i): $v(\ell) < \infty$, $v(r) < \infty$, $v_b(\ell) = \infty$, $v_b(r) = \infty$.

Case (3)(ii): $v(\ell) < \infty$, $v(r) < \infty$, $v_b(\ell) = \infty$, $v_b(r) < \infty$.

Case (3)(iii): $v(\ell) < \infty$, $v(r) < \infty$, $v_b(\ell) < \infty$, $v_b(r) = \infty$.

Note that in all the above seven subcases, $P(\zeta < \infty) > 0$, and it means that there exists a sufficiently large $T^* \in [0, \infty)$, such that $P(\zeta \leqslant T^*) > 0$. Now we analyze each of the above subcases in detail.

Both Case (1)(i) and Case (1)(ii) share the conditions "$v(\ell) < \infty$, $v(r) = \infty$, $v_b(\ell) = \infty$". If $v(r) = \infty$, then $P(\zeta < \infty, \lim_{t\to\zeta} Y_t = r) = 0$. Then $P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) = 0$.

From the decomposition (2.22) substituting $T = T^*$

$$\begin{aligned}
&P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty\right) \\
&= P(\zeta = \infty) + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\quad + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&= P(\zeta = \infty) + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right). \qquad (2.23)
\end{aligned}$$

Now we analyze the second term in (2.23)

$$\begin{aligned}
&P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&= P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\quad + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&= P\left(\int_0^{\zeta} b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\quad + P\left(\int_0^{T^*} b^2(Y_u)du < \infty, T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right). \qquad (2.24)
\end{aligned}$$

If $v_b(\ell) = \infty$, then from Proposition 2.3.1, $P\left(\int_0^\zeta b^2(Y_u)du < \infty, \lim_{t\to\zeta} Y_t = l\right) = 0$. Then $P\left(\int_0^\zeta b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = l\right) = 0$. Use this equality in (2.24)

$$\begin{aligned}&P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du<\infty, \zeta<\infty, \lim_{t\to\zeta} Y_t=l\right)\\&=P\left(\int_0^{T^*} b^2(Y_u)du<\infty, T^*<\zeta<\infty, \lim_{t\to\zeta} Y_t=l\right).\end{aligned}$$

Then

$$\begin{aligned}&P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du<\infty\right)\\&=P(\zeta=\infty)+P\left(\int_0^{T^*} b^2(Y_u)du<\infty, T^*<\zeta<\infty, \lim_{t\to\zeta} Y_t=l\right)\\&\leqslant P(\zeta=\infty)+P(T^*<\zeta<\infty)\\&=1-P(\zeta\leqslant T^*)\\&<1.\end{aligned}$$

The proof of Case (2) is similar to Case (1) by switching the roles of ℓ and r, and is thus omitted.

Consider Case (3), and recall that $P(\zeta<\infty)>0$, which means that there exists a sufficiently large $T_1^*\in[0,\infty)$, such that $P(\zeta\leqslant T_1^*)>0$. All the subcases in Case (3) share the conditions "$v(\ell)<\infty$, $v(r)<\infty$". From Feller's test of explosions, they are equivalent respectively to $P(\zeta<\infty, \lim_{t\to\zeta} Y_t=\ell)>0$ and $P(\zeta<\infty, \lim_{t\to\zeta} Y_t=r)>0$. These two conditions mean that there exist two sufficiently large $T_2^*\in[0,\infty)$ and $T_3^*\in[0,\infty)$ such that $P(\zeta\leqslant T_2^*, \lim_{t\to\zeta} Y_t=\ell)>0$ and $P(\zeta\leqslant T_3^*, \lim_{t\to\zeta} Y_t=r)>0$ under P. Choose $T^*=\max(T_1^*, T_2^*, T_3^*)\in[0,\infty)$. With this newly constructed T^*, we aim to show that $P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du<\infty\right)<1$.

Both Case (3)(i) and Case (3)(ii) share the condition $v_b(\ell)=\infty$. If $v_b(\ell)=\infty$, then from Proposition 2.3.1, $P\left(\int_0^{\zeta} b^2(Y_u)du<\infty, \lim_{t\to\zeta} Y_t=l\right)=0$. Then $P\left(\int_0^{\zeta} b^2(Y_u)du<\infty, \zeta\leqslant T^*<\infty, \lim_{t\to\zeta} Y_t=l\right)=0$.

On the set $\{T^*<\zeta<\infty\}$, from Lemma 2.2.1, $P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du<\infty\right)=$

1. Then

$$P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) = P(T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = l),$$

and similarly

$$P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) = P(T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = r).$$

Recall the decomposition (2.22) substituting $T = T^*$

$$\begin{aligned}
&P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty\right) \\
&= P(\zeta = \infty) + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\quad + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&= P(\zeta = \infty) + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\quad + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\quad + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&\quad + P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty, T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&= P(\zeta = \infty) + P\left(\int_0^{\zeta} b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = l\right) \\
&\quad + P(T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = \ell) + P\left(\int_0^{\zeta} b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&\quad + P(T^* < \zeta < \infty, \lim_{t\to\zeta} Y_t = r) \\
&= P(\zeta = \infty) + P(T^* < \zeta < \infty) + P\left(\int_0^{\zeta} b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = r\right).
\end{aligned} \tag{2.25}$$

Consider the two subcases separately:

Case (3)(i): with $v_b(r) = \infty$, from Proposition 2.3.1,
$P\left(\int_0^\zeta b^2(Y_u)du < \infty, \lim_{t\to\zeta} Y_t = r\right) = 0$, so

$$P\left(\int_0^\zeta b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = r\right) = 0.$$

Then use this equality in (2.25)

$$\begin{aligned}
&P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty\right) \\
&= P(\zeta = \infty) + P(T^* < \zeta < \infty) \\
&\quad + P\left(\int_0^\zeta b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&= P(\zeta = \infty) + P(T^* < \zeta < \infty) \\
&= 1 - P(\zeta \leqslant T^*) \\
&< 1.
\end{aligned}$$

And the last strict inequality holds because $T^* \geqslant T_1^*$ P-a.s., and $P(\zeta \leqslant T^*) \geqslant P(\zeta \leqslant T_1^*) > 0$.

Case (3)(ii): With $v_b(r) < \infty$, from Proposition 2.3.1

$$P\left(\int_0^\zeta b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = r\right) = P(\zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = r).$$

Use this equality in (2.25)

$$
\begin{aligned}
&P\left(\int_0^{\zeta\wedge T^*} b^2(Y_u)du < \infty\right) \\
&= P(\zeta = \infty) + P(T^* < \zeta < \infty) \\
&\quad + P\left(\int_0^{\zeta} b^2(Y_u)du < \infty, \zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&= P(\zeta = \infty) + P(T^* < \zeta < \infty) + P(\zeta \leqslant T^* < \infty, \lim_{t\to\zeta} Y_t = r) \\
&= 1 - P(\zeta \leqslant T^*, \lim_{t\to\zeta} Y_t = \ell) \\
&< 1.
\end{aligned}
$$

And the last strict inequality holds because $T^* \geqslant T_2^*$, P-a.s., and $P(\zeta \leqslant T^*, \lim_{t\to\zeta} Y_t = \ell) \geqslant P(\zeta \leqslant T_2^*, \lim_{t\to\zeta} Y_t = \ell) > 0$.

The remaining Case (3)(iii) can be proved similarly as Case (3)(ii) by switching the roles of ℓ and r. Also note that $T^* \geqslant T_3^*$, P-a.s., and $P(\zeta \leqslant T^*, \lim_{t\to\zeta} Y_t = r) \geqslant P(\zeta \leqslant T_3^*, \lim_{t\to\zeta} Y_t = r) > 0$. This completes the proof of the necessity part. □

Similarly there is a corollary to Theorem 2.3.2 under $\widetilde{P}$. Its proof is similar to that of Theorem 2.3.2 and is thus omitted.

Corollary 2.3.2. *Assume the conditions* (2.2) *and* (2.9), *then for all* $T \in [0, \infty)$, $\widetilde{P}(\varphi_{\zeta\wedge T} < \infty) = \widetilde{P}\left(\int_0^{\zeta\wedge T} b^2(Y_u)du < \infty\right) = 1$ *if and only if at least one of the following conditions is satisfied:*

(a) $\widetilde{v}(\ell) = \widetilde{v}(r) = \infty$,

(b) $\widetilde{v}_b(r) < \infty$ *and* $\widetilde{v}(\ell) = \infty$,

(c) $\widetilde{v}_b(\ell) < \infty$ *and* $\widetilde{v}(r) = \infty$,

(d) $\widetilde{v}_b(r) < \infty$ *and* $\widetilde{v}_b(\ell) < \infty$.

2.4 Generalization of some results in Mijatović and Urusov

In this section, we generalize the main results in Mijatović and Urusov (2012b, 2012c) and provide new unified proofs without the concepts of "separating times". Note that Mijatović and Urusov (2012b, 2012c) work in the $\rho = 1$ case, and we generalize it to the arbitrary correlation case.

Consider the stochastic exponential Z defined in (2.7). The following proposition provides the necessary and sufficient condition for Z_T to be a P-martingale for all $T \in [0, \infty)$, when $-1 \leqslant \rho \leqslant 1$. Note that Theorem 2.1 in Mijatović and Urusov (2012c) is a special case of the following proposition when $\rho = 1$.

Proposition 2.4.1. *Assume the conditions* (2.2) *and* (2.9)*, then for all* $T \in [0, \infty)$, $\mathbb{E}^P[Z_T] = 1$ *if and only if at least one of the conditions (1)-(4) below is satisfied:*

(1) $\widetilde{v}(\ell) = \widetilde{v}(r) = \infty$,

(2) $\widetilde{v}_b(r) < \infty$ *and* $\widetilde{v}(\ell) = \infty$,

(3) $\widetilde{v}_b(\ell) < \infty$ *and* $\widetilde{v}(r) = \infty$,

(4) $\widetilde{v}_b(r) < \infty$ *and* $\widetilde{v}_b(\ell) < \infty$.

Proof. From Proposition 2.2.4, for all $T \in [0, \infty)$, $\mathbb{E}^P[Z_T] = 1$ if and only if $\widetilde{P}(\int_0^{\zeta \wedge T} b^2(Y_u)du < \infty) = 1$. Then the statement follows from Corollary 2.3.2. This completes the proof. □

We have the following necessary and sufficient condition for Z to be a uniformly integrable P-martingale on $[0, \infty]$, when $-1 \leqslant \rho \leqslant 1$. Note that Theorem 2.3 of Mijatović and Urusov (2012c) is a special case of the following proposition when $\rho = 1$.

Proposition 2.4.2. *Assume the conditions* (2.2) *and* (2.9)*, then* $\mathbb{E}^P[Z_\infty] = 1$ *if and only if at least one of the conditions* $(A') - (D')$ *below is satisfied:*

(A') $b = 0$ *a.e. on* J *with respect to the Lebesgue measure,*

(B') $\widetilde{v}_b(r) < \infty$ *and* $\widetilde{s}(\ell) = -\infty$,

(C') $\widetilde{v}_b(\ell) < \infty$ *and* $\widetilde{s}(r) = \infty$,

(D') $\widetilde{v}_b(r) < \infty$ *and* $\widetilde{v}_b(\ell) < \infty$.

Proof. Condition (A') is a trivial case and it is easy to verify. From Corollary 2.3.1 and the classification in Table 2.2, $\mathbb{E}^P[Z_\infty] = 1$ if and only if at least one of the conditions (B'), (C') or (D') holds. This completes the proof. □

Remark 2.4.1. *Financial bubbles have recently attracted some attention in the literature, see Cox and Hobson (2005), Ekström and Tysk (2007), Heston, Lowenstein and Willard (2007), Jarrow, Protter and Shimbo (2007), Madan and Yor (2006), and Pal and Protter (2010). For a survey on the mathematical theory behind the financial bubbles, refer to Protter (2012). Under the risk-neutral measure P, let the (discounted) stock price be modeled as a non-negative local martingale Z. Using the notation in Protter (2012), assume a complete financial market, let $(S_t)_{t\in[0,T^*]}, T^* \in [0,\infty]$ be the underlying risky stock price with life up to a stopping time τ. Let $\Delta \in \mathcal{F}_\tau$ be the time τ terminal payoff or liquidation value of the stock. Assume that $\Delta \geqslant 0$, and $Z_\tau = \Delta \mathbb{1}_{\tau \leqslant T^*}$. Assume that the stock pays no dividends and the risk-free spot interest rate is equal to 0. The fundamental value of the stock is defined as $Z_t^* = E^P[\Delta \mathbb{1}_{\tau\leqslant T^*} \mid \mathcal{F}_t]$ for $t \in [0, T^*]$ Then the financial bubble β is defined as*

$$\beta_t := Z_t - Z_t^*, \quad 0 \leqslant t \leqslant T^*$$

Intuitively the bubble is equal to the difference between the current market stock price and the fundamental price(conditional expected value of the stock's cash flows under the risk-neutral measure P).

Clearly we see that for Z being a non-negative local martingale(thus a non-negative supermartingale), the bubble $\beta_t \geqslant 0, 0 \leqslant t \leqslant T^*$ always holds. Also we call that a bubble "bursts" if $\beta_u = 0$ for some time $0 \leqslant u \leqslant T^*$. Since Z is a supermartingale, if $\beta_u = 0$, then $\beta_t = 0$ for $0 \leqslant u \leqslant t \leqslant T^*$. Intuitively this means that if a bubble bursts, it can never start again. For detailed classification of bubbles based on whether Z is a uniformly *integrable martingale, a martingale or a strict local martingale, refer to Jarrow, Protter and Shimbo (2007).*

Here we generalize some results in Mijatović and Urusov (2012b) to the arbitrary correlation case and provide new proofs without the concept of *separating times*. Precisely, Theorem 2.1 of Mijatović and Urusov (2012b) is a special case of the following proposition when $\rho = 1$.

Proposition 2.4.3. *Assume the conditions* (2.2) *and* (2.9)*, then for all* $T \in [0, \infty)$*,* $Z_T > 0$ *P-a.s. if and only if at least one of the conditions*[7] *(1)-(4) below is satisfied:*

(1) $v(\ell) = v(r) = \infty$*,*

(2) $v_b(r) < \infty$ *and* $v(\ell) = \infty$*,*

(3) $v_b(\ell) < \infty$ *and* $v(r) = \infty$*,*

(4) $v_b(r) < \infty$ *and* $v_b(\ell) < \infty$*.*

Proof. From Lemma 2.2.3, for all $T \in [0, \infty)$, $Z_T > 0$, P-a.s. if and only if $P\left(\int_0^{\zeta \wedge T} b^2(Y_u) du < \infty\right) = 1$. Then the statement follows from Theorem 2.3.2. This completes the proof. □

Note that Theorem 2.3 of Mijatović and Urusov (2012b) is a special case of the following proposition when $\rho = 1$.

Proposition 2.4.4. *Let the functions* μ*,* σ *and* b *satisfy conditions* (2.1), (2.3) *and* (2.5) [8] *in Mijatović and Urusov (2012b), and let* Y *be a (possibly explosive) solution of the SDE* (2.1) *under* P*, with* Z *defined in* (2.7)*, Then* $Z_\infty > 0$*, P-a.s. if and only if at least one of the conditions (I)-(IV) below is satisfied:*

(I) $b = 0$ *a.e. on* J *with respect to the Lebesgue measure,*

(II) $v_b(r) < \infty$ *and* $s(\ell) = -\infty$*,*

[7] Note that conditions (1)-(4) in Proposition 2.4.3 do not depend on the correlation ρ, which means that the positivity of the (discounted) stock price does not depend on the correlation. Similar remarks hold for Proposition 2.4.4 and Proposition 2.4.5.

[8] These conditions are the same as the conditions (2.2) and (2.9) in this chapter. Similar remark holds for Proposition 2.4.5.

(III) $v_b(\ell) < \infty$ *and* $s(r) = \infty$*,*

(IV) $v_b(r) < \infty$ *and* $v_b(\ell) < \infty$*.*

Proof. Condition (I) is a trivial case and it is easy to verify. From Lemma 2.2.3, $Z_\infty > 0$, P-a.s. if and only if $P\left(\int_0^\zeta b^2(Y_s)ds < \infty\right) = 1$. Then the proof follows from Theorem 2.3.1 and the classification in Table 2.1. This completes the proof. □

Note that Theorem 2.5 of Mijatović and Urusov (2012b) is a special case of the following proposition when $\rho = 1$.

Proposition 2.4.5. *Let the functions* μ*,* σ *and* b *satisfy conditions* (2.1), (2.3) *and* (2.5) *of Mijatović and Urusov (2012b), and let* Y *be a (possibly explosive) solution of the SDE* (2.1) *under* P*, with* Z *defined in* (2.7). *Then* $Z_\infty = 0$*,* P*-a.s. if and only if both conditions (i) and (ii) below are satisfied:*

(i) b *is not identically zero with respect to the Lebesgue measure,*

(ii) $v_b(\ell) = v_b(r) = \infty$*.*

Proof. Condition (i) is a trivial case and it is easy to verify. From Lemma 2.2.3, $Z_\infty = 0$, P-a.s. if and only if $P\left(\int_0^\zeta b^2(Y_u)du = \infty\right) = P(\varphi_\zeta = \infty) = 1$. From Theorem 2.3.1 (iii), this is equivalent to checking the condition (ii) here. This completes the proof. □

2.5 Examples of correlated stochastic volatility models

In this section, we apply the results in Section 2.4 to the study of martingale properties of (discounted) stock prices in four popular correlated stochastic volatility models: the Heston, the 3/2, the Schöbel-Zhu and the Hull-White models. We consider the arbitrary correlation case in the following. All our results are consistent with the literature. Throughout this section, we work in the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ as constructed in Section 2.2.

2.5.1 Heston stochastic volatility model

Under P, the (correlated) Heston stochastic volatility model has the following diffusive dynamics

$$\begin{aligned} dS_t &= S_t\sqrt{Y_t}\mathbb{1}_{t\in[0,\zeta)}dW_t^{(1)}, \quad S_0 = 1. \\ dY_t &= \kappa(\theta - Y_t)\mathbb{1}_{t\in[0,\zeta)}dt + \xi\sqrt{Y_t}\mathbb{1}_{t\in[0,\zeta)}dW_t, \quad Y_0 = x_0, \end{aligned} \tag{2.26}$$

with $\mathbb{E}^P[dW_t^{(1)}dW_t] = \rho dt$, $-1 \leqslant \rho \leqslant 1$, $\kappa > 0, \theta > 0, \sigma > 0$. The natural state space for Y is $J = (\ell, r) = (0, \infty)$. ζ is the possible exit time of the process Y from its state space J. The model in (2.26) belongs to the general stochastic volatility model considered in (2.8) with $\mu(x) = \kappa(\theta - x)$, $\sigma(x) = \xi\sqrt{x}$, and $b(x) = \sqrt{x}$. Clearly $\sigma(x) = \xi\sqrt{x} \neq 0, x \in J$, $\frac{1}{\sigma^2(x)} = \frac{1}{\xi^2 x} \in L^1_{loc}(J)$, $\frac{\mu(x)}{\sigma^2(x)} = \frac{\kappa(\theta-x)}{\xi^2 x} \in L^1_{loc}(J)$, and $\frac{b^2(x)}{\sigma^2(x)} = \frac{1}{\xi^2} \in L^1_{loc}(J)$ are satisfied. This implies that the conditions (2.2) and (2.9) are satisfied.

Remark 2.5.1. *In the literature, the Heston model is often equipped with a reflecting boundary at* 0. *The model we consider here for convenience assumes an absorbing boundary at* 0, *which is less common.*

From Proposition 2.2.3, under $\widetilde{P}$, the diffusion Y satisfies the following SDE

$$dY_t = \widetilde{\kappa}(\widetilde{\theta} - Y_t)\mathbb{1}_{t\in[0,\zeta)}dt + \xi\sqrt{Y_t}\mathbb{1}_{t\in[0,\zeta)}d\widetilde{W}_t, \quad Y_0 = x_0,$$

where $\widetilde{\kappa} = \kappa - \rho\xi$ and $\widetilde{\theta} = \frac{\kappa\theta}{\kappa - \rho\xi}$.

For a constant $c \in J$, the scale functions of the SDE (2.1) and SDE (2.13) are respectively

$$\begin{aligned} s(x) &= e^{\frac{2\kappa c}{\xi^2}} c^{\frac{2\kappa\theta}{\xi^2}} \int_c^x y^{-\frac{2\kappa\theta}{\xi^2}} e^{-\frac{2\kappa y}{\xi^2}} dy = C_1 \int_c^x y^{-\alpha} e^{-\beta y} dy, \\ \widetilde{s}(x) &= e^{\frac{2\widetilde{\kappa} c}{\xi^2}} c^{\frac{2\widetilde{\kappa}\widetilde{\theta}}{\xi^2}} \int_c^x y^{-\frac{2\widetilde{\kappa}\widetilde{\theta}}{\xi^2}} e^{-\frac{2\widetilde{\kappa} y}{\xi^2}} dy = C_2 \int_c^x y^{-\alpha} e^{-\gamma y} dy, \end{aligned} \tag{2.27}$$

with $\alpha = \frac{2\kappa\theta}{\xi^2}$, $\beta = \frac{2\kappa}{\xi^2} > 0$, $\gamma = \frac{2\kappa}{\xi^2} - \frac{2\rho}{\xi}$, and the constant terms are $C_1 = e^{\frac{2\kappa c}{\xi^2}} c^{\frac{2\kappa\theta}{\xi^2}} > 0$ and $C_2 = e^{\frac{2\kappa c}{\xi^2} - \frac{2\rho c}{\xi}} c^{\frac{2\kappa\theta}{\xi^2}} > 0$. Under $\widetilde{P}$, we have the following test function for $x \in \bar{J}$

$$\widetilde{v}(x) = \frac{2}{\xi^2}\int_c^x \frac{\int_y^x z^{-\alpha}e^{-\gamma z}dz}{y^{1-\alpha}e^{-\gamma y}}dy, \tag{2.28}$$

and

$$\widetilde{v}_b(x) = \frac{2}{\xi^2}\int_c^x \frac{\int_y^x z^{-\alpha}e^{-\gamma z}dz}{y^{-\alpha}e^{-\gamma y}}dy. \tag{2.29}$$

Proposition 2.5.1. *For*[9] *the Heston model in* (2.26)*, the underlying (discounted) stock price* $(S_t)_{0\leqslant t\leqslant T}, T \in [0,\infty)$ *is a true* P*-martingale.*

Proof. We aim at checking the conditions of Proposition 2.4.1.

Case (1): $\alpha > 1$. From the property of the gamma function

$$\widetilde{s}(\infty)\begin{cases} < \infty, & \text{if } \gamma \geqslant 0, \\ = \infty, & \text{if } \gamma < 0. \end{cases}$$

We now aim to check the finiteness of $\widetilde{v}(r)$. We divide the discussion into three cases.

(i) When $\alpha > 1$ and $\gamma < 0$, $\widetilde{s}(\infty) = \infty$, then $\widetilde{v}(\infty) = \infty$ and $\widetilde{v}_b(\infty) = \infty$.

(ii) When $\alpha > 1$ and $\gamma = 0$, then

$$\begin{aligned}\widetilde{v}(\infty) &= \frac{2}{\xi^2}\int_c^\infty y^{\alpha-1}\left(\int_y^\infty z^{-\alpha}dz\right)dy. \\ &= \frac{2}{\xi^2}\int_c^\infty y^{\alpha-1}\frac{(-y^{1-\alpha})}{1-\alpha}dy \\ &= \infty,\end{aligned}$$

[9]Proposition 2.5.1 is a special case of Proposition 2.5, p34 of Andersen and Piterbarg (2007), also see Remark 4.2, p2052 of Del Baño Rollin et al. (2010).

and

$$\begin{aligned}
\widetilde{v}_b(\infty) &= \frac{2}{\xi^2}\int_c^\infty y^\alpha \left(\int_y^\infty z^{-\alpha}dz\right)dy. \\
&= \frac{2}{\xi^2}\int_c^\infty y^\alpha \frac{(-y^{1-\alpha})}{1-\alpha}dy \\
&= \infty,
\end{aligned}$$

(iii) When $\alpha > 1$ and $\gamma > 0$, then $\lim_{y\to\infty} y^{-\alpha}e^{-\gamma y} = 0$. From L'Hôpital's rule

$$\begin{aligned}
\lim_{y\to\infty}\frac{\int_y^\infty z^{-\alpha}e^{-\gamma z}dz}{y^{-\alpha}e^{-\gamma y}} &= \lim_{y\to\infty}\frac{-y^{-\alpha}e^{-\gamma y}}{(-\gamma y^{-\alpha}-\alpha y^{-\alpha-1})e^{-\gamma y}} \\
&= \lim_{y\to\infty}\frac{1}{\gamma+\alpha/y} \\
&= \frac{1}{\gamma} > 0.
\end{aligned}$$

Thus as $y \to \infty$

$$\int_y^\infty z^{-\alpha}e^{-\gamma z}dz \sim \frac{1}{\gamma}y^{-\alpha}e^{-\gamma y}, \tag{2.30}$$

and there exists $M > c > 0$, such that for $y > M$, $\int_y^\infty z^{-\alpha}e^{-\gamma z}dz > \frac{1}{2\gamma}y^{-\alpha}e^{-\gamma y}$. Substitute this into equation (2.28)

$$\begin{aligned}
\widetilde{v}(\infty) &= \frac{2}{\xi^2}\int_c^\infty y^{\alpha-1}e^{\gamma y}\left(\int_y^\infty z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&\geqslant \frac{2}{\xi^2}\int_M^\infty y^{\alpha-1}e^{\gamma y}\left(\int_y^\infty z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&> \frac{2}{\xi^2}\int_M^\infty y^{\alpha-1}e^{\gamma y}\frac{1}{2\gamma}y^{-\alpha}e^{-\gamma y}dy \\
&= \frac{1}{\gamma\xi^2}\int_M^\infty y^{-1}dy \\
&= \infty,
\end{aligned}$$

and similarly substitute $\int_y^\infty z^{-\alpha}e^{-\gamma z}dz > \frac{1}{2\gamma}y^{-\alpha}e^{-\gamma y}$ into (2.29), then

$$\begin{aligned}
\widetilde{v}_b(\infty) &= \frac{2}{\xi^2}\int_c^\infty y^\alpha e^{\gamma y}\left(\int_y^\infty z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&\geqslant \frac{2}{\xi^2}\int_M^\infty y^\alpha e^{\gamma y}\left(\int_y^\infty z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&> \frac{2}{\xi^2}\int_M^\infty y^\alpha e^{\gamma y}\frac{1}{2\gamma}y^{-\alpha}e^{-\gamma y}dy \\
&= \frac{1}{\gamma\xi^2}\int_M^\infty dy \\
&= \infty.
\end{aligned}$$

To summarize, when $\alpha > 1$, $\widetilde{v}(r) = \infty$ and $\widetilde{v}_b(r) = \infty$ always hold for $\gamma \in \mathbb{R}$.

Case (2): $\alpha \leqslant 1$. Similarly divide into two cases based on γ:

(i) If $\gamma \leqslant 0$, then $e^{-\gamma y} \geqslant 1$ and

$$\begin{aligned}
\widetilde{s}(\infty) &= C_2\int_c^\infty y^{-\alpha}e^{-\gamma y}dy \\
&\geqslant C_2\int_c^\infty y^{-\alpha}dy \\
&= \infty, \qquad (2.31)
\end{aligned}$$

and consequently $\widetilde{v}(\infty) = \infty$ and $\widetilde{v}_b(\infty) = \infty$.

(ii) If $\gamma > 0$, note that $\lim_{y\to\infty} y^{-\alpha}e^{-\gamma y} = 0$ still holds, then we can apply the L'Hôpital's rule similar as Case (1) (iii), and we can conclude that $\widetilde{v}(r) = \infty$ and $\widetilde{v}_b(r) = \infty$ always hold.

To summarize, in Case (2), $\widetilde{v}(r) = \infty$ and $\widetilde{v}_b(r) = \infty$ always hold for $\gamma \in \mathbb{R}$.

To check similar conditions for ℓ, recall

$$\widetilde{s}(0) = C_2\int_c^0 y^{-\alpha}e^{-\gamma y}dy = -C_2\int_0^c y^{-\alpha}e^{-\gamma y}dy.$$

From the properties of the gamma function

$$\widetilde{s}(0)\begin{cases} > -\infty, & \text{if } \alpha < 1, \\ = -\infty, & \text{if } \alpha \geqslant 1. \end{cases}$$

Case (1): if $\alpha \geqslant 1$ holds, then $\widetilde{s}(0) = -\infty$, and $\widetilde{v}(0) = \infty$, $\widetilde{v}_b(0) = \infty$ hold.

Case (2): if $\alpha < 1$, $\lim\limits_{y\to 0} y^{1-\alpha}e^{-\gamma y} = 0$ holds, then from the L'Hôpital's rule

$$\begin{aligned}
\lim_{y\to 0}\frac{\int_0^y z^{-\alpha}e^{-\gamma z}dz}{y^{1-\alpha}e^{-\gamma y}} &= \lim_{y\to 0}\frac{y^{-\alpha}e^{-\gamma y}}{(1-\alpha)y^{-\alpha}e^{-\gamma y} + y^{1-\alpha}(-\gamma)e^{-\gamma y}} \\
&= \lim_{y\to 0}\frac{1}{1-\alpha-\gamma y} \\
&= \frac{1}{1-\alpha} \\
&> 0.
\end{aligned} \tag{2.32}$$

As $y \to 0$

$$\int_0^y z^{-\alpha}e^{-\gamma z}dz \sim \frac{1}{1-\alpha}y^{1-\alpha}e^{-\gamma y}. \tag{2.33}$$

Thus there exists $0 < \varepsilon < c$, such that for $0 < y < \varepsilon$, $\int_0^y z^{-\alpha}e^{-\gamma z}dz < \frac{2}{(1-\alpha)}y^{1-\alpha}e^{-\gamma y}$. Substitute this into equation (2.28), then

$$\begin{aligned}
\widetilde{v}(0) &= \frac{2}{\xi^2}\int_0^c y^{\alpha-1}e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&= \frac{2}{\xi^2}\int_0^\varepsilon y^{\alpha-1}e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy + \frac{2}{\xi^2}\int_\varepsilon^c y^{\alpha-1}e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&< \frac{2}{\xi^2}\int_0^\varepsilon y^{\alpha-1}e^{\gamma y}\left(\frac{2}{(1-\alpha)}y^{1-\alpha}e^{-\gamma y}\right)dy + \frac{2}{\xi^2}\int_\varepsilon^c y^{\alpha-1}e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&= \frac{4\varepsilon}{(1-\alpha)\xi^2} + \frac{2}{\xi^2}\int_\varepsilon^c y^{\alpha-1}e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&< \infty.
\end{aligned} \tag{2.34}$$

Similarly substitute $\int_0^y z^{-\alpha}e^{-\gamma z}dz < \frac{2}{(1-\alpha)}y^{1-\alpha}e^{-\gamma y}$ into (2.29), then

$$\begin{aligned}
\widetilde{v}_b(0) &= \frac{2}{\xi^2}\int_0^c y^\alpha e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&= \frac{2}{\xi^2}\int_0^\varepsilon y^\alpha e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy + \frac{2}{\xi^2}\int_\varepsilon^c y^\alpha e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&< \frac{2}{\xi^2}\int_0^\varepsilon y^\alpha e^{\gamma y}\left(\frac{2}{(1-\alpha)}y^{1-\alpha}e^{-\gamma y}\right)dy + \frac{2}{\xi^2}\int_\varepsilon^c y^\alpha e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&= \frac{2\varepsilon^2}{(1-\alpha)\xi^2} + \frac{2}{\xi^2}\int_\varepsilon^c y^{\alpha-1}e^{\gamma y}\left(\int_0^y z^{-\alpha}e^{-\gamma z}dz\right)dy \\
&< \infty. \qquad (2.35)
\end{aligned}$$

We summarize the above results in Table 2.3, and from Proposition 2.4.1, $(S_t)_{0\leqslant t\leqslant T}, T \in [0,\infty)$ is a true martingale. This completes the proof. □

Case	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\alpha \geqslant 1$	∞	∞	∞	∞
$\alpha < 1$	$< \infty$	∞	$< \infty$	∞

Table 2.3: Classification table for the Heston model

Proposition 2.5.2. *For the Heston model in* (2.26)*, the underlying (discounted) stock price* $(S_t)_{0\leqslant t\leqslant\infty}$ *is a uniformly integrable* P*-martingale if and only if* $2\kappa\theta < \xi^2$ *and* $\kappa \leqslant \rho\xi$[10]*.*

Proof. From the proof in Proposition 2.5.1, we have the following classification:

If $\alpha \leqslant 1$, then

$$\widetilde{s}(\infty)\begin{cases} = \infty, & \text{if } \gamma \leqslant 0, \\ < \infty, & \text{if } \gamma > 0. \end{cases}$$

[10]Note that $\kappa \leqslant \rho\xi$ implies that $\rho > 0$.

If $\alpha > 1$, then

$$\widetilde{s}(\infty) \begin{cases} = \infty, & \text{if } \gamma < 0, \\ < \infty, & \text{if } \gamma \geqslant 0. \end{cases}$$

We also have

$$\widetilde{s}(0) \begin{cases} > -\infty, & \text{if } \alpha < 1, \\ = -\infty, & \text{if } \alpha \geqslant 1. \end{cases}$$

This, combined with the classification in Table 2.3, gives us the classification in Table 2.4. From Table 2.4 and Proposition 2.4.2, we have that $(S_t)_{0 \leqslant t \leqslant \infty}$ is a uniformly integrable P-martingale if and only if $\alpha < 1$ and $\gamma \leqslant 0$, which is equivalent to $2\kappa\theta < \xi^2$ and $\kappa \leqslant \rho\xi$. This completes the proof. □

Case	$\widetilde{s}(\ell)$	$\widetilde{s}(r)$	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\alpha > 1, \gamma < 0$	$-\infty$	∞	∞	∞	∞	∞
$\alpha > 1, \gamma = 0$	$-\infty$	$< \infty$	∞	∞	∞	∞
$\alpha > 1, \gamma > 0$	$-\infty$	$< \infty$	∞	∞	∞	∞
$\alpha = 1, \gamma < 0$	$-\infty$	∞	∞	∞	∞	∞
$\alpha = 1, \gamma = 0$	$-\infty$	∞	∞	∞	∞	∞
$\alpha = 1, \gamma > 0$	$-\infty$	$< \infty$	∞	∞	∞	∞
$\alpha < 1, \gamma < 0$	$> -\infty$	∞	$< \infty$	∞	$< \infty$	∞
$\alpha < 1, \gamma = 0$	$> -\infty$	∞	$< \infty$	∞	$< \infty$	∞
$\alpha < 1, \gamma > 0$	$> -\infty$	$< \infty$	$< \infty$	∞	$< \infty$	∞

Table 2.4: Second classification table for the Heston model

Under P, we have the following result on the positivity of the (discounted) stock price in the Heston model.

Proposition 2.5.3. *For the Heston model in* (2.26)*, we have:*

(1) $P(S_T > 0) = 1$ *for all* $T \in [0, \infty)$*,*

(2) $P(S_\infty > 0) < 1$*.*

Proof. Similar to the proofs of Proposition 2.5.1 and Proposition 2.5.2 with γ replaced by β and C_2 by C_1, we have the classification in Table 2.5. Based on Table 2.5, from Proposition 2.4.3 and Proposition 2.4.4, we have the desired results. This completes the proof. □

Case	$s(\ell)$	$s(r)$	$v(\ell)$	$v(r)$	$v_b(\ell)$	$v_b(r)$
$\alpha > 1$	$-\infty$	$< \infty$	∞	∞	∞	∞
$\alpha = 1$	$-\infty$	$< \infty$	∞	∞	∞	∞
$\alpha < 1$	$> -\infty$	$< \infty$	$< \infty$	∞	$< \infty$	∞

Table 2.5: Third classification table for the Heston model

2.5.2 3/2 stochastic volatility model

Under P, the (correlated) 3/2 stochastic volatility model has the following diffusive dynamics

$$\begin{aligned} dS_t &= S_t\sqrt{Y_t}\mathbb{1}_{t\in[0,\zeta)}dW_t^{(1)}, \quad S_0 = 1, \\ dY_t &= (\omega Y_t - \theta Y_t^2)\mathbb{1}_{t\in[0,\zeta)}dt + \xi Y_t^{3/2}\mathbb{1}_{t\in[0,\zeta)}dW_t, \quad Y_0 = x_0. \end{aligned} \tag{2.36}$$

where $\mathbb{E}^P[dW_t^{(1)}dW_t] = \rho dt$, $-1 \leqslant \rho \leqslant 1$, $\omega > 0, \xi > 0, \theta \in \mathbb{R}$.

The natural state space is given by $J = (\ell, r) = (0, \infty)$. ζ is the possible exit time of the process Y from its state space J. The model in (2.36) belongs to the general stochastic volatility model considered in (2.8) with $\mu(x) = \omega x - \theta x^2, \sigma(x) = \xi x^{3/2}$, and $b(x) = \sqrt{x}$. Clearly $\sigma(x) = \xi x^{3/2} \neq 0, x \in J$, $\frac{1}{\sigma^2(x)} = \frac{1}{\xi^2 x^3} \in L^1_{loc}(J)$, $\frac{\mu(x)}{\sigma^2(x)} = \frac{\omega - \theta x}{\xi^2 x^2} \in L^1_{loc}(J)$, and $\frac{b^2(x)}{\sigma^2(x)} = \frac{1}{\xi^2 x^2} \in L^1_{loc}(J)$ are satisfied. This implies that the conditions (2.2) and (2.9) are satisfied.

From Proposition 2.2.3, under $\widetilde{P}$, the diffusion Y satisfies the following SDE

$$dY_t = (\omega Y_t - \widetilde{\theta} Y_t^2)\mathbb{1}_{t\in[0,\zeta)}dt + \xi Y_t^{\frac{3}{2}}\mathbb{1}_{t\in[0,\zeta)}d\widetilde{W}_t, \quad Y_0 = x_0,$$

where $\widetilde{\theta} = \theta - \rho\xi$.

For a constant $c \in J$, the scale functions of the SDE (2.1) and SDE (2.13) are respectively

$$\begin{aligned} s(x) &= \frac{b}{c^a} \int_c^x y^a \exp\left(\frac{d}{y}\right) dy, \\ \widetilde{s}(x) &= \frac{b}{c^{\widetilde{a}}} \int_c^x y^{\widetilde{a}} \exp\left(\frac{d}{y}\right) dy, \quad x \in \bar{J}, \end{aligned} \tag{2.37}$$

where $a = \frac{2\theta}{\xi^2}$, $b = \exp\left(-\frac{2\omega}{c\xi^2}\right)$, $d = \frac{2\omega}{\xi^2}$ and $\widetilde{a} = a - \frac{2\rho}{\xi}$. Since the only difference between $s(.)$ and $\widetilde{s}(.)$ is in the parameters a and $\widetilde{a}$, the analysis under $\widetilde{P}$ is similar to the analysis under P, except with a change of the parameter from a to $\widetilde{a}$. Thus we only focus on the study under P. We have the following test functions

$$\begin{aligned} v(x) &= \frac{2}{\xi^2} \int_c^x \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_y^x z^a \exp\left(\frac{d}{z}\right) dz \right) dy, \\ v_b(x) &= \frac{2}{\xi^2} \int_c^x \frac{1}{y^{a+2} \exp\left(\frac{d}{y}\right)} \left(\int_y^x z^a \exp\left(\frac{d}{z}\right) dz \right) dy. \end{aligned}$$

Then

$$v(\infty) = \frac{2}{\xi^2} \int_c^\infty \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_y^\infty z^a \exp\left(\frac{d}{z}\right) dz \right) dy, \tag{2.38}$$

and

$$v_b(\infty) = \frac{2}{\xi^2} \int_c^\infty \frac{1}{y^{a+2} \exp\left(\frac{d}{y}\right)} \left(\int_y^\infty z^a \exp\left(\frac{d}{z}\right) dz \right) dy. \tag{2.39}$$

Lemma 2.5.1. *With $\omega > 0$, we have*

$$\begin{aligned} a < -1 &\iff v(r) < \infty, \\ \widetilde{a} < -1 &\iff \widetilde{v}(r) < \infty. \end{aligned}$$

$$\forall a \in \mathbb{R}, \quad v_b(r) = \infty, \quad \forall \widetilde{a} \in \mathbb{R}, \quad \widetilde{v}_b(r) = \infty.$$

$$\forall a \in \mathbb{R}, \quad v(\ell) = \infty, \quad \forall \widetilde{a} \in \mathbb{R}, \quad \widetilde{v}(\ell) = \infty.$$

$$\forall a \in \mathbb{R}, \quad v_b(\ell) = \infty, \quad \forall \widetilde{a} \in \mathbb{R}, \quad \widetilde{v}_b(\ell) = \infty.$$

Proof. We aim to check the conditions in Proposition 2.4.1. For the right boundary r, divide into two cases:

(i) When $a < -1$, $\lim_{y\to\infty} y^{a+1} \exp\left(\frac{d}{y}\right) = 0$. From L'Hôpital's rule

$$\begin{aligned}
\lim_{y\to\infty} \frac{\int_y^\infty z^a \exp\left(\frac{d}{z}\right) dz}{y^{a+1} \exp\left(\frac{d}{y}\right)} &= \lim_{y\to\infty} \frac{-y^a \exp\left(\frac{d}{y}\right)}{((a+1)y^a - y^{a-1}d) \exp\left(\frac{d}{y}\right)} \\
&= \lim_{y\to\infty} \frac{1}{y^{-1}d - (a+1)} \\
&= -\frac{1}{a+1}.
\end{aligned}$$

As $y \to \infty$

$$\int_y^\infty z^a \exp\left(\frac{d}{z}\right) dz \sim -\frac{1}{a+1} y^{a+1} \exp\left(\frac{d}{y}\right). \tag{2.40}$$

Note that $-\frac{1}{a+1} > 0$. Since $\int_y^\infty z^a \exp\left(\frac{d}{z}\right) dz$ is decreasing in y, there exists $M > c > 0$, such that for $y > M$

$$\int_y^\infty z^a \exp\left(\frac{d}{z}\right) dz < \frac{-2}{a+1} y^{a+1} \exp\left(\frac{d}{y}\right). \tag{2.41}$$

Substitute (2.41) into (2.38)

$$\begin{aligned}
v(\infty) &= \frac{2}{\xi^2} \int_c^\infty \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_y^\infty z^a \exp\left(\frac{d}{z}\right) dz \right) dy \\
&= \frac{2}{\xi^2} \int_c^M \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_y^\infty z^a \exp\left(\frac{d}{z}\right) dz \right) dy \\
&\quad + \frac{2}{\xi^2} \int_M^\infty \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_y^\infty z^a \exp\left(\frac{d}{z}\right) dz \right) dy
\end{aligned}$$

$$
\begin{aligned}
&< \frac{2}{\xi^2}\int_c^M \frac{1}{y^{a+3}\exp\left(\frac{d}{y}\right)}\left(\int_y^\infty z^a \exp\left(\frac{d}{z}\right)dz\right)dy \\
&\quad + \frac{2}{\xi^2}\int_M^\infty \frac{1}{y^{a+3}\exp\left(\frac{d}{y}\right)}\frac{-2}{a+1}y^{a+1}\exp\left(\frac{d}{y}\right)dy \\
&= \frac{2}{\xi^2}\int_c^M \frac{1}{y^{a+3}\exp\left(\frac{d}{y}\right)}\left(\int_y^\infty z^a \exp\left(\frac{d}{z}\right)dz\right)dy + \frac{-4}{(a+1)\xi^2}\int_M^\infty \frac{1}{y^2}dy \\
&= \frac{2}{\xi^2}\int_c^M \frac{1}{y^{a+3}\exp\left(\frac{d}{y}\right)}\int_y^\infty z^a \exp\left(\frac{d}{z}\right)dzdy + \frac{-4}{(a+1)\xi^2 M} \\
&< \infty.
\end{aligned}
$$

From (2.40), there exists $M' > c > 0$, such that for $y > M'$

$$
\int_y^\infty z^a \exp\left(\frac{d}{z}\right)dz > \frac{-1}{2(a+1)}y^{a+1}\exp\left(\frac{d}{y}\right). \tag{2.42}
$$

Similarly substitute (2.42) into (2.39)

$$
\begin{aligned}
v_b(\infty) &= \frac{2}{\xi^2}\int_c^\infty \frac{1}{y^{a+2}\exp\left(\frac{d}{y}\right)}\left(\int_y^\infty z^a \exp\left(\frac{d}{z}\right)dz\right)dy \\
&\geqslant \frac{2}{\xi^2}\int_{M'}^\infty \frac{1}{y^{a+2}\exp\left(\frac{d}{y}\right)}\left(\int_y^\infty z^a \exp\left(\frac{d}{z}\right)dz\right)dy \\
&> \frac{2}{\xi^2}\int_{M'}^\infty \frac{1}{y^{a+2}\exp\left(\frac{d}{y}\right)}\left(\frac{-1}{2(a+1)}y^{a+1}\exp\left(\frac{d}{y}\right)\right)dy \\
&= \frac{-1}{\xi^2(a+1)}\int_{M'}^\infty \frac{1}{y}dy \\
&= \infty.
\end{aligned}
$$

(ii) When $a \geqslant -1$, since $d > 0$, we have that $\exp\left(\frac{d}{y}\right) \geqslant 1$, for $y > c > 0$. Then

$$
s(\infty) = \frac{b}{c^a}\int_c^\infty y^a \exp\left(\frac{d}{y}\right)dy \geqslant \frac{b}{c^a}\int_c^\infty y^a dy = \infty.
$$

Thus $v(\infty) = \infty$ and $v_b(\infty) = \infty$ in this case. To summarize, $v(r) < \infty$ if and only if $a < -1$, and $v_b(r) = \infty$ for $a \in \mathbb{R}$.

For the left endpoint ℓ

$$v(0) = \frac{2}{\xi^2} \int_c^0 \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_y^0 z^a \exp\left(\frac{d}{z}\right) dz \right) dy$$
$$= \frac{2}{\xi^2} \int_0^c \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_0^y z^a \exp\left(\frac{d}{z}\right) dz \right) dy, \tag{2.43}$$

and

$$v_b(0) = \frac{2}{\xi^2} \int_0^c \frac{1}{y^{a+2} \exp\left(\frac{d}{y}\right)} \left(\int_0^y z^a \exp\left(\frac{d}{z}\right) dz \right) dy. \tag{2.44}$$

For $0 \leqslant z \leqslant y$, we have $e^{\frac{d}{z}} \geqslant e^{\frac{d}{y}}$, and plug this inequality into (2.43)

$$\begin{aligned} v(0) &= \frac{2}{\xi^2} \int_0^c \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_0^y z^a \exp\left(\frac{d}{z}\right) dz \right) dy \\ &\geqslant \frac{2}{\xi^2} \int_0^c \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_0^y z^a dz \right) \exp\left(\frac{d}{y}\right) dy \\ &= \frac{2}{(a+1)\xi^2} \int_0^c \frac{1}{y^2} dy \\ &= \infty. \end{aligned}$$

Similarly plug this inequality into (2.44)

$$\begin{aligned} v_b(0) &= \frac{2}{\xi^2} \int_0^c \frac{1}{y^{a+2} \exp\left(\frac{d}{y}\right)} \left(\int_0^y z^a \exp\left(\frac{d}{z}\right) dz \right) dy \\ &\geqslant \frac{2}{\xi^2} \int_0^c \frac{1}{y^{a+2} \exp\left(\frac{d}{y}\right)} \left(\int_0^y z^a dz \right) \exp\left(\frac{d}{y}\right) dy \\ &= \frac{2}{(a+1)\xi^2} \int_0^c \frac{1}{y} dy = \infty. \end{aligned}$$

To summarize, $v(\ell) = \infty$ and $v_b(\ell) = \infty$ for $a \in \mathbb{R}$. From (2.37), the above proofs also work for the case of $\widetilde{v}$ by substituting a for $\widetilde{a}$. The results in Lemma 2.5.1 can be summarized in Table 2.6. This completes the proof. □

Case	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\widetilde{a} < -1$	∞	$< \infty$	∞	∞
$\widetilde{a} \geqslant -1$	∞	∞	∞	∞

Table 2.6: Classification table for the 3/2 model

Proposition 2.5.4. *For*[11] *the* 3/2 *model in* (2.36)*, the underlying (discounted) stock price* $(S_t)_{0 \leqslant t \leqslant T}, T \in [0, \infty)$ *is a true* P*-martingale if and only if* $\xi^2 - 2\rho\xi + 2\theta \geqslant 0$*.*

Proof. From Lemma 2.5.1 and Table 2.6, combined with Proposition 2.4.1, we have that $(S_t)_{0 \leqslant t \leqslant T}, T \in [0, \infty)$ is a true P-martingale if and only if $\widetilde{a} \geqslant -1$, which is equivalent to $\xi^2 - 2\rho\xi + 2\theta \geqslant 0$ after some simplifications. This completes the proof. □

Proposition 2.5.5. *For the* 3/2 *model in* (2.36)*, the underlying (discounted) stock price* $(S_t)_{0 \leqslant t \leqslant \infty}$ *is not a uniformly integrable* P*-martingale.*

Proof. From Table 2.6, for all $\widetilde{a} \in \mathbb{R}$, $\widetilde{v}_b(\ell) = \infty$ and $\widetilde{v}_b(\ell) = \infty$ hold. From Proposition 2.4.2, $(S_t)_{0 \leqslant t \leqslant \infty}$ is not a uniformly integrable P-martingale. This completes the proof. □

Under P, we have the following result on the positivity of the (discounted) stock price in the 3/2 model.

Proposition 2.5.6. *For the* 3/2 *model in* (2.36)*, we have:*

(1) $P(S_T > 0) = 1$ *for all* $T \in [0, \infty)$ *if and only if* $\xi^2 + 2\theta \geqslant 0$*,*

(2) $P(S_\infty > 0) < 1$*.*

[11] Theorem 3, p110 of Carr and Sun (2007) proves *sufficiency*. See also Lewis (2000) and Drimus (2012).

Proof. Similar to the proofs of Proposition 2.5.4 and Proposition 2.5.5 with $\widetilde{a}$ replaced by a, we have the classification in Table 2.7. Based on Table 2.7, from Proposition 2.4.3 and Proposition 2.4.4, we have the desired results. Note that $a \geqslant -1$ is equivalent to $\xi^2 + 2\theta \geqslant 0$. This completes the proof. □

Case	$v(\ell)$	$v(r)$	$v_b(\ell)$	$v_b(r)$
$a < -1$	∞	$< \infty$	∞	∞
$a \geqslant -1$	∞	∞	∞	∞

Table 2.7: Second classification table for the 3/2 model

2.5.3 Schöbel-Zhu stochastic volatility model

Under P, the correlated Schöbel-Zhu stochastic volatility model[12] (see Schöbel and Zhu (1999)) can be described by the following diffusive dynamics

$$\begin{aligned} dS_t &= S_t Y_t \mathbb{1}_{t\in[0,\zeta)} dW_t^{(1)}, \quad S_0 = 1, \\ dY_t &= -\kappa(Y_t - \theta)\mathbb{1}_{t\in[0,\zeta)} dt + \gamma \mathbb{1}_{t\in[0,\zeta)} dW_t^{(2)}, \quad Y_0 = x_0. \end{aligned} \tag{2.45}$$

where $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$, $-1 \leqslant \rho \leqslant 1$, $\kappa > 0$, $\theta > 0$, $\gamma > 0$. The process Y is an Ornstein-Uhlenbeck process, and this implies that its natural state space is $J = (\ell, r) = (-\infty, \infty)$. ζ is the possible exit time of the process Y from its state space J. The model (2.45) belongs to the general stochastic volatility model considered in (2.8) with $\mu(x) = \kappa(\theta - x)$, $\sigma(x) = \gamma$, and $b(x) = x$. Clearly $\sigma(x) = \gamma \neq 0, x \in J$, then $\frac{1}{\sigma(x)^2} = \frac{1}{\gamma^2} \in L^1_{loc}(J)$, $\frac{\mu(x)}{\sigma(x)^2} = \frac{\kappa(\theta - x)}{\gamma^2} \in L^1_{loc}(J)$, and $\frac{b^2(x)}{\sigma^2(x)} = \frac{x^2}{\gamma^2} \in L^1_{loc}(J)$ are satisfied. This implies that the conditions (2.2) and (2.9) are satisfied.

[12]It is the correlated version of the Stein-Stein (1991) model. In Rheinländer (2005), the minimal entropy martingale measure is studied in detail for this model, and its Proposition 3.1 gives a necessary and sufficient condition such that the associated stochastic exponential is a true martingale. Here we provide deterministic criteria.

From Proposition 2.2.3, under $\widetilde{P}$, the diffusion Y satisfies the following SDE

$$dY_t = (\kappa\theta - (\kappa - \rho\gamma)Y_t)\mathbb{1}_{t\in[0,\zeta)}dt + \gamma\mathbb{1}_{t\in[0,\zeta)}d\widetilde{W}_t, \quad Y_0 = x_0.$$

For a positive constant $c \in J$, denote $\alpha = \kappa - \rho\gamma$, and compute the scale functions respectively of the SDE (2.1) and SDE (2.13)

$$s(x) = \int_c^x e^{\frac{\kappa(y-\theta)^2-\kappa(c-\theta)^2}{\gamma^2}}dy = C_1\int_c^x e^{\frac{\kappa(y-\theta)^2}{\gamma^2}}dy,$$
$$\widetilde{s}(x) = \int_c^x e^{\frac{\alpha y^2-2\kappa\theta y+2\kappa\theta c-\alpha c^2}{\gamma^2}}dy = \begin{cases} C_2\int_c^x e^{\frac{\alpha(y-\frac{\kappa\theta}{\alpha})^2}{\gamma^2}}dy, & \text{if } \alpha \neq 0, \\ C_3\left(e^{-\frac{2\kappa\theta c}{\gamma^2}} - e^{-\frac{2\kappa\theta}{\gamma^2}x}\right), & \text{if } \alpha = 0, \end{cases}$$

with constants $C_1 = e^{-\kappa(c-\theta)^2/\gamma^2} > 0$, $C_2 = e^{(-\kappa^2\theta^2/\alpha+2\kappa\theta c-\alpha c^2)/\gamma^2} > 0$ for $\alpha \neq 0$, and the constant $C_3 = e^{2\kappa\theta c/\gamma^2}\frac{\gamma^2}{2\kappa\theta} > 0$ for $\alpha = 0$. Since $\kappa > 0$ by assumption, $e^{\frac{\kappa(y-\theta)^2}{\gamma^2}} \geqslant 1$ for any $y \in [c, x]$, with $c \in J, x \in \bar{J}$, then we have that $s(r) = s(\infty) = \infty$ always holds, and consequently $v(r) = v(\infty) = \infty$.

Proposition 2.5.7. *For the Schöbel-Zhu model in* (2.45)*, the underlying (discounted) stock price* $(S_t)_{0\leqslant t\leqslant T}, T \in [0, \infty)$ *is a true* P*-martingale.*

Proof. We aim to check the conditions in Proposition 2.4.1. For the case of the right endpoint r, depending on the sign of $\alpha = \kappa - \rho\gamma$, we have the following classification

$$\widetilde{s}(\infty) \begin{cases} < \infty, & \text{if } \alpha \leqslant 0, \\ = \infty, & \text{if } \alpha > 0. \end{cases}$$

Divide into three cases:

(i) When $\alpha > 0$, $\widetilde{s}(\infty) = \infty$, then $\widetilde{v}(\infty) = \infty$ and $\widetilde{v}_b(\infty) = \infty$.

(ii) When $\alpha = 0$

$$\begin{aligned}\widetilde{v}(x) &= \frac{1}{\kappa\theta}\int_c^x \left(1 - e^{-\frac{2\kappa\theta}{\gamma^2}(x-y)}\right) dy \\ &= \frac{1}{\kappa\theta}\left(x + \frac{\gamma^2}{2\kappa\theta}e^{\frac{2\kappa\theta}{\gamma^2}(c-x)} - c - \frac{\gamma^2}{2\kappa\theta}\right).\end{aligned}$$

Then $\widetilde{v}(\infty) = \infty$. Similarly we can compute

$$\begin{aligned}\widetilde{v}_b(x) &= \frac{1}{\kappa\theta}\int_c^x y^2\left(1 - e^{-\frac{2\kappa\theta}{\gamma^2}(x-y)}\right) dy \\ &= \frac{1}{3\kappa\theta}x^3 - e^{-\frac{2\kappa\theta}{\gamma^2}x}\int_c^x y^2 e^{\frac{2\kappa\theta}{\gamma^2}y}dy - \frac{c^3}{3\kappa\theta}.\end{aligned}$$

Since $\int_c^x y^2 e^{\frac{2\kappa\theta}{\gamma^2}y}dy \leqslant \int_c^x x^2 e^{\frac{2\kappa\theta}{\gamma^2}y}dy$, then

$$\begin{aligned}\widetilde{v}_b(x) &\geqslant \frac{1}{3\kappa\theta}x^3 - e^{-\frac{2\kappa\theta}{\gamma^2}x}\int_c^x x^2 e^{\frac{2\kappa\theta}{\gamma^2}y}dy - \frac{c^3}{3\kappa\theta} \\ &= \frac{1}{3\kappa\theta}x^3 - \frac{\gamma^2}{2\kappa\theta}x^2(1 - e^{\frac{2\kappa\theta}{\gamma^2}(c-x)}) - \frac{c^3}{3\kappa\theta}. \qquad (2.46)\end{aligned}$$

Then $\widetilde{v}_b(\infty) = \infty$ can be verified, because the right hand side of (2.46) tends to ∞ as $x \to \infty$.

(iii) When $\alpha < 0$, the test function is

$$\widetilde{v}(x) = \frac{2}{\gamma^2}\int_c^x \frac{\int_y^x \frac{\alpha}{\gamma^2}\left(z - \frac{\kappa\theta}{\alpha}\right)^2 dz}{e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}}dy = \frac{2}{\gamma^2}\int_c^x e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_y^x \frac{\alpha}{\gamma^2}\left(z - \frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy.$$

Then

$$\widetilde{v}(\infty) = \frac{2}{\gamma^2}\int_c^\infty e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_y^\infty \frac{\alpha}{\gamma^2}\left(z - \frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy. \qquad (2.47)$$

Since $\alpha < 0$ is assumed here, then $\lim_{y\to\infty} y^{-1}e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2} = 0$, and we can apply L'Hôpital's rule

$$\begin{aligned}
\lim_{y\to\infty} \frac{\int_y^\infty \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz}{y^{-1}e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}} &= \lim_{y\to\infty} \frac{-\frac{\alpha}{\gamma^2}\left(y-\frac{\kappa\theta}{\alpha}\right)^2}{e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}(-\frac{1}{y^2}+\frac{2\alpha}{\gamma^2}(1-\frac{\kappa\theta}{\alpha y}))} \\
&= \lim_{y\to\infty} \frac{1}{\frac{1}{y^2}-\frac{2\alpha}{\gamma^2}(1-\frac{\kappa\theta}{\alpha y})} \\
&= \frac{-\gamma^2}{2\alpha} > 0.
\end{aligned}$$

So as $y \to \infty$

$$\int_y^\infty \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz \sim \frac{-\gamma^2}{2\alpha} y^{-1} e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}.$$

Thus there exists $M > c > 0$, such that for $y > M$

$$\int_y^\infty \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz > \frac{-\gamma^2}{4\alpha} y^{-1} e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}. \tag{2.48}$$

Substitute (2.48) into (2.47)

$$\begin{aligned}
\widetilde{v}(\infty) &= \frac{2}{\gamma^2}\int_c^\infty e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_y^\infty \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy \\
&\geqslant \frac{2}{\gamma^2}\int_M^\infty e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_y^\infty \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy \\
&> \frac{2}{\gamma^2}\int_M^\infty e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\frac{-\gamma^2}{4\alpha} y^{-1} e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\right) dy \\
&= \frac{-1}{2\alpha}\int_M^\infty y^{-1} dy \\
&= \infty.
\end{aligned}$$

Thus $\widetilde{v}(\infty) = \infty$ in this case.

Similarly we can compute

$$\widetilde{v}_b(\infty) = \frac{2}{\gamma^2}\int_c^\infty y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_y^\infty \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right)dy. \tag{2.49}$$

With the same M as above, substitute (2.48) into (2.49)

$$\begin{aligned}
\widetilde{v}_b(\infty) &= \frac{2}{\gamma^2}\int_c^\infty y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_y^\infty \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right)dy\\
&\geqslant \frac{2}{\gamma^2}\int_M^\infty y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_y^\infty \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right)dy\\
&> \frac{2}{\gamma^2}\int_c^\infty y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\frac{-\gamma^2}{4\alpha}y^{-1}e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\right)dy\\
&= \frac{-1}{2\alpha}\int_M^\infty y dy\\
&= \infty.
\end{aligned}$$

Thus $\widetilde{v}_b(\infty) = \infty$ in this case.

Now we consider the case of the left endpoint ℓ. From the definition of $\widetilde{s}(.)$, we have that $\widetilde{s}(0) > -\infty$ for $\alpha \in \mathbb{R}$.

Similar as above, we divide into the following two cases:

(i) When $\alpha = 0$

$$\widetilde{v}(0) = \frac{1}{\kappa\theta}\left(\frac{\gamma^2}{2\kappa\theta}e^{\frac{2\kappa\theta}{\gamma^2}(c)} - c - \frac{\gamma^2}{2\kappa\theta}\right) < \infty.$$

(ii) When $\alpha \neq 0$

$$\widetilde{v}(0) = \frac{2}{\gamma^2}\int_0^c e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right)dy. \tag{2.50}$$

Since $\lim_{y\to 0} ye^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2} = 0$, we can apply L'Hôpital's rule

$$\begin{aligned}
\lim_{y\to 0} \frac{\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz}{ye^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}} &= \lim_{y\to 0} \frac{\frac{\alpha}{\gamma^2}\left(y-\frac{\kappa\theta}{\alpha}\right)^2}{e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}(1+\frac{2\alpha}{\gamma^2}y(y-\frac{\kappa\theta}{\alpha}))} \\
&= \lim_{y\to 0} \frac{1}{1+\frac{2\alpha}{\gamma^2}y(y-\frac{\kappa\theta}{\alpha}))} \\
&= 1.
\end{aligned}$$

So as $y \to 0$

$$\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz \sim ye^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}.$$

Thus there exists $0 < \varepsilon < c$, such that for $0 \leqslant y < \varepsilon$

$$\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz < 2ye^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}. \tag{2.51}$$

Substitute (2.51) into (2.50)

$$\begin{aligned}
\widetilde{v}(0) &= \frac{2}{\gamma^2}\int_0^c e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy \\
&= \frac{2}{\gamma^2}\int_0^\varepsilon e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy \\
&\quad + \frac{2}{\gamma^2}\int_\varepsilon^c e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy \\
&< \frac{2}{\gamma^2}\int_0^\varepsilon e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(2ye^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\right) dy \\
&\quad + \frac{2}{\gamma^2}\int_\varepsilon^c e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy
\end{aligned}$$

$$= \frac{2}{\gamma^2}\int_0^\varepsilon 2y dy + \frac{2}{\gamma^2}\int_\varepsilon^c e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy$$
$$= \frac{2\varepsilon^2}{\gamma^2} + \frac{2}{\gamma^2}\int_\varepsilon^c e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy$$
$$< \infty. \tag{2.52}$$

To summarize, $\widetilde{v}(\ell) < \infty$ for $\alpha \in \mathbb{R}$.

Similarly, when $\alpha = 0$

$$\widetilde{v}_b(0) = \int_0^c y^2 e^{\frac{2\kappa\theta}{\gamma^2}y} dy - \frac{c^3}{3\kappa\theta} < \infty.$$

When $\alpha \neq 0$

$$\widetilde{v}_b(0) = \frac{2}{\gamma^2}\int_0^c y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy. \tag{2.53}$$

Substitute (2.51) into (2.53), and use the same ε as above. For $0 \leqslant y < \varepsilon$

$$\widetilde{v}_b(0) = \frac{2}{\gamma^2}\int_0^c y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy$$
$$= \frac{2}{\gamma^2}\int_0^\varepsilon y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy$$
$$+ \frac{2}{\gamma^2}\int_\varepsilon^c y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy$$
$$< \frac{2}{\gamma^2}\int_0^\varepsilon y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(2y e^{\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\right) dy$$
$$+ \frac{2}{\gamma^2}\int_\varepsilon^c y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy$$

$$
\begin{aligned}
&= \frac{2}{\gamma^2}\int_0^{\varepsilon} 2y^3 dy + \frac{2}{\gamma^2}\int_{\varepsilon}^{c} y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy \\
&= \frac{\varepsilon^4}{\gamma^2} + \frac{2}{\gamma^2}\int_{\varepsilon}^{c} y^2 e^{-\frac{\alpha}{\gamma^2}(y-\frac{\kappa\theta}{\alpha})^2}\left(\int_0^y \frac{\alpha}{\gamma^2}\left(z-\frac{\kappa\theta}{\alpha}\right)^2 dz\right) dy \\
&< \infty. \qquad (2.54)
\end{aligned}
$$

To summarize, $\widetilde{v}_b(\ell) < \infty$, for $\alpha \in \mathbb{R}$.

Above all, we can summarize the results in Table 2.8. From Proposition 2.4.1 (3), for $T \in [0,\infty)$, $(S_t)_{0\leqslant t\leqslant T}$ is a true P-martingale. This completes the proof. □

Case	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\alpha \in \mathbb{R}$	$< \infty$	∞	$< \infty$	∞

Table 2.8: Classification table for the Schöbel-Zhu model

Proposition 2.5.8. *For the Schöbel-Zhu model in* (2.45)*, the underlying (discounted) stock price* $(S_t)_{0\leqslant t\leqslant\infty}$ *is a uniformly integrable* P*-martingale if and only if* $\kappa > \rho\gamma$*.*

Proof. From the proof in Proposition 2.5.7, we have the following classification:

$$
\widetilde{s}(\infty)\begin{cases} < \infty, & \text{if } \alpha \leqslant 0, \\ = \infty, & \text{if } \alpha > 0, \end{cases}
$$

and

$$
\widetilde{s}(0) > -\infty, \qquad \text{for } \alpha \in \mathbb{R}.
$$

This, combined with the classification in Table 2.8, gives us the classification in Table 2.9. From Table 2.9 and Proposition 2.4.2, we have that $(S_t)_{0\leqslant t\leqslant\infty}$ is a uniformly integrable P-martingale if and only if $\alpha > 0$, or equivalently $\kappa > \rho\gamma$. This completes the proof. □

Case	$\widetilde{s}(\ell)$	$\widetilde{s}(r)$	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\alpha \leqslant 0$	$> -\infty$	$< \infty$	$< \infty$	∞	$< \infty$	∞
$\alpha > 0$	$> -\infty$	∞	$< \infty$	∞	$< \infty$	∞

Table 2.9: Second classification table for the Schöbel-Zhu model

Under P, we have the following result on the positivity of the (discounted) stock price in the Schöbel-Zhu model.

Proposition 2.5.9. *For the Schöbel-Zhu model in* (2.45)*, we have:*

(1) $P(S_T > 0) = 1$ *for all* $T \in [0, \infty)$*,*

(2) $P(S_\infty > 0) = 1$.

Proof. Similar to the proofs of Proposition 2.5.7 and Proposition 2.5.8 with α replaced by $\kappa > 0$, we have the classification given in Table 2.10. From Table 2.10 and Proposition 2.4.3 and Proposition 2.4.4, we have the desired results. This completes the proof. □

Case	$s(\ell)$	$s(r)$	$v(\ell)$	$v(r)$	$v_b(\ell)$	$v_b(r)$
$\alpha > 0$	$> -\infty$	∞	$< \infty$	∞	$< \infty$	∞

Table 2.10: Third classification table for the Schöbel-Zhu model

2.5.4 Hull-White stochastic volatility model

Under P, the correlated Hull-White stochastic volatility model (see Hull and White (1987)) can be described by the following diffusive dynamics

$$\begin{aligned} dS_t &= S_t \sqrt{Y_t} \mathbb{1}_{t \in [0,\zeta)} dW_t^{(1)}, \quad S_0 = 1, \\ dY_t &= \mu Y_t \mathbb{1}_{t \in [0,\zeta)} dt + \sigma Y_t \mathbb{1}_{t \in [0,\zeta)} dW_t^{(2)}, \quad Y_0 = x_0, \end{aligned} \tag{2.55}$$

where $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$, $-1 \leqslant \rho \leqslant 1$, $\mu > 0$, and $\sigma > 0$. The process Y is a geometric Brownian motion process, and this implies that its natural state space is $J = (\ell, r) = (0, \infty)$. ζ is the possible exit time of the process Y from its state space J. The model (2.55) belongs to the general stochastic volatility model considered in (2.8) with $\mu(x) = \mu x$, $\sigma(x) = \sigma x$, and $b(x) = \sqrt{x}$. Clearly $\sigma(x) = \sigma x \neq 0, x \in J$, $\frac{1}{\sigma(x)^2} = \frac{1}{\sigma^2 x^2} \in L^1_{loc}(J)$, $\frac{\mu(x)}{\sigma(x)^2} = \frac{\mu}{\sigma^2 x} \in L^1_{loc}(J)$, and $\frac{b^2(x)}{\sigma^2(x)} = \frac{1}{\sigma^2 x} \in L^1_{loc}(J)$ are satisfied. This implies that the conditions (2.2) and (2.9) are satisfied.

From Proposition 2.2.3, under $\widetilde{P}$, the diffusion Y satisfies the following SDE

$$dY_t = (\mu Y_t + \rho\sigma Y_t^{\frac{3}{2}})\mathbb{1}_{t\in[0,\zeta)}dt + \sigma Y_t \mathbb{1}_{t\in[0,\zeta)}d\widetilde{W}_t, \quad Y_0 = x_0, \tag{2.56}$$

Denote $\alpha = \frac{4\mu}{\sigma^2} - 1$ and $\gamma = \frac{2\rho}{\sigma}$. For a constant $c \in J$, compute the scale functions of the SDE (2.13)

$$\begin{aligned} \widetilde{s}(x) &= \int_c^x e^{-\int_c^y \frac{2\mu u + \rho\sigma u^{3/2}}{\sigma^2 u^2}du} dy \\ &= C_1 \int_c^x y^{-\frac{2\mu}{\sigma^2}} e^{-\frac{2\rho}{\sigma}\sqrt{y}} dy, \\ &= C_1 \int_c^x y^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{y}} dy, \quad x \in \bar{J}, \end{aligned} \tag{2.57}$$

where $C_1 = c^{\frac{2\mu}{\sigma^2}} e^{\frac{2\rho}{\sigma}\sqrt{c}}$ is a positive constant.

From the definition in (2.21) and the scale function in (2.57)

$$\begin{aligned} \widetilde{v}(x) &= \int_c^x \frac{2(\widetilde{s}(x) - \widetilde{s}(y))}{\widetilde{s}'(y)\widetilde{\sigma}^2(y)} dy \\ &= \frac{2}{\sigma^2} \int_c^x \frac{\int_y^x z^{-\frac{2\mu}{\sigma^2}} e^{-\frac{2\rho}{\sigma}\sqrt{z}} dz}{y^{2-\frac{2\mu}{\sigma^2}} e^{-\frac{2\rho}{\sigma}\sqrt{y}}} dy \\ &= \frac{2}{\sigma^2} \int_c^x y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^x z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy, \end{aligned} \tag{2.58}$$

and

$$\widetilde{v}_b(x) = \frac{2}{\sigma^2} \int_c^x y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_y^x z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy. \tag{2.59}$$

Proposition 2.5.10. *For*[13] *the Hull-White model in* (2.55)*, the underlying (discounted) stock price* $(S_t)_{0\leqslant t\leqslant T}, T \in [0,\infty)$ *is a true* P*-martingale if and only if* $\rho \leqslant 0$*.*

Proof. We distinguish three situations:

(I) $\mu > \frac{1}{2}\sigma^2$. Apply a change of variable $z = \sqrt{y}$. Then $y = z^2$, $dy = 2zdz$, and

$$\begin{aligned}\widetilde{s}(x) &= 2C_1 \int_{\sqrt{c}}^{\sqrt{x}} z^{1-\frac{4\mu}{\sigma^2}} e^{-\frac{2\rho}{\sigma} z} dz \\ &= 2C_1 \int_{\sqrt{c}}^{\sqrt{x}} z^{-\alpha} e^{-\gamma z} dz, \quad x \in \bar{J}. \end{aligned} \tag{2.60}$$

Note that the function in (2.60) is similar to the scale function in (2.27), except that there is a $\sqrt{x}$ in place of x. From (2.60)

$$\widetilde{s}(\infty) = 2C_1 \int_{\sqrt{c}}^{\infty} z^{-\alpha} e^{-\gamma z} dz.$$

From the property of the gamma function

$$\widetilde{s}(\infty) \begin{cases} < \infty, & \text{if } \gamma \geqslant 0, \\ = \infty, & \text{if } \gamma < 0. \end{cases}$$

Divide into three cases based on γ:

(i) When $\gamma < 0$, $\widetilde{s}(\infty) = \infty$, then $\widetilde{v}(\infty) = \infty$ and $\widetilde{v}_b(\infty) = \infty$.

(ii) When $\gamma = 0$

$$\begin{aligned}\widetilde{v}(\infty) &= \frac{2}{\sigma^2} \int_c^{\infty} y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^{\infty} z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy \\ &= \frac{2}{\sigma^2} \int_c^{\infty} y^{\frac{\alpha-3}{2}} \left(\int_y^{\infty} z^{-\frac{\alpha+1}{2}} dz \right) dy\end{aligned}$$

[13] Proposition 2.5.10 is equivalent to Theorem 1 of Jourdain (2004), and a special case of Proposition 2.5., p34 of Andersen and Piterbarg (2007).

$$= \frac{4}{\sigma^2(\alpha-1)} \int_c^\infty y^{-1} dy$$
$$= \infty,$$

and

$$\begin{aligned}
\widetilde{v}_b(\infty) &= \frac{2}{\sigma^2} \int_c^\infty y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy \\
&= \frac{2}{\sigma^2} \int_c^\infty y^{\frac{\alpha-1}{2}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} dz \right) dy \\
&= \int_c^\infty \frac{4}{\sigma^2(\alpha-1)} dy \\
&= \infty.
\end{aligned}$$

(iii) When $\gamma > 0$, from (2.58)

$$\widetilde{v}(\infty) = \frac{2}{\sigma^2} \int_c^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy, \tag{2.61}$$

and

$$\widetilde{v}_b(\infty) = \frac{2}{\sigma^2} \int_c^\infty y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy. \tag{2.62}$$

Since $\alpha > 1$, then $\lim_{y\to\infty} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}} = 0$, and from L'Hôpital's rule

$$\lim_{y\to\infty} \frac{\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz}{y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}} = \lim_{y\to\infty} \frac{1}{\frac{\alpha}{2} y^{-1/2} + \frac{\gamma}{2}} = \frac{2}{\gamma} > 0.$$

As $y \to \infty$

$$\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \sim \frac{2}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}. \tag{2.63}$$

From (2.63), there exists $0 < M < \infty$, such that for $y > M$

$$\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz < \frac{4}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}. \tag{2.64}$$

Substitute (2.64) into (2.61)

$$\begin{aligned}
\widetilde{v}(\infty) &= \frac{2}{\sigma^2} \int_c^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy \\
&= \frac{2}{\sigma^2} \int_c^M y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy \\
&\quad + \frac{2}{\sigma^2} \int_M^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy \\
&< \frac{2}{\sigma^2} \int_c^M y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy \\
&\quad + \frac{2}{\sigma^2} \int_M^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\frac{4}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}} \right) dy \\
&= \frac{2}{\sigma^2} \int_c^M y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy \\
&\quad + \frac{16}{\sqrt{M}\gamma\sigma^2} \\
&< \infty.
\end{aligned}$$

Then $\widetilde{v}(\infty) < \infty$, for $\gamma > 0$.

From (2.63), there exists $0 < c < M' < \infty$, such that for $y > M'$

$$\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz > \frac{1}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}. \tag{2.65}$$

Substitute (2.65) into (2.62)

$$\begin{aligned}
\widetilde{v}(\infty) &= \frac{2}{\sigma^2} \int_c^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy \\
&\geqslant \frac{2}{\sigma^2} \int_{M'}^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy
\end{aligned}$$

$$
\begin{aligned}
&> \frac{2}{\sigma^2} \int_{M'}^{\infty} y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\frac{1}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}} \right) dy \\
&= \frac{2}{\gamma\sigma^2} \int_{M'}^{\infty} y^{-1} dy \\
&= \infty.
\end{aligned}
$$

Then $\widetilde{v}_b(\infty) = \infty$, for $\gamma > 0$.

We now look at the case of the left boundary ℓ. From (2.60)

$$\widetilde{s}(0) = -2C_1 \int_0^{\sqrt{c}} z^{-\alpha} e^{-\gamma z} dz.$$

When $\gamma > 0$, since $\alpha > 1$, from the property of the gamma function, we have $\widetilde{s}(0) = -\infty$. When $\gamma \leqslant 0$, then $e^{-\gamma z} \geqslant 1$, and

$$\widetilde{s}(0) = -2C_1 \int_0^{\sqrt{c}} z^{-\alpha} e^{-\gamma z} dz \leqslant -2C_1 \int_0^{\sqrt{c}} z^{-\alpha} dz = -\infty.$$

To summarize, $\widetilde{s}(0) = -\infty$ for $\gamma \in \mathbb{R}$. Then $\widetilde{v}(0) = \infty$ and $\widetilde{v}_b(0) = \infty$ hold.

Above all, when $\alpha > 1$, we have the following Table 2.11. The results in Table 2.11, combined with Proposition 2.4.1, imply that, for $\alpha > 1$, $(S_t)_{0 \leqslant t \leqslant T}, T \in [0, \infty)$ is a true P-martingale if and only if $\widetilde{v}(r) = \infty$. This is equivalent to $\gamma \leqslant 0$, and further equivalent to $\rho \leqslant 0$ from the definition of γ. This completes the proof of situation (I).

Case	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\gamma \leqslant 0$	∞	∞	∞	∞
$\gamma > 0$	∞	$< \infty$	∞	∞

Table 2.11: Classification table for the Hull-White model when $2\mu/\sigma^2 > 1$

(II) $\mu = \frac{1}{2}\sigma^2$. We consider the case when $\alpha = 1$. Then

$$\widetilde{s}(\infty) = 2C_1 \int_{\sqrt{c}}^{\infty} z^{-1} e^{-\gamma z} dz,$$

Divide into two cases based on the value of γ. If $\gamma \leqslant 0$, then $e^{-\gamma z} \geqslant 1$, and

$$\widetilde{s}(\infty) \geqslant 2C_1 \int_{\sqrt{c}}^{\infty} z^{-1} dz = \infty.$$

Then in this case, $\widetilde{v}(r) = \infty$ and $\widetilde{v}_b(r) = \infty$.

If $\gamma > 0$, from properties of the gamma function, $\widetilde{s}(\infty) < \infty$. To summarize, when $\alpha = 1$

$$\widetilde{s}(\infty) \begin{cases} = \infty, & \text{if } \gamma \leqslant 0, \\ < \infty, & \text{if } \gamma > 0. \end{cases}$$

Similarly for the case of the left boundary ℓ. If $\gamma > 0$, from the properties of the gamma function, $\widetilde{s}(0) = -\infty$. If $\gamma \leqslant 0$, then $e^{-\gamma z} \geqslant 1$, and

$$\widetilde{s}(0) \leqslant -2C_1 \int_0^{\sqrt{c}} z^{-1} dz = -\infty.$$

To summarize, when $\alpha = 1$, we have $\widetilde{s}(\ell) = -\infty$, then $\widetilde{v}(\ell) = \infty$ and $\widetilde{v}_b(\ell) = \infty$.

Consider the case when $\alpha = 1$ and $\gamma > 0$, from the above result, there is $\widetilde{s}(\infty) < \infty$, and we aim to study the properties of $\widetilde{v}(\infty)$ and $\widetilde{v}_b(\infty)$. From the definition in (2.58)

$$\widetilde{v}(\infty) = \frac{2}{\sigma^2} \int_c^{\infty} y^{-1} e^{\gamma\sqrt{y}} \left(\int_y^{\infty} z^{-1} e^{-\gamma\sqrt{z}} dz \right) dy. \tag{2.66}$$

Since $\gamma > 0$, then $\lim_{y\to\infty} y^{-\frac{1}{2}} e^{-\gamma\sqrt{y}} = 0$, and from L'Hôpital's rule

$$\lim_{y\to\infty} \frac{\int_y^{\infty} z^{-1} e^{-\gamma\sqrt{z}} dz}{y^{-\frac{1}{2}} e^{-\gamma\sqrt{y}}} = \lim_{y\to\infty} \frac{1}{\frac{1}{2} y^{-1/2} + \frac{\gamma}{2}} = \frac{2}{\gamma} > 0.$$

As $y \to \infty$

$$\int_y^{\infty} z^{-1} e^{-\gamma\sqrt{z}} dz \sim \frac{2}{\gamma} y^{-\frac{1}{2}} e^{-\gamma\sqrt{y}}. \tag{2.67}$$

Then there exists $M < \infty$, such that for $y > M$

$$\int_y^\infty z^{-1} e^{-\gamma\sqrt{z}} dz < \frac{4}{\gamma} y^{-\frac{1}{2}} e^{-\gamma\sqrt{y}}. \tag{2.68}$$

Substitute (2.68) into (2.66)

$$\begin{aligned}
\widetilde{v}(\infty) &= \frac{2}{\sigma^2} \int_c^\infty y^{-1} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-1} e^{-\gamma\sqrt{z}} dz \right) dy \\
&= \frac{2}{\sigma^2} \int_c^M y^{-1} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-1} e^{-\gamma\sqrt{z}} dz \right) dy \\
&\quad + \frac{2}{\sigma^2} \int_M^\infty y^{-1} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-1} e^{-\gamma\sqrt{z}} dz \right) dy \\
&< \frac{2}{\sigma^2} \int_c^M y^{-1} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-1} e^{-\gamma\sqrt{z}} dz \right) dy \\
&\quad + \frac{2}{\sigma^2} \int_M^\infty y^{-1} e^{\gamma\sqrt{y}} \left(\frac{4}{\gamma} y^{-\frac{1}{2}} e^{-\gamma\sqrt{y}} \right) dy \\
&= \frac{2}{\sigma^2} \int_c^M y^{-1} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-1} e^{-\gamma\sqrt{z}} dz \right) dy \\
&\quad + \frac{8}{\gamma\sigma^2} \int_M^\infty y^{-\frac{3}{2}} dy \\
&< \infty.
\end{aligned}$$

From the definition in (2.58)

$$\widetilde{v}_b(\infty) = \frac{2}{\sigma^2} \int_c^\infty e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-1} e^{-\gamma\sqrt{z}} dz \right) dy, \tag{2.69}$$

From (2.67), there exits $M' > c > 0$, such that for $y > M'$

$$\int_y^\infty z^{-1} e^{-\gamma\sqrt{z}} dz > \frac{1}{\gamma} y^{-\frac{1}{2}} e^{-\gamma\sqrt{y}}. \tag{2.70}$$

Substitute (2.70) into (2.69)

$$\begin{aligned}
\widetilde{v}_b(\infty) &= \frac{2}{\sigma^2}\int_c^\infty e^{\gamma\sqrt{y}}\left(\int_y^\infty z^{-1}e^{-\gamma\sqrt{z}}dz\right)dy \\
&\geqslant \frac{2}{\sigma^2}\int_{M'}^\infty e^{\gamma\sqrt{y}}\left(\int_y^\infty z^{-1}e^{-\gamma\sqrt{z}}dz\right)dy \\
&> \frac{2}{\sigma^2}\int_{M'}^\infty e^{\gamma\sqrt{y}}\left(\frac{1}{\gamma}y^{-\frac{1}{2}}e^{-\gamma\sqrt{y}}\right)dy \\
&= \frac{2}{\sigma^2}\int_{M'}^\infty y^{-\frac{1}{2}}dy \\
&= \infty.
\end{aligned}$$

When $\alpha = 1$, the results are summarized in Table 2.12.

Case	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\gamma \leqslant 0$	∞	∞	∞	∞
$\gamma > 0$	∞	$< \infty$	∞	∞

Table 2.12: Classification table for the Hull-White model when $2\mu/\sigma^2 = 1$

The results in Table 2.12, combined with Proposition 2.4.1, imply that, for $\alpha = 1$, $(S_t)_{0\leqslant t\leqslant T}, T \in [0,\infty)$ is a true P-martingale if and only if $\widetilde{v}(r) = \infty$. This is equivalent to $\gamma \leqslant 0$, and further equivalent to $\rho \leqslant 0$ from the definition of γ. This completes the proof of situation (II).

(III) $\mu < \frac{1}{2}\sigma^2$. We consider the case when $\alpha < 1$. Since $-\frac{\alpha+1}{2} > -1$, then from the property of the gamma function

$$\widetilde{s}(0) = -C_1\int_0^c y^{-\frac{\alpha+1}{2}}e^{-\gamma\sqrt{y}}dy > -\infty.$$

From (2.60), we have $\widetilde{s}(\infty) = 2C_1\int_{\sqrt{c}}^\infty z^{-\alpha}e^{-\gamma z}dz$, and divide into three cases. If $\gamma > 0$, then from the property of gamma function, $\widetilde{s}(\infty) < \infty$. If $\gamma \leqslant 0$, then $e^{-\gamma z} \geqslant 1$, and $\widetilde{s}(\infty) \geqslant 2C_1\int_{\sqrt{c}}^\infty z^{-\alpha}dz = \infty$. To summarize, when $\alpha < 1$

$$\widetilde{s}(\infty) \begin{cases} = \infty, & \text{if } \gamma \leqslant 0, \\ < \infty, & \text{if } \gamma > 0. \end{cases}$$

We first look at $\widetilde{v}(0)$ and $\widetilde{v}_b(0)$. From the definition in (2.58)

$$\widetilde{v}(0) = \frac{2}{\sigma^2} \int_0^c y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy, \tag{2.71}$$

and

$$\widetilde{v}_b(0) = \frac{2}{\sigma^2} \int_0^c y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy. \tag{2.72}$$

Divide into two cases based on γ. When $\gamma \leqslant 0$, $e^{-\gamma\sqrt{z}} \geqslant 1$, then

$$\begin{aligned} \widetilde{v}(0) &= \frac{2}{\sigma^2} \int_0^c y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \right) dy \\ &\geqslant \frac{2}{\sigma^2} \int_0^c y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} dz \right) dy \\ &= \frac{2}{\sigma^2} \int_0^c y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\frac{2}{1-\alpha} y^{\frac{1-\alpha}{2}} \right) dy \\ &= \frac{4}{\sigma^2(1-\alpha)} \int_0^c y^{-1} e^{\gamma\sqrt{y}} dy. \end{aligned}$$

Apply a change of variable $z = \sqrt{y}$, then

$$\begin{aligned} \widetilde{v}(0) &\geqslant \frac{4}{\sigma^2(1-\alpha)} \int_0^c y^{-1} e^{\gamma\sqrt{y}} dy \\ &= \frac{8}{\sigma^2(1-\alpha)} \int_0^{\sqrt{c}} z^{-1} e^{\gamma z} dz \\ &= \infty. \end{aligned}$$

The last equality is from the property of the gamma function. Then $\widetilde{v}(0) = \infty$ holds when $\gamma \leqslant 0$.

Since $\gamma \leqslant 0$ is assumed, then $e^{-\gamma\sqrt{z}} \leqslant e^{-\gamma\sqrt{y}}$ for $0 \leqslant z \leqslant y$, and

$$\begin{aligned}
\widetilde{v}_b(0) &= \frac{2}{\sigma^2}\int_0^c y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&\leqslant \frac{2}{\sigma^2}\int_0^c y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{y}} dz\right) dy \\
&= \frac{2}{\sigma^2}\int_0^c y^{\frac{\alpha-1}{2}} \left(\frac{2}{1-\alpha} y^{\frac{1-\alpha}{2}}\right) dy \\
&= \frac{4c}{\sigma^2(1-\alpha)} \\
&< \infty.
\end{aligned}$$

Then $\widetilde{v}_b(0) < \infty$ holds when $\gamma \leqslant 0$.

When $\gamma > 0$, $e^{-\gamma\sqrt{z}} > e^{-\gamma\sqrt{y}}$ for $0 \leqslant z \leqslant y$, then

$$\begin{aligned}
\widetilde{v}(0) &= \frac{2}{\sigma^2}\int_0^c y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&> \frac{2}{\sigma^2}\int_0^c y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{y}} dz\right) dy \\
&= \frac{2}{\sigma^2}\int_0^c y^{\frac{\alpha-3}{2}} \left(\frac{2}{1-\alpha} y^{\frac{1-\alpha}{2}}\right) dy \\
&= \frac{4}{\sigma^2(1-\alpha)} \int_0^c y^{-1} dy \\
&= \infty.
\end{aligned}$$

Then $\widetilde{v}(0) = \infty$ holds when $\gamma > 0$.

When $\gamma > 0$, $e^{-\gamma\sqrt{z}} < 1$ for $0 \leqslant z \leqslant y$, then

$$\begin{aligned}
\widetilde{v}_b(0) &= \frac{2}{\sigma^2}\int_0^c y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&< \frac{2}{\sigma^2}\int_0^c y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_0^y z^{-\frac{\alpha+1}{2}} dz\right) dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\sigma^2}\int_0^c y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\frac{2}{1-\alpha} y^{\frac{1-\alpha}{2}}\right) dy \\
&= \frac{4}{\sigma^2(1-\alpha)}\int_0^c e^{\gamma\sqrt{y}} dy \\
&< \infty.
\end{aligned}
$$

Then $\widetilde{v}_b(0) < \infty$ holds when $\gamma > 0$.

To summarize, we have that $\widetilde{v}(0) = \infty$ and $\widetilde{v}_b(0) < \infty$ hold when $\alpha < 1$.

Consider the case when $\alpha < 1$ and $\gamma > 0$. From the definition in (2.58)

$$
\widetilde{v}(\infty) = \frac{2}{\sigma^2}\int_c^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy, \tag{2.73}
$$

and

$$
\widetilde{v}_b(\infty) = \frac{2}{\sigma^2}\int_c^\infty y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy. \tag{2.74}
$$

Since $\gamma > 0$ is assumed, then $\lim_{y\to\infty} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}} = 0$, and we can apply L'Hôpital's rule

$$
\lim_{y\to\infty} \frac{\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz}{y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}} = \lim_{y\to\infty} \frac{1}{\frac{\alpha}{2} y^{-\frac{1}{2}} + \frac{\gamma}{2}} = \frac{2}{\gamma} > 0.
$$

As $y \to \infty$

$$
\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz \sim \frac{2}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}. \tag{2.75}
$$

From (2.75), there exists $M > 0$, such that for $y > M$

$$
\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz < \frac{4}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}. \tag{2.76}
$$

Substitute (2.76) into (2.73)

$$\begin{aligned}
\widetilde{v}(\infty) &= \frac{2}{\sigma^2}\int_c^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&= \frac{2}{\sigma^2}\int_c^M y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&\quad + \frac{2}{\sigma^2}\int_M^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&< \frac{2}{\sigma^2}\int_c^M y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&\quad + \frac{2}{\sigma^2}\int_M^\infty y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\frac{4}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}\right) dy \\
&= \frac{2}{\sigma^2}\int_c^M y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&\quad + \frac{8}{\gamma\sigma^2}\int_M^\infty y^{-\frac{3}{2}} dy \\
&= \frac{2}{\sigma^2}\int_c^M y^{\frac{\alpha-3}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&\quad + \frac{16}{\sqrt{M}\gamma\sigma^2} \\
&< \infty.
\end{aligned}$$

Then $\widetilde{v}(\infty) < \infty$, for $\alpha < 1$ and $\gamma > 0$.

From (2.75), there exists $M' > c > 0$, such that for $y > M'$

$$\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz > \frac{1}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}. \tag{2.77}$$

Substitute (2.77) into (2.74)

$$\widetilde{v}_b(\infty) = \frac{2}{\sigma^2}\int_c^\infty y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}} \left(\int_y^\infty z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy$$

$$
\begin{aligned}
&\geqslant \frac{2}{\sigma^2}\int_{M'}^{\infty} y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}}\left(\int_{y}^{\infty} z^{-\frac{\alpha+1}{2}} e^{-\gamma\sqrt{z}} dz\right) dy \\
&> \frac{2}{\sigma^2}\int_{M'}^{\infty} y^{\frac{\alpha-1}{2}} e^{\gamma\sqrt{y}}\left(\frac{1}{\gamma} y^{-\frac{\alpha}{2}} e^{-\gamma\sqrt{y}}\right) dy \\
&= \frac{2}{\gamma\sigma^2}\int_{M'}^{\infty} y^{-\frac{1}{2}} dy \\
&= \infty.
\end{aligned}
$$

Then $\widetilde{v}_b(\infty) = \infty$, for $\alpha < 1$ and $\gamma > 0$.

When $\alpha < 1$, the results are summarized in Table 2.13.

Case	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\gamma \leqslant 0$	∞	∞	$< \infty$	∞
$\gamma > 0$	∞	$< \infty$	$< \infty$	∞

Table 2.13: Classification table for the Hull-White model when $2\mu/\sigma^2 < 1$

The results in Table 2.13, combined with Proposition 2.4.1, imply that, for $\alpha < 1$, $(S_t)_{0\leqslant t\leqslant T}, T \in [0,\infty)$ is a true P-martingale if and only if $\widetilde{v}(r) = \infty$. This is equivalent to $\gamma \leqslant 0$, and further equivalent to $\rho \leqslant 0$ from the definition of γ. This completes the proof of situation (III). □

Proposition 2.5.11. *For the Hull-White model in* (2.55)*, the underlying (discounted) stock price* $(S_t)_{0\leqslant t\leqslant \infty}$ *is a uniformly integrable* P*-martingale if and only if* $\mu < \frac{1}{2}\sigma^2$ *and* $\rho \leqslant 0$*.*

Proof. From the proof in Proposition 2.5.10, we divide into the following three cases:

(I) $\mu > \frac{1}{2}\sigma^2$. Then we have the following classification:

$$
\widetilde{s}(\infty)\begin{cases} < \infty, & \text{if } \gamma \geqslant 0, \\ = \infty, & \text{if } \gamma < 0, \end{cases}
$$

and

$$\widetilde{s}(0) = -\infty, \qquad \text{for } \gamma \in \mathbb{R}.$$

This, combined with the classification in Table 2.11, gives us the classification in Table 2.14. From Table 2.14 and Proposition 2.4.2, we have that when $\mu > \frac{1}{2}\sigma^2$, $(S_t)_{0 \leqslant t \leqslant \infty}$ is not a uniformly integrable P-martingale.

Case	$\widetilde{s}(\ell)$	$\widetilde{s}(r)$	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\gamma > 0$	$-\infty$	$< \infty$	∞	$< \infty$	∞	∞
$\gamma = 0$	$-\infty$	$< \infty$	∞	∞	∞	∞
$\gamma < 0$	$-\infty$	∞	∞	∞	∞	∞

Table 2.14: Second classification table: Hull-White model when $2\mu/\sigma^2 > 1$

(II) $\mu = \frac{1}{2}\sigma^2$. Then we have the following classification:

$$\widetilde{s}(\infty) \begin{cases} < \infty, & \text{if } \gamma > 0, \\ = \infty, & \text{if } \gamma \leqslant 0, \end{cases}$$

and

$$\widetilde{s}(0) = -\infty, \qquad \text{for } \gamma \in \mathbb{R}.$$

This, combined with the classification in Table 2.12, gives us the classification in Table 2.15. From Table 2.15 and Proposition 2.4.2, we have that when $\mu = \frac{1}{2}\sigma^2$, $(S_t)_{0 \leqslant t \leqslant \infty}$ is not a uniformly integrable P-martingale.

Case	$\widetilde{s}(\ell)$	$\widetilde{s}(r)$	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\gamma > 0$	$-\infty$	$< \infty$	∞	$< \infty$	∞	∞
$\gamma = 0$	$-\infty$	∞	∞	∞	∞	∞
$\gamma < 0$	$-\infty$	∞	∞	∞	∞	∞

Table 2.15: Second classification table: Hull-White model when $2\mu/\sigma^2 = 1$

(III) $\mu < \frac{1}{2}\sigma^2$. Then we have the following classification:

$$\widetilde{s}(\infty)\begin{cases} < \infty, & \text{if } \gamma > 0, \\ = \infty, & \text{if } \gamma \leqslant 0, \end{cases}$$

and

$$\widetilde{s}(0) > -\infty, \quad \text{for } \gamma \in \mathbb{R}.$$

This, combined with the classification in Table 2.13, gives us the classification in Table 2.16. From Table 2.16 and Proposition 2.4.2, we have that when $\mu < \frac{1}{2}\sigma^2$, $(S_t)_{0\leqslant t\leqslant\infty}$ is a uniformly integrable P-martingale if and only if $\gamma \leqslant 0$, or equivalently $\rho \leqslant 0$. This completes the proof. □

Case	$\widetilde{s}(\ell)$	$\widetilde{s}(r)$	$\widetilde{v}(\ell)$	$\widetilde{v}(r)$	$\widetilde{v}_b(\ell)$	$\widetilde{v}_b(r)$
$\gamma > 0$	$> -\infty$	$< \infty$	∞	$< \infty$	$< \infty$	∞
$\gamma = 0$	$> -\infty$	∞	∞	∞	$< \infty$	∞
$\gamma < 0$	$> -\infty$	∞	∞	∞	$< \infty$	∞

Table 2.16: Second classification table: Hull-White model when $2\mu/\sigma^2 < 1$

Under P, we have the following result on the positivity of the (discounted) stock price in the Hull-White model.

Proposition 2.5.12. *For the Hull-White model in* (2.55)*, we have:*

(1) $P(S_T > 0) = 1$ *for all* $T \in [0, \infty)$*,*

(2) $P(S_\infty > 0) = 1$ *if and only if* $\frac{2\mu}{\sigma^2} < 1$.

Proof. Similar to the proofs of Proposition 2.5.10 and Proposition 2.5.11 with $\gamma = 0$, we have the classification in Table 2.17. From Table 2.17 and Proposition 2.4.3 and Proposition 2.4.4, we have the desired results. This completes the proof. □

Case	$s(\ell)$	$s(r)$	$v(\ell)$	$v(r)$	$v_b(\ell)$	$v_b(r)$
$2\mu/\sigma^2 > 1$	$-\infty$	$< \infty$	∞	∞	∞	∞
$2\mu/\sigma^2 = 1$	$-\infty$	∞	∞	∞	∞	∞
$2\mu/\sigma^2 < 1$	$> -\infty$	∞	∞	∞	$< \infty$	∞

Table 2.17: Third classification table for the Hull-White model

2.6 Stochastic time-change transformation

From Proposition 2.2.4 and Proposition 2.2.5, to determine whether a stochastic exponential is a true P-martingale on $[0, T]$ (or a uniformly integrable P-martingale on $[0, \infty]$) or not, the goal is to find deterministic necessary and sufficient conditions for $\widetilde{P}(\varphi_{\zeta\wedge T} < \infty) = 1$ (or $\widetilde{P}(\varphi_\zeta < \infty) = 1$) to hold. To find criteria for the convergence or divergence of these integral functionals of diffusions, we introduce the stochastic time-change approach in this section. As an application, we provide an alternative simple proof to the Engelbert-Schmidt type zero-one law with slightly stronger assumptions.

In and only in this section, we make a stronger assumption and assume that $\lambda(x \in (\ell, r) : b^2(x) = 0) = 0$, which means that the function $b(.)$ is positive.

Theorem 2.6.1. *Assume that the conditions* (2.2) (2.9) *are satisfied, and* $\lambda(x \in (\ell, r) : b^2(x) = 0) = 0$.

(i) Under[14] $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$*, define*

$$T_t := \begin{cases} \inf\{u \geqslant 0 : \varphi_{u\wedge\zeta} > t\}, & \text{on } \{0 \leqslant t < \varphi_\zeta\}, \\ \infty, & \text{on } \{\varphi_\zeta \leqslant t < \infty\}. \end{cases} \tag{2.78}$$

Define a new filtration $\mathcal{G}_t = \mathcal{F}_{T_t}, t \in [0, \infty)$*, and a new process* $X_t := Y_{T_t}$*, on* $\{0 \leqslant t < \varphi_\zeta\}$*. Then* X_t *is* $\mathcal{G}_t$*-adapted and we have the stochastic*

[14]The statements (i) and (ii) in Theorem 2.6.1 are consequences of well-known results on stochastic time-change, see section III 21, p277 of Rogers and Williams (1994), p1248 of Cissé, Patie and Tanré (2012). For completeness, we provide a proof here.

representation

$$Y_t = X_{\int_0^t b^2(Y_s)ds} = X_{\varphi_t},\ P\text{-a.s.} \qquad on\ \{0 \leqslant t < \zeta\}, \tag{2.79}$$

and the process X is a time-homogeneous diffusion, which solves the following SDE under P

$$dX_t = \frac{\mu(X_t)}{b^2(X_t)} \mathbb{1}_{t\in[0,\varphi_\zeta)} dt + \frac{\sigma(X_t)}{b(X_t)} \mathbb{1}_{t\in[0,\varphi_\zeta)} dB_t, \quad X_0 = x_0, \tag{2.80}$$

where B_t is the $\mathcal{G}_t$-adapted Dambis-Dubins-Schwartz Brownian motion under P defined in the proof. Similar results hold under $\widetilde{P}$.

(ii) Define $\zeta^X := \inf\{u > 0 : X_u \notin J\}$, then $\zeta^X = \varphi_\zeta = \int_0^\zeta b^2(Y_s)ds$, P-a.s., and we can rewrite the SDE (2.80) *under P as*

$$dX_t = \frac{\mu(X_t)}{b^2(X_t)} \mathbb{1}_{t\in[0,\zeta^X)} dt + \frac{\sigma(X_t)}{b(X_t)} \mathbb{1}_{t\in[0,\zeta^X)} dB_t, \quad X_0 = x_0. \tag{2.81}$$

Similar results hold under $\widetilde{P}$.

(iii) The event $\left\{\limsup_{t\to\zeta} Y_t = r\right\}$ is identical to $\left\{\limsup_{t\to\zeta^X} X_t = r\right\}$. Similarly for the case of the left boundary ℓ, the case of lim inf, lim *and also the case under $\widetilde{P}$.*

Proof. Since $\lambda(x \in (\ell, r) : b^2(x) = 0) = 0$, φ_s is an increasing function on $[0,\zeta]$. From Problem 3.4.5 (ii)[15], p174 of Karatzas and Shreve (1991), $\varphi_{T_t\wedge\zeta} = t \wedge \varphi_\zeta$, P-a.s. for $0 \leqslant t < \infty$. On $\{0 \leqslant t < \varphi_\zeta\}$, take $u = \zeta$, then $\varphi_{\zeta\wedge\zeta} = \varphi_\zeta > t$ holds P-a.s. according to the assumption. Then $T_t \leqslant \zeta$, P-a.s. because of the definition (2.78), $T_t := \inf\{u \geqslant 0 : \varphi_{u\wedge\zeta} > t\}$. Thus $\varphi_{T_t} = t$, P-a.s. on $\{0 \leqslant t < \varphi_\zeta\}$.

Choose $t = \varphi_s$ on $\{0 \leqslant s < \zeta\}$, then $0 \leqslant t < \varphi_\zeta$, P-a.s.. After substituting this t into the definition of the process X, we have $X_{\varphi_s} = X_t := Y_{T_t} = Y_{T_{\varphi_s}} = Y_s$, P-a.s.. For the last equality, recall the definition and $T_{\varphi_s} = \inf\{u \geqslant 0 : \varphi_{u\wedge\zeta} > \varphi_s\} = \inf\{u \geqslant 0 : u \wedge \zeta > s\} = s$, P-a.s., on

[15] See Section 2.8 for the statement and proof of this Problem 3.4.5.

$\{0 \leqslant s < \zeta\}$. Then we have proved the representation $Y_s = X_{\varphi_s}$, P-a.s. on $\{0 \leqslant s < \zeta\}$, and the next goal is to determine the coefficients of the SDE satisfied by X under P.

For X satisfying the relation (2.79), we aim to show that X satisfies the following SDE under P

$$dX_t = \frac{\mu(X_t)}{b^2(X_t)} \mathbb{1}_{t \in [0, \varphi_\zeta)} dt + \frac{\sigma(X_t)}{b(X_t)} \mathbb{1}_{t \in [0, \varphi_\zeta)} dB_t, \quad X_0 = Y_0 = x_0. \tag{2.82}$$

where B is the Dambis-Dubins-Schwartz Brownian motion adapted to $\mathcal{G}_t$ constructed as follows:

Note that $M_{t\wedge\zeta} = \int_0^{t\wedge\zeta} b(Y_u) dW_u, t \in [0, \infty)$ is a continuous local martingale with quadratic variation $\varphi_{t\wedge\zeta} = \int_0^{t\wedge\zeta} b^2(Y_u) du, t \in [0, \infty)$. Then $\lim_{t\to\infty} \varphi_{t\wedge\zeta} = \varphi_\zeta$, P-a.s. due to the left continuity of φ_s at $s = \zeta$ (see Remark 2.3.2).

From the Dambis-Dubins-Schwartz theorem (Ch.V, Theorem 1.6 and Theorem 1.7 of Revuz and Yor (1999)), there exists an enlargement $(\overline{\Omega}, \overline{\mathcal{G}_t})$ of $(\Omega, \mathcal{G}_t)$ and a standard Brownian motion $\overline{\beta}$ on $\overline{\Omega}$ independent of M with $\overline{\beta}_0 = 0$, such that the process

$$B_t := \begin{cases} \int_0^{T_t} b(Y_u) dW_u, & \text{on } \{t < \varphi_\zeta\}, \\ \int_0^{\zeta} b(Y_u) dW_u + \widetilde{\beta}_{t-\varphi_\zeta}, & \text{on } \{t \geqslant \varphi_\zeta\}. \end{cases} \tag{2.83}$$

is a standard linear Brownian motion. Our construction of $T_t, t \in [0, \infty)$ agrees with that in Problem 3.4.5, p174 of Karatzas and Shreve (1991). From Problem 3.4.5 (ii) and the construction (2.83), $B_{\varphi_s} = M_s$, P-a.s. on $\{0 \leqslant s < \zeta\}$, and on $\{s = \zeta\}$, $B_{\varphi_\zeta} := \int_0^\zeta b(Y_u) dW_u =: M_\zeta$, P-a.s.. Thus $B_{\varphi_t} = M_t$, P-a.s. on $\{0 \leqslant t \leqslant \zeta\}$.

For the convenience of exposition, denote $\mu_1(.) = \mu(.)/b^2(.)$, and $\sigma_1(.) =$

$\sigma(.)/b(.)$. Integrate the SDE in (2.1) under P from 0 to $t \wedge \zeta$

$$\begin{aligned} Y_{t\wedge\zeta} - Y_0 &= \int_0^{t\wedge\zeta} \mu(Y_u)du + \int_0^{t\wedge\zeta} \sigma(Y_u)dW_u \\ &= \int_0^{t\wedge\zeta} \mu_1(Y_u)b^2(Y_u)du + \int_0^{t\wedge\zeta} \sigma_1(Y_u)b(Y_u)dW_u. \end{aligned} \tag{2.84}$$

Apply the change of variables formula similar to Problem 3.4.5 (vi)[16], p174 of Karatzas and Shreve (1991), and note the relation (2.79)

$$\int_0^{t\wedge\zeta} \mu_1(Y_u)b^2(Y_u)du = \int_0^{t\wedge\zeta} \mu_1(X_{\varphi_u})d\varphi_u = \int_0^{\varphi_{t\wedge\zeta}} \mu_1(X_u)du, \tag{2.85}$$

and similarly

$$\int_0^{t\wedge\zeta} \sigma_1(Y_u)b(Y_u)dW_u = \int_0^{t\wedge\zeta} \sigma_1(X_{\varphi_u})dB_{\varphi_u} = \int_0^{\varphi_{t\wedge\zeta}} \sigma_1(X_u)dB_u, \tag{2.86}$$

where the first equality in (2.86) is due to the relationship $B_{\varphi_u} = M_u = \int_0^u b(V_s)dW_s$, P-a.s., on $\{0 \leqslant u \leqslant t \wedge \zeta\}$, which we have established above. Also notice the representation $Y_{t\wedge\zeta} = X_{\varphi_{t\wedge\zeta}}$, P-a.s., and $Y_0 = X_0$, then

$$X_{\varphi_{t\wedge\zeta}} - X_0 = \int_0^{\varphi_{t\wedge\zeta}} \mu_1(X_u)du + \int_0^{\varphi_{t\wedge\zeta}} \sigma_1(X_u)dB_u. \tag{2.87}$$

Then on $\{0 \leqslant s \leqslant \varphi_{t\wedge\zeta}\}$

$$X_s - X_0 = \int_0^s \mu_1(X_u)du + \int_0^s \sigma_1(X_u)dB_u. \tag{2.88}$$

Note that for $0 \leqslant t < \infty$, we have $s \in [0, \varphi_\zeta]$, P-a.s. From (2.88), and recall the definition of $\mu_1(.)$ and $\sigma_1(.)$, we have the following SDE for X under P:

$$dX_s = \frac{\mu(X_s)}{m^2(X_s)}\mathbb{1}_{s\in[0,\varphi_\zeta)}ds + \frac{\sigma(X_s)}{m(X_s)}\mathbb{1}_{s\in[0,\varphi_\zeta)}dB_s, \quad X_0 = Y_0 = x_0.$$

[16] See Section 2.8 in Chapter 2 for the statement and the proof.

This completes the proof of statement (i).

Statement (ii) is a direct consequence of the stochastic representation $Y_{t\wedge\zeta} = X_{\varphi_{t\wedge\zeta}}$, P-a.s. in statement (i), because φ_t is an increasing function in t.

For statement (iii), denote $f(t) = Y_t$ on $\{0 \leqslant t < \zeta\}$ and $g(t) = X_t$ on $\left\{0 \leqslant t < \zeta^X\right\}$. From statement (i), $g(\varphi_t) = X_{\varphi_t} = Y_t = f(t)$, P-a.s. on $\{0 \leqslant t < \zeta\}$. They are two real-valued functions linked by an increasing and continuous function φ_t. From statement (ii), $\varphi_\zeta = \zeta^X$, P-a.s. This means that $\limsup_{t\to\zeta} Y_t = \limsup_{t\to\zeta} f(t) = \limsup_{t\to\zeta} g(\varphi_t) = \limsup_{t\to\zeta^X} g(t) = \limsup_{t\to\zeta^X} X_t$, P-a.s., and the equivalence of the two events holds. Similarly for the cases of lim inf, lim and the case of $\widetilde{P}$. This completes the proof. □

Remark 2.6.1. *The SDE of Y and the SDE of X have the same scale functions under P, because $(\mu(.)/b^2(.))/(\sigma^2(.)/b^2(.)) = \mu(.)/\sigma^2(.)$. For the SDE* (2.81) *of the process X, we can check that $\frac{1}{(\sigma(.)/b(.))^2} = \frac{b^2(.)}{\sigma^2(.)}$ and $\frac{\mu(.)/b^2(.)}{(\sigma(.)/b(.))^2} = \frac{\mu(.)}{\sigma^2(.)}$. Thus the Engelbert-Schmidt condition* (2.2) *is satisfied, and the SDE* (2.81) *under P also has a unique in law weak solution in the sense of Definition 2.2.1 that possibly exits its state space(see Theorem* 5.5.15*, p*341 *of Karatzas and Shreve (1991)).*

Denote ζ_l $(resp.\ \ \zeta_r)$ as the possible exit time of the diffusion Y through the boundary l $(resp.\ \ r)$. Correspondingly, denote ζ_l^X $(resp.\ \ \zeta_r^X)$ as the possible exit time of the diffusion X through the boundary l $(resp.\ \ r)$. Define $\zeta = \min(\zeta_l,\ \ \zeta_r)$, $\ \zeta^X = \min(\zeta_l^X,\ \ \zeta_r^X)$, and similarly for the processes Y and X. From Theorem 2.6.1 (ii)

$$\begin{aligned}
&\zeta_l^X = \int_0^{\zeta_l} b^2(Y_s)ds, \quad \zeta_r^X = \int_0^{\zeta_r} b^2(Y_s)ds, \quad \zeta^X = \int_0^{\zeta} b^2(Y_s)ds,\ P\text{-a.s.}\\
&\zeta_l^X = \int_0^{\zeta_l} b^2(Y_s)ds, \quad \zeta_r^X = \int_0^{\zeta_r} b^2(Y_s)ds, \quad \zeta^X = \int_0^{\zeta} b^2(Y_s)ds,\ \widetilde{P}\text{-a.s.}
\end{aligned} \tag{2.89}$$

Denote the scale function of the SDE (2.1) under P and the SDE (2.81) under P as $s(.)$ defined in (2.10) , because they share the same scale function. Similarly can define $\widetilde{s}(.)$.

With a constant $c \in J$, for $x \in \bar{J}$, introduce the following test functions[17] respectively

$$v(x) \equiv \int_c^x (s(x) - s(y)) \frac{2}{s'(y)\sigma^2(y)} dy, \quad v_X(x) \equiv \int_c^x (s(x) - s(y)) \frac{2b^2(y)}{s'(y)\sigma^2(y)} dy.$$
$$\widetilde{v}(x) \equiv \int_c^x (\widetilde{s}(x) - \widetilde{s}(y)) \frac{2}{\widetilde{s}'(y)\sigma^2(y)} dy, \quad v_X(x) \equiv \int_c^x (\widetilde{s}(x) - \widetilde{s}(y)) \frac{2b^2(y)}{\widetilde{s}'(y)\sigma^2(y)} dy. \tag{2.90}$$

2.7 Alternative proof of the Engelbert-Schmidt type zero-one law

We complement the study of the Engelbert-Schmidt type zero-one law in Mijatović and Urusov (2012a) with a third new proof that circumvents theoretical tools such as the William's theorem (Ch.VII, Corollary 4.6, p317, Revuz and Yor (1999)), and the first Ray-Knight theorem (Ch.XI, Theorem 2.2, p455, Revuz and Yor (1999)). Our proof mainly relies on the stochastic time-change and the Feller's test of explosions for a one-dimensional time-homogeneous diffusion.

From Feller's test of explosions, we have the following results.

The process Y under P (resp. $\widetilde{P}$) may exit its state space J at the boundary point r, i.e. $P(\zeta < \infty, \lim_{t \to \zeta} Y_t = r) > 0$ (resp. $\widetilde{P}(\zeta < \infty, \lim_{t \to \zeta} Y_t = r) > 0$), if and only if

$$v(r) < \infty \quad (resp. \quad \widetilde{v}(r) < \infty) \tag{2.91}$$

The process X under P (resp. $\widetilde{P}$) may exit its state space J at the boundary point r, i.e. $P(\zeta^X < \infty, \lim_{t \to \zeta^X} X_t = r) > 0$ (resp. $\widetilde{P}(\zeta^X < \infty, \lim_{t \to \zeta^X} X_t =$

[17] Note that $v_X(x)$ and $\widetilde{v}_X(x)$ are exactly the same as $v_b(x)$ and $\widetilde{v}_b(x)$ defined in (2.21).

$r) > 0$), if and only if[18]

$$v_X(r) < \infty \quad (resp. \quad v_X(r) < \infty). \tag{2.92}$$

Similarly for the case of the endpoint ℓ.

For the ease of later discussions, we define the five possible events for the exit behaviors of Y at the boundaries of its state space J under $\widetilde{P}$

$$\begin{aligned}
A &= \left\{ \zeta = \infty, \limsup_{t\to\infty} Y_t = r, \liminf_{t\to\infty} Y_t = l \right\}, \\
B_r &= \left\{ \zeta = \infty, \lim_{t\to\infty} Y_t = r \right\}, \quad C_r = \left\{ \zeta < \infty, \lim_{t\to\zeta} Y_t = r \right\}, \\
B_l &= \left\{ \zeta = \infty, \lim_{t\to\infty} Y_t = l \right\}, \quad C_l = \left\{ \zeta < \infty, \lim_{t\to\zeta} Y_t = l \right\}.
\end{aligned}$$

Similarly for X under $\widetilde{P}$

$$\begin{aligned}
A^X &= \left\{ \zeta^X = \infty, \limsup_{t\to\infty} X_t = r, \liminf_{t\to\infty} X_t = l \right\}, \\
B_r^X &= \left\{ \zeta^X = \infty, \lim_{t\to\infty} X_t = r \right\}, \quad C_r^X = \left\{ \zeta^X < \infty, \lim_{t\to\zeta^X} X_t = r \right\}, \\
B_l^X &= \left\{ \zeta^X = \infty, \lim_{t\to\infty} X_t = l \right\}, \quad C_l^X = \left\{ \zeta^X < \infty, \lim_{t\to\zeta^X} X_t = l \right\}. \quad (2.93)
\end{aligned}$$

We first recall some results from Mijatović and Urusov (2012a) using our notation.

Proposition 2.7.1. *(Proposition 2.3, 2.4 and 2.5 on p4 of Mijatović and Urusov (2012a))*

(1) Either $P(A^X) = 1$ or $P(B_r^X \cup C_r^X \cup B_l^X \cup C_l^X) = 1$.

(2) (i) $P(B_r^X \cup C_r^X) = 0$ holds if and only if $s(r) = \infty$.

[18]In Mijatović and Urusov (2012c), with the same condition (2.92), they define the endpoint r to be *good*. Here we provide the probabilistic meaning: an endpoint is *good* if X may exit at it with positive probability. The *bad* endpoint can be similarly interpreted.

(ii) $P(B_l^X \cup C_l^X) = 0$ *holds if and only if* $s(\ell) = -\infty$.

(3) Assume that $s(r) < \infty$. *Then either* $P(B_r^X) > 0, P(C_r^X) = 0$ *or* $P(B_r^X) = 0, P(C_r^X) > 0$. *Similarly for the case of* ℓ.

Proof. For the proof, refer to Mijatović and Urusov (2012a). □

We have the following Engelbert-Schmidt type zero-one law for Y under $\widetilde{P}$.

Proposition 2.7.2. *Assume that the function* $f : J \to [0, \infty]$ *satisfies* $f/\sigma^2 \in L^1_{loc}(J)$, *and* $\lambda(x \in (\ell, r) : f(x) = 0) = 0$. *Let* $\widetilde{s}(r) < \infty$.

(i)If $\frac{(\widetilde{s}(r)-\widetilde{s})f}{\widetilde{s}'\sigma^2} \in L^1_{loc}(r-)$, *then* $\int_0^\zeta f(Y_u)du < \infty$, $\widetilde{P}$*-a.s. on* $\left\{\lim_{t\to\zeta} Y_t = r\right\}$.

(ii)If $\frac{(\widetilde{s}(r)-\widetilde{s})f}{\widetilde{s}'\sigma^2} \notin L^1_{loc}(r-)$, *then* $\int_0^\zeta f(Y_u)du = \infty$, $\widetilde{P}$*-a.s. on* $\left\{\lim_{t\to\zeta} Y_t = r\right\}$.

The analogous results on the set $\left\{\lim_{t\to\zeta} Y_t = l\right\}$ *can be similarly stated.*

Proof. To be consistent with our notation, define $b(y) = \sqrt{f(y)}$, since $f(.) \geqslant 0$. Denote $G = \{\lim_{t\to\zeta} \widetilde{Y}_t = r\}$, and from Theorem 2.6.1 (iii)

$$G = \{\lim_{t\to\zeta} Y_t = r\} = \{\lim_{t\to\zeta^X} X_t = r\} = B_r^X \cup C_r^X.$$

The result is trivial in the case $\widetilde{P}(G) = 0$, so we assume $\widetilde{P}(G) > 0$. Since the events B_r^X, C_r^X are disjoint

$$\widetilde{P}(G) = \widetilde{P}(B_r^X) + \widetilde{P}(C_r^X). \tag{2.94}$$

From Proposition 2.7.1, $\widetilde{s}(r) < \infty$ implies that either $\widetilde{P}(B_r^X) > 0, \widetilde{P}(C_r^X) = 0$ or $\widetilde{P}(B_r^X) = 0, \widetilde{P}(C_r^X) > 0$ holds.

For statement (i), $\frac{(\widetilde{s}(r)-\widetilde{s})f}{\widetilde{s}'\sigma^2} \in L^1_{loc}(r-)$, combined with $\widetilde{s}(r) < \infty$, implies $v_X(r) < \infty$. From equation (2.92), this is equivalent to $\widetilde{P}(\zeta^X < \infty, \lim_{t\to\zeta^X} X_t = r) > 0$, and from (2.93), it means $\widetilde{P}(C_r^X) > 0$. Thus

$\widetilde{P}(B_r^X) = 0, \widetilde{P}(C_r^X) > 0$ holds. This together with (2.94) implies

$$\begin{aligned}
\widetilde{P}(G) &= \widetilde{P}(C_r^X) = \widetilde{P}(\zeta^X < \infty, \lim_{t\to\zeta} Y_t = r) \\
&= \widetilde{P}\left(\int_0^\zeta b^2(Y_u)du < \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&= \widetilde{P}\left(\int_0^\zeta f(Y_u)du < \infty, \lim_{t\to\zeta} Y_t = r\right),
\end{aligned}$$

where the third equality follows from Theorem 2.6.1 (ii).

For statement (ii), $\frac{(\widetilde{s}(r)-\widetilde{s})f}{\widetilde{s}'\sigma^2} \notin L^1_{loc}(r-)$, combined with $\widetilde{s}(r) < \infty$, implies $\widetilde{v}_X(r) = \infty$. From equation (2.92), this is equivalent to $\widetilde{P}(\zeta^X < \infty, \lim_{t\to\zeta^X} X_t = r) = 0$, and from (2.93), it means $\widetilde{P}(C_r^X) = 0$. Thus $\widetilde{P}(B_r^X) > 0, \widetilde{P}(C_r^X) = 0$ holds. By a similar argument to that above

$$\begin{aligned}
\widetilde{P}(G) &= \widetilde{P}(B_r^X) = \widetilde{P}(\zeta^X = \infty, \lim_{t\to\zeta} Y_t = r) \\
&= \widetilde{P}\left(\int_0^\zeta b^2(Y_u)du = \infty, \lim_{t\to\zeta} Y_t = r\right) \\
&= \widetilde{P}\left(\int_0^\zeta f(Y_u)du = \infty, \lim_{t\to\zeta} Y_t = r\right).
\end{aligned}$$

The analogous results on the set $\{\lim_{t\to\zeta} Y_t = l\}$ can be similarly proved by switching the roles of r and ℓ in the above. This completes the proof. □

Clearly Proposition 2.7.2 has a corollary for the process Y under P, which is almost the same as the Theorem 2.12 of Mijatović and Urusov (2012a), but with a stronger assumption that $f(.)$ is positive. The proof is almost identical to that of Proposition 2.7.2 and is thus omitted.

Corollary 2.7.1. *(Engelbert-Schmidt type zero-one law for time-homogeneous diffusions, Theorem 2.12 of Mijatović and Urusov (2012a) with stronger assumption)*

Assume that the function $f : J \to [0, \infty]$ satisfies $f/\sigma^2 \in L^1_{loc}(J)$, and $\lambda(x \in (\ell, r) : f(x) = 0) = 0$. Let $s(r) < \infty$.

(i)If $\frac{(s(r)-s)f}{s'\sigma^2} \in L^1_{loc}(r-)$*, then* $\int_0^\zeta f(Y_u)du < \infty$*, P-a.s. on* $\left\{\lim_{t\to\zeta} Y_t = r\right\}$.

(ii)If $\frac{(s(r)-s)f}{s'\sigma^2} \notin L^1_{loc}(r-)$*, then* $\int_0^\zeta f(Y_u)du = \infty$*, P-a.s. on* $\left\{\lim_{t\to\zeta} Y_t = r\right\}$.

The analogous results on the set $\{\lim_{t\to\zeta} Y_t = l\}$ *can be similarly stated.*

2.8 A useful result from Karatzas and Shreve (1991)

Here we quote the statement and proof of Problem 3.4.5, on p174 of Karatzas and Shreve (1991), because it is useful in the proofs of Theorem 2.6.1 and also Theorem 3.2.1 in Chapter 3.

Proposition 2.8.1. *(Problem 3.4.5, p174 of Karatzas and Shreve (1991))*

Let $A = \{A(t); \quad 0 \leqslant t < \infty\}$ *be a continuous and nondecreasing function with* $A(0) = 0$*,* $S := A(\infty) \leqslant \infty$*, and define for* $0 \leqslant s < \infty$*:*

$$T(s) = \begin{cases} \inf\{t \geqslant 0; A(t) > s\}, & 0 \leqslant s < S, \\ \infty, & s \geqslant S. \end{cases}$$

The function $T = \{T(s); \quad 0 \leqslant s < \infty\}$ *has the following properties:*

(i) T *is nondecreasing and right-continuous on* $[0, S)$*, with values in* $[0, \infty)$*. If* $A(t) < S; \forall t \geqslant 0$*, then* $\lim_{s\uparrow S} T(s) = \infty$.

(ii) $A(T(s)) = s \wedge S; \quad 0 \leqslant s < \infty$.

(iii) $T(A(t)) = \sup\{\tau \geqslant t : A(\tau) = A(t)\}; \quad 0 \leqslant t < \infty$.

(iv) Suppose $\varphi : [0, \infty) \to \mathbb{R}$ *is continuous and has the property*

$A(t_1) = A(t)$ *for some* $0 \leqslant t_1 < t \Rightarrow \varphi(t_1) = \varphi(t)$.

Then $\varphi(T(s))$ *is continuous for* $0 \leqslant s < S$*, and*

$$\varphi(T(A(t))) = \varphi(t); \quad 0 \leqslant t < \infty. \tag{2.95}$$

(v) For $0 \leqslant t, s < \infty$*;* $s < A(t) \iff T(s) < t$ *and* $T(s) \leqslant t \Rightarrow s \leqslant A(t)$.

(vi) If G is a bounded, measurable, real-valued function defined on $[a, b] \subset [0, \infty)$, then

$$\int_a^b G(t)dA(t) = \int_{A(a)}^{A(b)} G(T(s))ds. \tag{2.96}$$

Proof. (i) The[19] nondecreasing character of T is obvious. Thus, for the right-continuity, we only need to show that $\lim_{\theta \downarrow s} T(\theta) \leqslant T(s)$, for $0 \leqslant s < S$. Set $t = T(s)$. The definition of $T(s)$ implies that for each $\varepsilon > 0$, we have $A(t+\varepsilon) > s$, and for $s < \theta < A(t+\varepsilon)$, we have $T(\theta) \leqslant t + \varepsilon$. Therefore, $\lim_{\theta \downarrow s} T(\theta) \leqslant t$.

(ii) The identity is trivial for $s \geqslant S$; if $s < S$, set $t = T(s)$ and choose $\varepsilon > 0$. We have $A(t+\varepsilon) > s$, and letting $\varepsilon \downarrow 0$, we see from the continuity of A that $A(T(s)) \geqslant s$. If $t = T(s) = 0$, we are done. If $t > 0$, then for $0 < \varepsilon < t$, the definition of $T(s)$ implies $A(t - \varepsilon) \leqslant s$. Letting $\varepsilon \downarrow 0$, we obtain $A(T(s)) \leqslant s$.

(iii) This follows immediately from the definition of $T(.)$.

(iv) Since, by (i), T is right-continuous, so is $\varphi(T(.))$. To show the left-continuity, take any $s \in [0, S)$, and any increasing sequence, $\{s_n\}$, such that $s_n \to s$. Since T is nondecreasing, $\{T(s_n)\}$ is a nondecreasing sequence of real numbers bounded from above by $T(s)$. Therefore $\lim_{n\to\infty} T(s_n)$ exists. Now we claim that $\varphi(\lim_{n\to\infty} T(s_n)) = \varphi(T(s))$. To see this, note that, by continuity of A and (ii), we have $A(\lim_{n\to\infty} T(s_n)) = \lim_{n\to\infty} A(T(s_n)) = \lim_{n\to\infty} s_n = s$. This, together with the property (iii), proves our claim. Finally, by the continuity of φ, it follows that $\lim_{n\to\infty} \varphi(T(s_n)) = \varphi(\lim_{n\to\infty} T(s_n)) = \varphi(T(s))$. Hence, $\varphi(T(.))$ is continuous. Finally, to prove statement (iv), note that, by (ii), we have $A(T(A(t))) = A(t) \wedge S = A(t)$. Now (iv) follows from the property (iii) of φ.

(v) This is a direct consequence of the definition of T, and the continuity

[19]Here we state the proof provided on p231 of Karatzas and Shreve (1991), and add details where necessary.

of A.

(vi) For $a \leqslant t_1 < t_2 \leqslant b$, let $G(t) = \mathbb{1}_{[t_1,t_2)}(t)$. According to statement (v), $t_1 \leqslant T(s) < t_2$ if and only if $A(t_1) \leqslant s < A(t_2)$, so

$$\int_a^b G(t)dA(t) = A(t_2) - A(t_1) = \int_{A(a)}^{A(b)} G(T(s))ds. \tag{2.97}$$

The linearity of the integral and the monotone convergence theorem imply that the collection of sets $C \in \mathcal{B}([a,b])$ for which

$$\int_a^b \mathbb{1}_C(t)dA(t) = \int_{A(a)}^{A(b)} \mathbb{1}_C(T(s))ds \tag{2.98}$$

forms a Dynkin system. Since it contains all intervals of the form $[t_1, t_2) \subset [a,b]$, and these are closed under finite intersection and generate $\mathcal{B}([a,b])$, from the Dynkin System Thoerem (Theorem 2.1.3, p49 of Karatzas and Shreve (1991)), we have (2.98) for every $C \in \mathcal{B}([a,b])$. The proof of (vi) follows. This completes the proof. □

2.9 Conclusion of Chapter 2

This chapter provides a unification of results on the convergence or divergence properties of integral functionals of time-homogeneous diffusions. We also generalize some results of Mijatović and Urusov (2012b, 2012c) from the $\rho = 1$ case to the case of arbitrary correlation, and provide new unified proofs without using the concept of *separating times*.

Part II

Probabilistic pricing methods

Chapter 3

First hitting times of integrated time-homogeneous diffusions

3.1 Introduction and financial motivations

The time integrals of stochastic processes are of interest in both applied probability and mathematical finance. The integrated geometric Brownian motion is a key component in the payoff of the arithmetic Asian option in mathematical finance, and it has been extensively studied by many authors, such as Dufresne (2001) and Yor (1992) (2001). The integrated geometric Brownian motion is also an important component in the equity linked insurance products. Recently, there is some interest in the study of the first hitting time of the integral of a stochastic process to a fixed level. In Metzler (2013), for the case of the integrated geometric Brownian motion, he provides a closed-form formula for the Laplace transform of the first hitting time.

In April 2007, Société Générale Corporate and Investment Banking (SG CIB) started to sell a new type of option that allows buyers to specify the level of volatility used to price the instrument, which is named the "timer option". Consider the underlying asset S, with strike K and denote by ℓ the "variance budget" that is chosen by the investor. Denote τ as the random maturity time of the option, which is defined as the first hitting time of the realized variance to the variance budget ℓ: $\tau = \inf\left\{u > 0 : \int_0^u V_s ds = \ell\right\}$. The payoff of a timer call option is $\max(S_\tau - K, 0)$ at time τ.

The financial meaning of the first hitting time considered in this chapter actually corresponds to the "random maturity time" of the "timer option". Bernard and Cui (2011) propose an efficient Monte Carlo method for pricing the "timer option". In Saunders (2009), Li and Mercurio (2013a) (2013b), they propose asymptotic expansion methods to price the timer option in a general stochastic volatility model similar to (3.1). For further literature on the "timer option", please refer to Cui (2010), Li (2013) or the Ph.D. thesis of Li (2010) and the references therein.

In this chapter we give a detailed study of the first hitting time of an integral functional of a time-homogeneous diffusion to a fixed level. Furthermore, we construct a link between this first hitting time and the time integral of another time-homogeneous diffusion. Some new probabilistic results related to this hitting time are obtained.

As a first application, we extend the work of Metzler (2013) from the geometric Brownian motion to the setting of time-homogeneous diffusions. We also show a novel method to price the arithmetic Asian option under a time-homogeneous diffusion model and provide an explicit triple integral formula for the price in the Black-Scholes setting.

In this chapter, the new results, which contribute to the current literature, are as follows: Theorem 3.2.1, Lemma 3.3.1, Proposition 3.3.1, Proposition 3.3.2, Proposition 3.3.3, Proposition 3.3.4, Lemma 3.4.1, Proposition 3.4.1, Proposition 3.4.2, Proposition 3.4.3, and Proposition 3.5.1.

The chapter is organized as follows. Section 3.2 presents the main results of the chapter, which is the joint distributions of the first hitting time and place of an integral functional of a time-homogeneous diffusion to a fixed level. Section 3.3 studies the Laplace transform of the first hitting time of an integral functional of the geometric Brownian motion. Section 3.4 studies the first hitting time of an integral functional of three other time-homogeneous diffusions that are commonly used in mathematical finance. Section 3.5 studies the pricing of arithmetic Asian options when the stock prices are modeled as time-homogeneous diffusions. Section 3.6 concludes the chapter.

3.2 Main result

In this section, we give the probabilistic setup and state the main results.

3.2.1 Theoretical joint distribution of (τ, V_τ)

Given a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with state space $J = (\ell, r), -\infty \leqslant l < r \leqslant \infty$, and assume that the J-valued diffusion $V = (V_t)_{t\in[0,\infty)}$ satisfies the SDE

$$dV_t = \mu(V_t)\,dt + \sigma(V_t)\,dW_t, \quad V_0 = v_0 \in J. \tag{3.1}$$

where W is a $\mathcal{F}_t$-Brownian motion and $\mu, \sigma : J \to \mathbb{R}$ are Borel functions satisfying the Engelbert-Schmidt conditions

$$\forall x \in J,\ \sigma(x) \neq 0, \quad \text{and} \quad \frac{1}{\sigma^2(\cdot)},\quad \frac{\mu(\cdot)}{\sigma^2(\cdot)} \in L^1_{loc}(J). \tag{3.2}$$

$L^1_{loc}(J)$ denotes the class of locally integrable functions, i.e. the functions $J \to \mathbb{R}$ that are integrable on compact subsets of J. This condition (3.2) guarantees that the SDE (3.1) has a unique in law weak solution that possibly exits its state space J (see Theorem 5.5.15, p341, Karatzas and Shreve (1991)).

In what follows, λ denotes the Lebesgue measure on $B(\mathbb{R})$. Let m be a Borel function such that $\lambda(x \in (l, r) : m^2(x) = 0) = 0$, and assume the following local integrability conditions

$$\forall x \in J,\ \sigma(x) \neq 0, \quad \text{and} \quad \frac{m^2(\cdot)}{\sigma^2(\cdot)} \in L^1_{loc}(J). \tag{3.3}$$

Denote the possible exit time of V from its state space by ζ, i.e. $\zeta = \inf\{u > 0, V_u \notin J\}$, P-a.s., which means that P-a.s. on $\{\zeta = \infty\}$ the trajectories of V do not exit J, and P-a.s. on $\{\zeta < \infty\}$, $\lim\limits_{t\to\zeta} V_t = r$ or $\lim\limits_{t\to\zeta} V_t = l$. V is defined such that it stays at its exit point, which means that ℓ and r are absorbing boundaries. The following terminology is used: "V may exit the state space J at r" means $P(\zeta < \infty, \lim\limits_{t\to\zeta} V_t = r) > 0$.

The following is about stochastic time-change.

Theorem 3.2.1. *Assume the conditions* (3.2)*,* (3.3)*, and* $\lambda(x \in (l, r) : m^2(x) = 0) = 0$.

(i) Define[1]

$$\tau(t) := \tau_t := \begin{cases} \inf\{u \geqslant 0 : \varphi_{u\wedge\zeta} > t\}, & on\ \{0 \leqslant t < \varphi_\zeta\}, \\ \infty, & on\ \{\varphi_\zeta \leqslant t < \infty\}. \end{cases} \tag{3.4}$$

Define a new filtration $\mathcal{G}_t = \mathcal{F}_{\tau_t}, t \in [0,\infty)$*, and a new* $\mathcal{G}_t$*-adapted process* $X_t := V_{\tau_t}$*, on* $\{0 \leqslant t < \varphi_\zeta\}$*. Then we have the stochastic representation*

$$V_t = X_{\int_0^t m^2(V_s)ds} = X_{\varphi_t}, \quad P-a.s., \quad on\ \{0 \leqslant t < \zeta\}. \tag{3.5}$$

and the process X *is a time-homogeneous diffusion, which solves the following SDE under* P

$$dX_t = \frac{\mu(X_t)}{m^2(X_t)}\mathbb{1}_{t\in[0,\varphi_\zeta)}dt + \frac{\sigma(X_t)}{m(X_t)}\mathbb{1}_{t\in[0,\varphi_\zeta)}dB_t, \quad X_0 = v_0. \tag{3.6}$$

where B_t *is the* $\mathcal{G}_t$*-adapted Dambis-Dubins-Schwartz Brownian motion defined in the proof.*

(ii) Define $\zeta^X := \inf\{u > 0 : X_u \notin J\}$*, then* $\zeta^X = \varphi_\zeta = \int_0^\zeta m^2(V_s)ds$*, P-a.s., and we can rewrite the SDE* (3.6) *as*

$$dX_t = \frac{\mu(X_t)}{m^2(X_t)}\mathbb{1}_{t\in[0,\zeta^X)}dt + \frac{\sigma(X_t)}{m(X_t)}\mathbb{1}_{t\in[0,\zeta^X)}dB_t, \quad X_0 = v_0. \tag{3.7}$$

(iii) Define the first hitting time of the integrated diffusion process to a fixed level $a \in [0,\infty)$ *as* $\tau(a)$ *It is well-defined, and on* $\{0 \leqslant a < \varphi_\zeta\}$*, we have*

$$(\tau(a), V_{\tau(a)}) = \left(\int_0^a \frac{1}{m^2(X_s)}ds, X_a\right),\ P\text{-a.s.}$$

Proof. Similar as Remark 2.3.2, since $\lambda(x \in (\ell,r) : m^2(x) = 0) = 0$, φ_s is an increasing and continuous function on $[0,\zeta]$. From Problem 3.4.5 (ii), p174 of Karatzas and Shreve (1991), $\varphi_{\tau_t\wedge\zeta} = t \wedge \varphi_\zeta$, P-a.s. for $0 \leqslant t < \infty$. On $\{0 \leqslant t < \varphi_\zeta\}$, when $u = \zeta$, $\varphi_{\zeta\wedge\zeta} = \varphi_\zeta > t$ holds P-a.s. according to

[1] This theorem is almost identical to Theorem 2.6.1 except part (iii), and the different assumption $\lambda(x \in (l,r) : m^2(x) = 0) = 0$. For the consistency of the development of this chapter, we repeat the statement and the proof.

the assumption. Then $\tau_t \leqslant \zeta$, P-a.s. due to the definition in (3.4). Thus $\varphi_{\tau_t} = t$, P-a.s. on $\{0 \leqslant t < \varphi_\zeta\}$.

On $\{0 \leqslant s < \zeta\}$, choose $t = \varphi_s$, then $0 \leqslant t < \varphi_\zeta$, P-a.s. Substituting this t into the definition of the process X, $X_{\varphi_s} = X_t := V_{\tau_t} = V_{\tau_{\varphi_s}} = V_s$, P-a.s. For the last equality, note that $\tau_{\varphi_s} = \inf\{u \geqslant 0 : \varphi_{u\wedge\zeta} > \varphi_s\} = \inf\{u \geqslant 0 : u \wedge \zeta > s\} = s$, P-a.s., on $\{0 \leqslant s < \zeta\}$. Then we have proved the representation $V_s = X_{\varphi_s}$, on $\{0 \leqslant s < \zeta\}$.

For X satisfying the relation (3.5), we aim to show that X satisfies the SDE (3.6), where B is the Dambis-Dubins-Schwartz Brownian motion adapted to $\mathcal{G}_t$ constructed as follows:

Note that $M_{t\wedge\zeta} = \int_0^{t\wedge\zeta} m(V_u)dW_u, t \in [0,\infty)$ is a continuous local martingale, with quadratic variation $\varphi_{t\wedge\zeta} = \int_0^{t\wedge\zeta} m^2(V_u)du, t \in [0,\infty)$. Then $\lim_{t\to\infty} \varphi_{t\wedge\zeta} = \varphi_\zeta$, P-a.s. due to the left continuity of φ_s at $s = \zeta$. From the Dambis-Dubins-Schwartz theorem (Ch.V, Theorem 1.6 and Theorem 1.7 of Revuz and Yor (1999)), there exists an enlargement $(\bar{\Omega}, \bar{\mathcal{G}}_t, \bar{P})$ of $(\Omega, \mathcal{G}_t, P)$ and a standard Brownian motion $\bar{\beta}$ on $\bar{\Omega}$ independent of M with $\bar{\beta}_0 = 0$, such that the process

$$B_t := \begin{cases} \int_0^{\tau_t} m(V_u)dW_u, & \text{on } \{t < \varphi_\zeta\}, \\ \int_0^{\zeta} m(V_u)dW_u + \widetilde{\beta}_{t-\varphi_\zeta}, & \text{on } \{t \geqslant \varphi_\zeta\}. \end{cases} \tag{3.8}$$

is a standard linear $\mathcal{G}_t$-Brownian motion. Our construction of $\tau_t, t \in [0,\infty)$ agrees with that in Problem 3.4.5[2], p174 of Karatzas and Shreve (1991). From Problem 3.4.5 (ii) and the construction (3.8), $B_{\varphi_s} = M_s$, P-a.s. on $\{0 \leqslant s < \zeta\}$. On $\{s = \zeta\}$, $B_{\varphi_\zeta} := \int_0^\zeta m(V_u)dW_u + \widetilde{\beta}_0 = \int_0^\zeta m(V_u)dW_u =: M_\zeta$, P-a.s. Thus $B_{\varphi_t} = M_t$, P-a.s. on $\{0 \leqslant t \leqslant \zeta\}$.

For the convenience of exposition, denote $\mu_1(.) = \mu(.)/m^2(.)$, and

[2]See Section 2.8 in Chapter 2 for the statement and proof of this result.

$\sigma_1(.) = \sigma(.)/m(.)$. Integrate the SDE in (3.1) from 0 to $t \wedge \zeta$

$$\begin{aligned} V_{t\wedge\zeta} - V_0 &= \int_0^{t\wedge\zeta} \mu(V_u)du + \int_0^{t\wedge\zeta} \sigma(V_u)dW_u \\ &= \int_0^{t\wedge\zeta} \mu_1(V_u)m^2(V_u)du + \int_0^{t\wedge\zeta} \sigma_1(V_u)m(V_u)dW_u. \end{aligned} \tag{3.9}$$

Apply the change of variables formula similar to Problem 3.4.5 (vi)[3], p174 of Karatzas and Shreve (1991), and note the relation (3.5)

$$\int_0^{t\wedge\zeta} \mu_1(V_u)m^2(V_u)du = \int_0^{t\wedge\zeta} \mu_1(X_{\varphi_u})d\varphi_u = \int_0^{\varphi_{t\wedge\zeta}} \mu_1(X_u)du, \tag{3.10}$$

and similarly

$$\int_0^{t\wedge\zeta} \sigma_1(V_u)m(V_u)dW_u = \int_0^{t\wedge\zeta} \sigma_1(X_{\varphi_u})dB_{\varphi_u} = \int_0^{\varphi_{t\wedge\zeta}} \sigma_1(X_u)dB_u \tag{3.11}$$

where the first equality in (3.11) is due to the relationship $B_{\varphi_u} = M_u = \int_0^u m(V_s)dW_s$, P-a.s. on $\{0 \leqslant u \leqslant t \wedge \zeta\}$, which we have established above. Also notice the representation $V_{t\wedge\zeta} = X_{\varphi_{t\wedge\zeta}}$, P-a.s. and $V_0 = X_0$, then

$$X_{\varphi_{t\wedge\zeta}} - X_0 = \int_0^{\varphi_{t\wedge\zeta}} \mu_1(X_u)du + \int_0^{\varphi_{t\wedge\zeta}} \sigma_1(X_u)dB_u \tag{3.12}$$

Then on $\{0 \leqslant s\varphi_{t\wedge\zeta}\}$

$$X_s - X_0 = \int_0^s \mu_1(X_u)du + \int_0^s \sigma_1(X_u)dB_u. \tag{3.13}$$

Note that for $0 \leqslant t < \infty$, we have $s \in [0, \varphi_\zeta]$, P-a.s. From (3.13), and recall the definition of $\mu_1(.)$ and $\sigma_1(.)$, we have the following SDE for X:

$$dX_s = \frac{\mu(X_s)}{m^2(X_s)}\mathbb{1}_{s\in[0,\varphi_\zeta)}ds + \frac{\sigma(X_s)}{m(X_s)}\mathbb{1}_{s\in[0,\varphi_\zeta)}dB_s, \quad X_0 = V_0 = v_0.$$

This completes the proof of statement (i).

[3]See Section 2.8 in Chapter 2 for the statement and the proof.

Statement (ii) is a direct consequence of the stochastic representation $V_{t\wedge\zeta} = X_{\varphi_{t\wedge\zeta}}$, P-a.s. in statement (i), because φ_t is an increasing function with respect to t.

For statement (iii), from Problem 3.4.5 (ii)[4], p174 of Karatzas and Shreve (1991), with similar reasoning as before, $\varphi_{\tau(a)} = a$, P-a.s. on $\{0 \leqslant a < \varphi_\zeta\}$. From the result in statement (i), $V_s = X_{\varphi_s}$, P-a.s. on $\{0 \leqslant s \leqslant \zeta\}$. On $\{0 \leqslant a < \varphi_\zeta\}$, $\tau(a) \leqslant \zeta$, P-a.s. Substitute $s = \tau(a)$, then $V_{\tau(a)} = X_{\varphi(\tau(a))} = X_a$, P-a.s. on $\{0 \leqslant a < \varphi_\zeta\}$.

By definition, on $\{0 \leqslant a < \varphi_\zeta\}$

$$\begin{aligned} \tau(a) &= \int_0^{\tau(a)} du = \int_0^{\tau(a)} \frac{1}{m^2(V_u)} d\varphi_u \\ &= \int_0^a \frac{1}{m^2(V_{\tau(s)})} d\varphi_{\tau(s)} \\ &= \int_0^a \frac{1}{m^2(X_s)} ds, \ P\text{-a.s.} \end{aligned} \tag{3.14}$$

Here we apply the change of variables formula in the above Stietljes integral similar to equation (5.5.24), p333 of Karatzas and Shreve (1991), see also Proposition 2.8.1 (vi). The last equality in (3.14) holds because $V_{\tau(s)} = X_s$, P-a.s. on $\{0 \leqslant s \leqslant a < \varphi_\zeta\}$ as proved above, and also because $\varphi_{\tau(s)} = s$, P-a.s. on $\{0 \leqslant s \leqslant a < \varphi_\zeta\}$. This completes the proof. □

The SDE (3.1) includes the Heston and the Hull-White stochastic volatility models as special cases. From Theorem 3.2.1, consider the case $m(x) = \sqrt{x}$, the joint distribution of (τ, V_τ) is calculated in the Heston model in Proposition 3.4.1 and in the Hull-White model in Proposition 3.3.1.

[4]See Section 2.8 in Chapter 2 for the statement and the proof.

3.3 Hitting times of integrated GBM

3.3.1 Joint distribution of the hitting time and place

Under P, the GBM model is governed by the SDE: $dV_t = \mu V_t dt + \sigma V_t dW_t$, $V_0 = v_0 > 0$, so that $\alpha(s) = \mu s$ and $\beta(s) = \sigma s$. We also have that $m(s) = \sqrt{s}$. Before applying the main result Theorem 3.2.1, we have to check the two conditions (3.2) and (3.3). The natural state space for the GBM is $J = (0, \infty)$, and the above conditions can be verified: $1/\sigma^2(x) = 1/(\sigma^2 x^2) \in L^1_{loc}(J)$, $\mu(x)/\sigma^2(x) = \mu/(\sigma^2 x) \in L^1_{loc}(J)$, and $m^2(x)/\sigma^2(x) = 1/(\sigma^2 x) \in L^1_{loc}(J)$. Denote ζ as the possible exit time of the process V from its natural state space J, and define $\varphi_t = \int_0^t V_s ds, t \in [0, \zeta]$.

We first prove a lemma.

Lemma 3.3.1. *Assume the conditions* (3.2), (3.3) *and* $\mu \geqslant \frac{1}{2}\sigma^2$, *then* $P(\varphi_\zeta = \infty) = 1$.

Proof. From the definition in (2.21), for the GBM process and a constant $c \in J$

$$v_b(x) = \frac{2}{\sigma^2} \int_c^x y^{2\mu/\sigma^2 - 1} \left(\int_y^x z^{-2\mu/\sigma^2} dz \right) dy.$$

Divide into two cases below

$$v_b(x) = \begin{cases} \frac{2}{\sigma^2} \int_c^x \ln\left(\frac{x}{y}\right) dy, & \text{if } \frac{2\mu}{\sigma^2} = 1, \\ \frac{2}{\sigma^2 - 2\mu} \int_c^x \left(\left(\frac{x}{y}\right)^{1 - \frac{2\mu}{\sigma^2}} - 1 \right) dy, & \text{if } \frac{2\mu}{\sigma^2} > 1. \end{cases}$$

Further simplify the above expression

$$v_b(x) = \begin{cases} \frac{2}{\sigma^2}(x - c\ln x + c\ln c - c), & \text{if } \frac{2\mu}{\sigma^2} = 1, \\ \frac{\sigma^2}{\mu(\sigma^2 - 2\mu)} \left((1 - \frac{2\mu}{\sigma^2})x - c^{\frac{2\mu}{\sigma^2}} x^{1 - \frac{2\mu}{\sigma^2}} + cq \right) dy, & \text{if } \frac{2\mu}{\sigma^2} > 1. \end{cases}$$

Then both $v_b(\infty) = \infty$ and $v_b(0) = \infty$ hold in the above two cases. From Lemma 2.3.1, in Chapter 2 of this thesis, $P(\varphi_\zeta = \infty) = 1$. This completes the proof. $\square$

Proposition 3.3.1. *Assuming $\mu \geqslant \frac{1}{2}\sigma^2$, for $0 \leqslant a < \infty$, we have $(\tau(a), V_{\tau(a)}) = \left(\frac{4}{\sigma^2}\int_0^a \frac{1}{X_s^2}ds, \frac{\sigma^2}{4}X_a\right)$, P-a.s., where X_t is governed by the SDE*

$$dX_t = \left(\frac{2\mu}{\sigma^2} - \frac{1}{2}\right)\frac{1}{X_t}dt + dB_t, \quad X_0 = \frac{2}{\sigma}\sqrt{v_0}, \tag{3.15}$$

where B is a standard $\mathcal{G}_t$-Brownian motion. Here X_t is a standard Bessel process (without drift) with index $\nu = \frac{2\mu}{\sigma^2} - 1 \geqslant 0$.

Proof. Since $\mu \geqslant \frac{1}{2}\sigma^2$, from Lemma 3.3.1, $P(\varphi_\zeta = \infty) = 1$. Then for $0 \leqslant t < \infty$, from Theorem 3.2.1, $V_t = Y_{\int_0^t V_s ds}$, P-a.s., $Y_0 = v_0$, where Y is governed by the following SDE

$$\begin{aligned} dY_t &= \frac{\mu Y_t}{Y_t}dt + \frac{\sigma Y_t}{\sqrt{Y_t}}dB_t, \\ &= \mu dt + \sigma\sqrt{Y_t}dB_t, \quad Y_0 = v_0. \end{aligned}$$

We recognize Y as the squared Bessel process $BESQ^\delta$ with $\delta = \frac{4\mu}{\sigma^2}$. From well-known properties of the trajectories of the squared Bessel processes(see Ch.X.I., Revuz and Yor (1999)), since $v_0 > 0$ and $\delta = \frac{4\mu}{\sigma^2} \geqslant 2$, the left boundary $\ell = 0$ is unattainable. Denote a new process $X_t = \frac{2}{\sigma}\sqrt{Y_t}$, and apply Itō's lemma[5], then $V_t = \frac{\sigma^2}{4}X_{\int_0^t V_s ds}$, P-a.s., with $X_0 = \frac{2}{\sigma}\sqrt{v_0}$, and the SDE of X is

$$dX_t = \left(\frac{2\mu}{\sigma^2} - \frac{1}{2}\right)\frac{1}{X_t}\mathbb{1}_{t\in[0,\zeta^X)}dt + \mathbb{1}_{t\in[0,\zeta^X)}dB_t, \quad X_0 = \frac{2}{\sigma}\sqrt{v_0}.$$

X is therefore a standard Bessel process without drift. The index of the Bessel process is $\nu = \frac{\delta}{2} - 1 = \frac{2\mu}{\sigma^2} - 1$. The joint representation follows from Theorem 3.2.1 (iii). Note that $\zeta^X := \inf\{u > 0 : X_u \notin J\}$ is the possible exit time of the process X from the state space J. From Theorem 3.2.1 (ii) combined with Lemma 3.3.1, $\zeta^X = \varphi_\zeta = \infty$, P-a.s. Thus we obtain

[5] Note that for $\delta < 2$, the squared Bessel process reaches 0, and the conditions needed to apply Itō's lemma are not satisfied; See p456 and p451 of Revuz and Yor (1999). Thus in the sequel we restrict our attention to the case when $\delta \geqslant 2$.

the SDE as given in (3.15). This completes the proof. □

Note that Proposition 3.3.1 represents the first hitting time of an integrated geometric Brownian motion to a fixed level as an integral functional of the reciprocal of a standard squared Bessel process, i.e. $\frac{4}{\sigma^2}\int_0^a \frac{1}{X_s^2} ds$. With the assumption $\mu \geqslant \frac{1}{2}\sigma^2$, the Bessel process X can not attain the left boundary $\ell = 0$.

3.3.2 The Laplace transform of hitting time

The main result of Metzler (2013) is stated as follows using our notation. The contribution here is to use the Proposition 3.3.1 to give an alternative probabilistic proof to his main result. The original proof of Theorem 1 in Metzler (2013) requires reducing the form of an ordinary differential equation to some ODE of special functions that we know, but the proof presented here is more straightforward. Because our proof is based on Proposition 3.3.1, we have to make a stronger assumption (i.e. $\mu \geqslant \frac{1}{2}\sigma^2$) compared to the statement and and proof of Theorem 1in Metzler (2013). This is because Bessel processes of non-negative indexes behave very differently from Bessel processes with negative indexes (refer to Section 3 of Metzler (2013) for a detailed discussion). The following proposition is Theorem 1, Metzler (2013) with a stronger assumption.

Proposition 3.3.2. *Assume $\mu \geqslant \frac{1}{2}\sigma^2$, for $0 \leqslant a < \infty$, $\alpha \geqslant 0$, the Laplace transform*

$$u(a, v_0, \alpha, \sigma, \mu) = \mathbb{E}[e^{-\alpha\tau(a)}]$$

is given by

$$u(a, v_0, \alpha, \sigma, \mu) = (2v_0/a\sigma^2)^\gamma \frac{\Gamma(\gamma + 2\mu/\sigma^2)}{\Gamma(2\gamma + 2\mu/\sigma^2)} M(\gamma, 2\gamma + 2\mu/\sigma^2; -2v_0/a\sigma^2), \tag{3.16}$$

where $\Gamma(x)$ is the gamma function, and $M(a, b; x)$ is the confluent hyperge-

ometric function(or Kummer function)[6] *and* $\gamma = \gamma(\alpha) = -\frac{\nu}{2} + \frac{1}{2}\sqrt{\nu^2 + \frac{8\alpha}{\sigma^2}}$ *is the larger root of*

$$\xi^2 + \nu\xi - 2\alpha/\sigma^2 = 0, \quad \xi \in \mathbb{R}, \tag{3.17}$$

with $\nu = \frac{2\mu}{\sigma^2} - 1 \geqslant 0$.

Proof. Since $\mu \geqslant \frac{1}{2}\sigma^2$, from Lemma 3.3.1, $P(\varphi_\zeta = \infty) = 1$. From Proposition 3.3.1, $\mathbb{E}[e^{-\alpha\tau(a)}] = \mathbb{E}[e^{-\frac{4\alpha}{\sigma^2}\int_0^a \frac{1}{X_s^2}ds}]$, for $0 \leqslant a < \infty$.

Consider a standard Bessel process X_t with index $\nu \geqslant 0$ and $X_0 = x_0 > 0$, from[7] formula (4.1.20.3) on p386 of Borodin and Salminen (2002):

$$\mathbb{E}[e^{-\frac{g^2}{2}\int_0^a \frac{1}{X_s^2}ds}] = \left(\frac{2a}{x_0^2}\right)^{(\nu+1)/2} \frac{\Gamma(1 + \frac{\nu}{2} + \frac{1}{2}\sqrt{\nu^2 + g^2})}{\Gamma(1 + \sqrt{\nu^2 + g^2})} e^{-x_0^2/4a} M_{-\frac{\nu}{2}-\frac{1}{2}, \frac{\sqrt{\nu^2+g^2}}{2}}\left(\frac{x_0^2}{2a}\right), \tag{3.18}$$

where $M_{\kappa,\mu}(z) = e^{-z/2}z^{\mu+1/2}M(\mu - \kappa + \frac{1}{2}, 1 + 2\mu; z)$ denotes the Whittaker function. We make the following substitutions: replace $\frac{g^2}{2}$ by $\frac{4\alpha}{\sigma^2}$, $\frac{x_0^2}{2a}$ by $\frac{2v_0}{a\sigma^2}$ and the larger root[8] of (3.17) by $\gamma = \gamma(\alpha)$. Then

$$\mathbb{E}[e^{-\frac{4\alpha}{\sigma^2}\int_0^a \frac{1}{X_s^2}ds}] = \left(\frac{a\sigma^2}{2v_0}\right)^{(\nu+1)/2} \frac{\Gamma(1 + \frac{\nu}{2} + \frac{1}{2}\sqrt{\nu^2 + \frac{8\alpha}{\sigma^2}})}{\Gamma(1 + \sqrt{\nu^2 + \frac{8\alpha}{\sigma^2}})} e^{-x_0^2/4a} M_{-\frac{\nu}{2}-\frac{1}{2}, \frac{1}{2}\sqrt{\nu^2+\frac{8\alpha}{\sigma^2}}}\left(\frac{2v_0}{a\sigma^2}\right),$$

or, since $\frac{\nu+1}{2} = \frac{\mu}{\sigma^2}$

$$\mathbb{E}[e^{-\frac{4\alpha}{\sigma^2}\int_0^a \frac{1}{X_s^2}ds}] = \left(\frac{a\sigma^2}{2v_0}\right)^{\mu/\sigma^2} \frac{\Gamma\left(\gamma + 2\frac{\mu}{\sigma^2}\right)}{\Gamma(2\gamma + 2\frac{\mu}{\sigma^2})} \exp\left(-\frac{v_0}{a\sigma^2}\right) M_{-\frac{\nu}{2}-\frac{1}{2}, \gamma+\frac{\nu}{2}}\left(\frac{2v_0}{a\sigma^2}\right). \tag{3.19}$$

We now use the relationships between the Whittaker function $M_{\kappa,\mu}(z)$ and the Kummer function $M(a, b, z)$ (see formula (13.1.327) on p505 of

[6]By definition $\Gamma(x) := \int_0^\infty u^{x-1}e^{-u}du$, $x > 0$, and $M(a, b; x) := 1 + \sum_{k=1}^\infty \frac{a(a+1)\ldots(a+k-1)x^k}{b(b+1)\ldots(b+k-1)k!}$.

[7]We make the following substitutions using our notation: $R_s^{(n)}$ becomes X_s, γ becomes g, t becomes a, and x becomes x_0.

[8]Namely, the larger root is $-\frac{\nu}{2} + \frac{1}{2}\sqrt{\nu^2 + \frac{8\alpha}{\sigma^2}} = -\frac{\nu}{2} + \frac{1}{2}\sqrt{\nu^2 + g^2}$.

Abramowitz and Stegun (1967)):

$$M_{\kappa,\mu}(z) = e^{\frac{z}{2}} z^{\mu+\frac{1}{2}} M\left(\frac{1}{2} + \mu + \kappa, 1 + 2\mu; -z\right), \qquad \text{since } M(a, b; z) = e^z M(b - a, b; -z).$$

Then

$$\begin{aligned} M_{-\frac{\nu}{2}-\frac{1}{2},\gamma+\frac{\nu}{2}}(\frac{2v_0}{a\sigma^2}) &= \exp(\frac{v_0}{a\sigma^2})\left(\frac{2v_0}{a\sigma^2}\right)^{\gamma+\frac{\nu+1}{2}} M\left(\gamma, 1 + 2\gamma + \nu, -\frac{2v_0}{a\sigma^2}\right) \\ &= \exp(\frac{v_0}{a\sigma^2})\left(\frac{2v_0}{a\sigma^2}\right)^{\gamma+\frac{\mu}{\sigma^2}} M\left(\gamma, 2\gamma + \frac{2\mu}{\sigma^2}, -\frac{2v_0}{a\sigma^2}\right), \qquad \text{since } \frac{\nu+1}{2} = \frac{\mu}{\sigma^2}. \end{aligned} \tag{3.20}$$

From (3.19), (3.20) and Proposition 3.3.1

$$\mathbb{E}[e^{-\alpha\tau(a)}] = \mathbb{E}[e^{-\frac{4\alpha}{\sigma^2}\int_0^a \frac{1}{X_s^2} ds}] = \left(\frac{2v_0}{a\sigma^2}\right)^{\gamma} \frac{\Gamma(\gamma + 2\frac{\mu}{\sigma^2})}{\Gamma(2\gamma + 2\frac{\mu}{\sigma^2})} M\left(\gamma, 2\gamma + \frac{2\mu}{\sigma^2}; -\frac{2V_0}{a\sigma^2}\right). \tag{3.21}$$

This expression (3.21) agrees with equation (3.16). This completes the proof. □

Although our proof requires a stronger assumption, the idea of the proof can be extended to other time-homogeneous diffusion processes. In the next section, we shall illustrate its application in obtaining the density function of the first hitting time of an integrated geometric Brownian motion.

3.3.3 Probability density of the hitting time of an integrated geometric Brownian motion

Here we shall derive some further results on the first hitting time of an integrated geometric Brownian motion to a fixed level.

Proposition 3.3.3. *Assume $\mu \geqslant \frac{1}{2}\sigma^2$, for $0 \leqslant a < \infty$, the probability density function of the first hitting time $\tau(a)$ defined in equation* (3.4) *for* *an integrated geometric Brownian motion is*

$$P(\tau(a) \in dy) = 2\left(\frac{a\sigma^2}{2v_0}\right)^{\mu/\sigma^2} \exp\left(-\frac{\nu^2\sigma^2}{8}y - \frac{v_0}{a\sigma^2}\right) m_{\sigma^2 y/2}(\frac{\nu+1}{2}, \frac{v_0}{a\sigma^2})dy,$$

where the special function $m_y(\mu, z)$ is defined on p645 of Borodin and Salminen (2002):

$$m_y(\mu, z) = \frac{8z^{3/2}\Gamma(\mu+\frac{3}{2})e^{\pi^2/4y}}{\pi\sqrt{2\pi y}} \int_0^\infty e^{-z\times ch(2u)-u^2/y} M\left(-\mu, \frac{3}{2}, 2z\times sh^2(u)\right) sh(2u)\sin\left(\frac{\pi u}{y}\right) du.$$

for $z > 0$. Here $ch(.)$ is the hyperbolic cosine function, and $sh(.)$ is the hyperbolic sine function[9]*.*

Proof. Since $\mu \geqslant \frac{1}{2}\sigma^2$, from Lemma 3.3.1, $P(\varphi_\zeta = \infty) = 1$. For $0 \leqslant a < \infty$, from Proposition 3.3.1, $(\tau(a), V_{\tau(a)}) = \left(\frac{4}{\sigma^2}\int_0^a \frac{1}{X_s^2}ds, \frac{\sigma^2}{4}X_a\right)$, P-a.s. Here X follows the SDE (3.15), and is a standard Bessel process(without drift) with index $\nu = \frac{2\mu}{\sigma^2} - 1 \geqslant 0$. Combine this with the formula (4.1.20.4) on p386 of Borodin and Salminen (2002), then

$$\begin{aligned} &P(\tau(a) \in dy) \\ &= P(\frac{4}{\sigma^2}\int_0^a \frac{1}{X_s^2}ds \in dy) = \frac{2(2a)^{\frac{\nu+1}{2}}}{x_0^{\nu+1}} \exp\left\{-\frac{\nu^2}{2}y\frac{\sigma^2}{4} - \frac{x_0^2}{4a}\right\} m_{\sigma^2 y/2}\left(\frac{\nu+1}{2}, \frac{x_0^2}{4a}\right) dy \\ &= 2\left(\frac{a\sigma^2}{2v_0}\right)^{(\nu+1)/2} \exp\left\{-\frac{\nu^2\sigma^2}{8}y - \frac{v_0}{a\sigma^2}\right\} m_{\sigma^2 y/2}\left(\frac{\nu+1}{2}, \frac{v_0}{a\sigma^2}\right) dy, \quad (3.22) \end{aligned}$$

since $x_0^2 = \frac{4}{\sigma^2}v_0$. This completes the proof. □

The following result gives the joint probability density of $(\tau(a), V_{\tau(a)})$ explicitly.

Proposition 3.3.4. *Assume $\mu \geqslant \frac{1}{2}\sigma^2$, for $0 \leqslant a < \infty$, the joint probability*

[9]By definition, $sh(x) := \frac{e^x - e^{-x}}{2}$, and $ch(x) := \frac{e^x + e^{-x}}{2}$.

density of $(\tau(a), V_{\tau(a)})$ *is*

$$P\left(\tau(a) \in dy, V_{\tau(a)} \in dz\right) = \frac{z^{\nu+1}}{2av_0^{\nu}} \exp\left\{-\frac{v_0^2+z^2}{2a} - \frac{\nu^2\sigma^2}{8y}\right\} i_{\sigma^2 y/8}\left(\frac{v_0 z}{a}\right) dydz,$$

for $z \geqslant 0, y \geqslant 0$*, where the special function* $i_y(z)$ *is given on* p644 *of Borodin and Salminen (2002):*

$$i_y(z) = \frac{ze^{\pi^2/4y}}{\pi\sqrt{\pi y}} \int_0^{\infty} e^{-z \times ch(u) - u^2/4y} sh(u) \sin(\pi u/2y) du, \quad \text{for } y > 0, z > 0. \tag{3.23}$$

Here $ch(.)$ *is the hyperbolic cosine function, and* $sh(.)$ *is the hyperbolic sine function.*

Proof. Since $\mu \geqslant \frac{1}{2}\sigma^2$, from Lemma 3.3.1, $P(\varphi_\zeta = \infty) = 1$. For $0 \leqslant a < \infty$, from Proposition 3.3.1, $(\tau(a), V_{\tau(a)}) = \left(\frac{4}{\sigma^2}\int_0^a \frac{1}{X_s^2} ds, \frac{\sigma^2}{4} X_a\right)$, P-a.s. Here X follows the SDE (3.15), and is a standard Bessel process(without drift) with index $\nu = \frac{2\mu}{\sigma^2} - 1 \geqslant 0$. Combine this with the formula (4.1.20.8) on p386 of Borodin and Salminen (2002), then

$$\begin{aligned} P\left(\tau(a) \in dy, V_{\tau(a)} \in dz\right) &= P\left(\frac{4}{\sigma^2}\int_0^a \frac{1}{X_s^2} ds \in dy, X_a \in dz\right) \\ &= \frac{z^{\nu+1}}{2av_0^{\nu}} \exp\left\{-\frac{v_0^2+z^2}{2a} - \frac{\nu^2\sigma^2}{8y}\right\} i_{\sigma^2 y/8}\left(\frac{v_0 z}{a}\right) dydz. \end{aligned}$$

Here we replace y by $\sigma^2 y/4$, and replace t by a in the original formula (4.1.20.8). This completes the proof. □

3.4 Hitting times of integrated diffusions

The representation of $\tau(a)$ as an integral functional of a time-homogeneous diffusion allows us to draw on existing results in the literature. Albanese and Lawi (2005) provide a classification scheme for integral functionals of diffusion processes, whose Laplace transforms or transition probability densities can be evaluated as integrals of hypergeometric functions against the spectral measures of certain operators. Recall $\mathbb{E}[e^{-\alpha\tau_a}] = \mathbb{E}\left[e^{-\alpha\int_0^a \frac{1}{m^2(X_s)}ds}\right]$, thus in order to compute the Laplace transform of the first hitting time, we only need to compute the corresponding Laplace transform of the integral functional of X, which is the key subject studied in Albanese and Lawi (2005). Hurd and Kuznetsov (2008) also provide closed-form formulae for the Laplace transforms of certain time integrals of stochastic processes, which, when combined with the results in this chapter, will lead to new formulae for the Laplace transforms of the hitting times. In the following, we show the applications of Theorem 3.2.1 to the first hitting times of integral functionals of time-homogeneous diffusions by linking the study to relevant literature.

3.4.1 CIR process

Under P, the CIR process is governed by the SDE: $dV_t = \kappa(\theta - V_t)dt + \sigma_v\sqrt{V_t}\,dW_t,\ \ V_0 = v_0$, so that $\alpha(s) = \kappa(\theta - s)$ and $\beta(s) = \sigma_v\sqrt{s}$, and we choose $m(s) = \sqrt{s}$. Before applying Theorem 3.2.1, we have to check the conditions (3.2) and (3.3). We assume the Feller condition $2\kappa\theta > \sigma^2$, then the natural state space is $J = (0, \infty)$. The above conditions can be verified: $1/\sigma^2(x) = 1/(\sigma_v^2 x) \in L^1_{loc}(J)$ and $\mu(x)/\sigma^2(x) = \kappa\theta/(\sigma_v^2 x^2) - \kappa/(\sigma_v^2 x) \in L^1_{loc}(J)$. Denote ζ as the possible exit time of V from its state space J and define $\varphi_t = \int_0^t V_s ds, t \in [0, \zeta]$.

We first prove a lemma.

Lemma 3.4.1. *Assume the conditions* (3.2), (3.3) *and the Feller condition* $2\kappa\theta > \sigma_v^2$*, then* $P(\varphi_\zeta = \infty) = 1$.

Proof. Since $2\kappa\theta > \sigma_v^2$, define $\alpha = \frac{2\kappa\theta}{\sigma_v^2}$, then $\alpha > 1$. From the proof of Proposition 2.5.1 in Chapter 2 of the thesis, $s(\ell) = s(0) = -\infty$, then $v_b(\ell) = \infty$ holds.

For the right endpoint r, define $\beta = \frac{2\kappa}{\sigma_v^2} > 0$, and from the definition in

(2.21)

$$v_b(x) = \frac{2}{\sigma_v^2} \int_c^x y^\alpha e^{\gamma y} \left(\int_y^x z^{-\alpha} e^{-\beta z} dz \right) dy.$$

Then

$$v_b(\infty) = \frac{2}{\sigma_v^2} \int_c^\infty y^\alpha e^{\beta y} \left(\int_y^\infty z^{-\alpha} e^{-\beta z} dz \right) dy. \tag{3.24}$$

Since $\alpha > 1$, then $\lim\limits_{y \to \infty} y^{-\alpha} e^{-\beta y} = 0$. Similar as the derivation below equation (2.28) in the proof of Proposition 2.5.1 in Chapter 2 of the thesis, from L'Hôpital's rule, as $y \to \infty$

$$\int_y^\infty z^{-\alpha} e^{-\beta z} dz \sim \frac{1}{\beta} y^{-\alpha} e^{-\beta y}.$$

Thus there exists $M > c > 0$, such that for $y > M$, we have $\int_y^\infty z^{-\alpha} e^{-\beta z} dz > \frac{1}{2\beta} y^{-\alpha} e^{-\beta y}$. Substituting this into equation (3.24)

$$\begin{aligned} v_b(\infty) &= \frac{2}{\sigma_v^2} \int_c^\infty y^\alpha e^{\beta y} \left(\int_y^\infty z^{-\alpha} e^{-\beta z} dz \right) dy \\ &\geqslant \frac{2}{\sigma_v^2} \int_M^\infty y^\alpha e^{\beta y} \left(\int_y^\infty z^{-\alpha} e^{-\beta z} dz \right) dy \\ &> \frac{2}{\sigma_v^2} \int_M^\infty y^\alpha e^{\beta y} \frac{1}{2\beta} y^{-\alpha} e^{-\beta y} dy \\ &= \infty, \end{aligned}$$

then we have $v_b(\infty) = \infty$.

To summarize, with $\alpha = \frac{2\kappa\theta}{\sigma_v^2} > 1$, we have both $v_b(\ell) = \infty$ and $v_b(r) = \infty$. From Lemma 2.3.1, in Chapter 2 of this thesis, $P(\varphi_\zeta = \infty) = 1$. This completes the proof. □

Proposition 3.4.1. ***Joint Representation of*** (τ, V_τ) ***for the CIR Process***

Assume the Feller condition $2\kappa\theta > \sigma_v^2$*. For* $0 \leqslant a < \infty$*, define* $\tau(a)$ *as* *in equation (3.4). Then* $(\tau(a), V_{\tau(a)}) = \left(\int_0^a \frac{1}{\sigma_v X_s} ds, \sigma_v X_a\right)$*, P-a.s., where* X_t *is governed by the SDE*

$$dX_t = \left(\frac{\kappa\theta}{\sigma_v^2 X_t} - \frac{\kappa}{\sigma_v}\right) dt + dB_t, \quad X_0 = \frac{v_0}{\sigma_v}, \tag{3.25}$$

where B_t *is a standard* $\mathcal{G}_t$*-Brownian motion. Here* X_t *is a standard Bessel process with drift* $\mu = -\kappa/\sigma_v$*, and index* $\nu = \kappa\theta/\sigma_v^2 - 1/2 > 0$*.*

Proof. Since $2\kappa\theta > \sigma_v^2$, from Lemma 3.4.1, $P(\varphi_\zeta = \infty) = 1$. For $0 \leqslant t < \infty$, from Theorem 3.2.1, we have the stochastic representation $V_t = Y_{\int_0^t V_s ds}$, P-a.s., $Y_0 = v_0$, where Y is governed by the following SDE

$$dY_t = \frac{\kappa(\theta - Y_t)}{Y_t} dt + \frac{\sigma_v \sqrt{Y_t}}{\sqrt{Y_t}} dB_t = \left(\frac{\kappa\theta}{Y_t} - \kappa\right) dt + \sigma_v dB_t.$$

Recognize Y as a squared Bessel process with drift. Since $2\kappa\theta > \sigma_v^2$ is assumed, the index of Y is $\delta = 2\kappa\theta/\sigma_v^2 + 1 > 2$, thus the left boundary 0 can not be attained (see Ch.X.I., Revuz and Yor (1999)). Denote a new process $X_t = Y_t/\sigma_v$ with $X_0 = v_0/\sigma_v$. From Itō's lemma, the SDE of X is

$$dX_t = \left(\frac{\kappa\theta}{\sigma_v^2 X_t} - \frac{\kappa}{\sigma_v}\right) \mathbb{1}_{t\in[0,\zeta^X)} dt + \mathbb{1}_{t\in[0,\zeta^X)} dB_t, \quad X_0 = \frac{v_0}{\sigma_v}. \tag{3.26}$$

The joint representation follows from Theorem 3.2.1 (iii). From Theorem 3.2.1 (ii) combined with Lemma 3.4.1, $\zeta^X = \varphi_\zeta = \infty$, P-a.s. Then the above SDE (3.26) agrees with the SDE (3.25). This completes the proof. □

Remark 3.4.1. *From Proposition 3.4.1, for* $0 \leqslant a < \infty$

$$P(\tau(a) \in dx) = P\left(\int_0^a \frac{1}{\sigma_v X_t} dt \in dx\right),$$

and

$$P(V_{\tau(a)} \in dx) = P(\sigma_v X_a \in dx),$$

where X_t is a standard Bessel process with drift $\mu = -\kappa/\sigma_v < 0$, and index $\nu = \kappa\theta/\sigma_v^2 - 1/2 > 0$. Then we can obtain the density function for $V_{\tau(a)}, 0 \leqslant a < \infty$, by referring to the spectral representation of the transition density of a Bessel process with constant drift (Proposition 1, p329 of Linetsky (2004)).

3.4.2 GARCH diffusion process

The GARCH diffusion is the continuous time limit of the discrete GARCH process, and it has been popular in the option pricing literature, see Duan (1995) and Lewis (2000). Here we use the GARCH diffusion to model the foreign exchange rate between the foreign currency and the domestic currency, and under P it has the following SDE

$$dV_t = k(\theta - V_t)dt + \varepsilon V_t dW_t, \quad V_0 = v_0.$$

Then $1/V$ denotes the exchange rate between the domestic currency and the foreign currency. Suppose the cost is denominated in the foreign currency, then $\int_0^t \frac{1}{V_s} ds$ represents the accumulated cost denominated in the foreign currency. Consider the following option which is exercised at the time when this accumulated cost reaches a fixed level. In the following, we study the Laplace transform of this first hitting time. Before applying Theorem 3.2.1, we have to check the conditions (3.2) and (3.3). With $\kappa, \theta, \varepsilon > 0$, from Feller's test of explosions, the natural state space is $J = (0, \infty)$, and it is easy to verify the above conditions: $1/\sigma^2(x) = 1/(\varepsilon^2 x^2) \in L^1_{loc}(J)$ and $\mu(x)/\sigma^2(x) = \kappa\theta/(\varepsilon^2 x^2) - \kappa/(\varepsilon^2 x) \in L^1_{loc}(J)$. Denote ζ as the possible exit time of V from its state space and define $\varphi_t = \int_0^t \frac{1}{V_s} ds, t \in [0, \zeta]$.

Proposition 3.4.2. *For $0 \leqslant a < \infty$, define $\tau(a)$ as in equation (3.4). Then for $\lambda \geqslant 0$*

$$u(\lambda) = \mathbb{E}[e^{-\lambda\tau(a)}] = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left(\frac{2}{\varepsilon^2 y_1}\right)^{\alpha} M\left(\alpha, \gamma; -\frac{2}{\varepsilon^2 y_1}\right), \qquad (3.27)$$

where $M(.)$ is the confluent hypergeometric function, and

$$\begin{aligned}
\alpha &= -\left(\frac{1}{2}+\frac{k}{\varepsilon^2}\right)+\sqrt{\left(\frac{1}{2}+\frac{k}{\varepsilon^2}\right)^2+\frac{2\lambda}{\varepsilon^2}},\\
\gamma &= 1+2\sqrt{\left(\frac{1}{2}+\frac{k}{\varepsilon^2}\right)^2+\frac{2\lambda}{\varepsilon^2}},\\
y_1 &= \frac{v_0(e^{k\theta a}-1)}{k\theta}.
\end{aligned} \tag{3.28}$$

Proof. Here $\alpha(s)=k(\theta-s)$, $\beta(s)=\varepsilon s$. For $0\leqslant a<\varphi_\zeta$, from Theorem 3.2.1, $(\tau(a),V_{\tau(a)}) = (\int_0^a X_s ds, X_a)$ P-a.s., where X is governed by the following SDE

$$dX_t = kX_t(\theta-X_t)\mathbb{1}_{t\in[0,\varphi_\zeta)}dt+\varepsilon X_t^{\frac{3}{2}}\mathbb{1}_{t\in[0,\varphi_\zeta)}dB_t, \quad X_0=v_0. \tag{3.29}$$

Recognize (3.29) as the SDE of the 3/2 stochastic volatility process. For $k>0$, $k>-\frac{\varepsilon^2}{2}$ always holds. Then the 3/2 stochastic volatility process does not explode at infinity, because it can be written as the reciprocal of a CIR process (see equation (67), p108 of Carr and Sun (2007)).

For $0\leqslant a<\infty$, the Laplace transform of $\tau(a)$ follows from the Laplace transform of $\int_0^a X_s ds$ as provided[10] in Theorem 3, p110 of Carr and Sun (2007):

$$u(\lambda)=\mathbb{E}[e^{-\lambda\tau(a)}]=\mathbb{E}[e^{-\lambda\int_0^a X_s ds}]=\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\left(\frac{2}{\varepsilon^2 y_1}\right)^{\alpha} M\left(\alpha,\gamma;-\frac{2}{\varepsilon^2 y_1}\right).$$

where α, γ and y_1 are given in (3.28). This completes the proof. □

Remark 3.4.2. *Note that in Proposition 3.4.2, the result in* (3.27) *holds also for $\lambda=0$ as long as the parameters of the model do not satisfy $\rho=1$ or $\frac{1}{2}+\frac{k}{\varepsilon^2}=\frac{1}{2\varepsilon}$. This is based on analytical continuation, and for its proof, refer to Lemma* 4.1.2. *on p72 of Gao (2012). Similarly, the result in Proposition 3.4.3 also holds when $\lambda=0$.*

[10] More specifically, we substitute $u=0$, $\lambda=-s$ in Theorem 3, p110 of Carr and Sun (2007).

3.4.3 Mean reverting geometric Brownian motion

Suppose that we want to model the electricity cost and choose the following mean reverting geometric Brownian motion model with SDE under P as

$$dV_t = kV_t(\theta - V_t)dt + \varepsilon V_t dW_t, \quad V_0 = v_0.$$

Then V_t represents the electricity cost, and $\int_0^t V_s ds$ represents the accumulated electricity cost. It is natural to define a digital option that pays off \$1 whenever the accumulated electricity cost reaches a certain fixed level. Thus it is important to study the first hitting time of $\int_0^t V_s ds$ to a fixed level, and the next result studies its Laplace transform. Before applying Theorem 3.2.1, we have to check the conditions (3.2) and (3.3). With $k, \theta, \varepsilon > 0$, from Feller's test of explosions, the natural state space is $J = (0, \infty)$. The above conditions can be verified: $1/\sigma^2(x) = 1/(\varepsilon^2 x^2) \in L^1_{loc}(J)$ and $\mu(x)/\sigma^2(x) = k\theta/(\varepsilon^2 x^2) - k/(\varepsilon^2 x) \in L^1_{loc}(J)$. Denote ζ as the possible exit time of V from its state space and define $\varphi_t = \int_0^t V_s ds, t \in [0, \zeta]$.

Proposition 3.4.3. *Assume $2k\theta > \varepsilon^2$, and for $0 \leqslant a < \infty$, define $\tau(a)$ as in equation (3.4). Then for $\lambda \geqslant 0$*

$$u(\lambda) = \mathbb{E}[e^{-\lambda\tau(a)}] = \frac{\Gamma(\gamma_2 - \alpha_2)}{\Gamma(\gamma_2)} \left(\frac{2}{\varepsilon^2 y_2}\right)^{\alpha_2} M\left(\alpha_2, \gamma_2; -\frac{2}{\varepsilon^2 y_2}\right),$$

where $M(.)$ is the confluent hypergeometric function, and

$$\begin{aligned} \alpha_2 &= \left(\frac{1}{2} - \frac{k\theta}{\varepsilon^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{k\theta}{\varepsilon^2}\right)^2 + \frac{2\lambda}{\varepsilon^2}}, \\ \gamma_2 &= 1 + 2\sqrt{\left(\frac{1}{2} - \frac{k\theta}{\varepsilon^2}\right)^2 + \frac{2\lambda}{\varepsilon^2}}, \\ y_2 &= \frac{e^{ka} - 1}{v_0 k}. \end{aligned} \tag{3.30}$$

Proof. Here $\alpha(s) = ks(\theta - s)$, $\beta(s) = \varepsilon s$. For $0 \leqslant a < \varphi_\zeta$, from Theorem 3.2.1(iii), $(\tau(a), V_{\tau(a)}) = (\int_0^a \frac{1}{X_s} ds, X_a)$, P-a.s., where X is governed by the following SDE

$$dX_t = k(\theta - X_t)\mathbb{1}_{t\in[0,\varphi_\varsigma)}dt + \varepsilon\sqrt{X_t}\mathbb{1}_{t\in[0,\varphi_\varsigma)}dB_t, X_0 = v_0. \tag{3.31}$$

Recognize X in (3.31) as the CIR process. Since $2k\theta > \varepsilon^2$ is assumed, the process X is positive P-a.s. Also note that $Y_t = 1/X_t$ follows the 3/2 stochastic process $dY_t = k_2Y_t(\theta_2 - Y_t)\mathbb{1}_{t\in[0,\varphi_\varsigma)}dt + \varepsilon_2 Y_t^{3/2}\mathbb{1}_{t\in[0,\varphi_\varsigma)}dW_t, \quad Y_0 = 1/v_0$, with new parameters $k_2 = k\theta - \varepsilon^2$, $\theta_2 = k/(k\theta - \varepsilon^2)$ and $\varepsilon_2 = -\varepsilon$. Thus the Laplace transform follows from Theorem 3, p110 of Carr and Sun (2007):

$$u(\lambda) = \mathbb{E}[e^{-\lambda\tau(a)}] = \mathbb{E}[e^{-\lambda\int_0^a Y_s ds}] = \frac{\Gamma(\gamma_2 - \alpha_2)}{\Gamma(\gamma_2)}\left(\frac{2}{\varepsilon^2 y_2}\right)^{\alpha_2} M\left(\alpha_2, \gamma_2; -\frac{2}{\varepsilon^2 y_2}\right),$$

where α_2, γ_2 and y_2 are given in (3.30). This completes the proof. □

3.5 Applications to the pricing of arithmetic Asian options

Arithmetic Asian options were introduced in Boyle and Emanuel (1980), and since then have constituted an important family of derivative contracts. In the Black-Scholes framework, the pricing of the arithmetic Asian option is closely linked to the integral of a geometric Brownian motion. The main theoretical difficulty is that this integral is not log normally distributed. In a pioneering work, Yor (1992) expresses the arithmetic Asian option price as a triple integral (equation (6.e), p528, Yor (1992)), and the method is based on the Hartman-Watson theory in Yor (1980).

The contribution of this section is to establish the link between the pricing of arithmetic Asian options and the first hitting times of integral functionals of diffusions. This provides new insights in the pricing of arithmetic Asian options in the time-homogeneous diffusion setting. In particular, in the Black-Scholes setting, we are able to derive a double integral formula for the price of the arithmetic Asian option.

Given a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t)$ with the risk-neutral measure Q. The stock price dynamic is

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s_0, \tag{3.32}$$

then we have the following representation of the stock price

$$S_T = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right\}. \tag{3.33}$$

The natural state space for S is $J = (\ell, r) = (0, \infty)$. Assume $r \geqslant \frac{1}{2}\sigma^2$, and from Feller's test of explosions, $S_t, t \geqslant 0$ never exits at the left boundary ℓ in finite time. If $r = \frac{1}{2}\sigma^2$, then $S_t, t \geqslant 0$ never exits at the right boundary in finite time. If $r > \frac{1}{2}\sigma^2$, then there is a positive probability that $S_t, t \geqslant 0$ may exit through the right boundary in finite time. Denote ζ as the possible exit time of the process S from its state space J. Define $\varphi_t = \int_0^t S_u du, t \in [0, \zeta]$.

Since we assume $r \geqslant \frac{1}{2}\sigma^2$, from Lemma 3.3.1, $Q(\varphi_\zeta = \infty) = 1$. In the following, we consider the pricing of the arithmetic Asian option for $T \in [0, \infty)$ (similar to equation (6.e), p528, Yor (1992)).

Proposition 3.5.1. *In the geometric Brownian motion model in* (3.32)*, assume $r \geqslant \frac{1}{2}\sigma^2$. For $T \in [0, \infty)$, the price of the arithmetic Asian option can be represented as*

$$C_0 = e^{-rT}\mathbb{E}^Q\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] = \int_0^\infty \int_0^T g(x, y) f(x, y) dx dy, \tag{3.34}$$

where $g(x, y) = y\frac{e^{-rx} - e^{-rT}}{rT}$, and

$$f(x, y) = \frac{y^{\nu+1}}{2KTS_0^\nu} \exp\left\{-\frac{S_0^2 + y^2}{2KT} - \frac{\nu^2\sigma^2}{8x}\right\} i_{\sigma^2 x/8}\left(\frac{S_0 y}{KT}\right) dx dy,$$

for $x, y \geqslant 0$, where $\nu = \frac{2r}{\sigma^2} - 1 \geqslant 0$, and the special function $i_y(z)$ is defined in (3.23).

Proof. For $T \in [0, \infty)$, the payoff of an arithmetic Asian option at the maturity T is $\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+$. From the risk-neutral valuation principle, its price can be expressed as

$$
\begin{aligned}
C_0 &= e^{-rT}\mathbb{E}^Q\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \\
&= e^{-rT}\mathbb{E}^Q\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)\mathbb{1}_{\left\{\frac{1}{T}\int_0^T S_t dt \geqslant K\right\}}\right] \\
&= e^{-rT}\mathbb{E}^Q\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)\mathbb{1}_{\left\{\int_0^T S_t dt \geqslant KT\right\}}\right] \qquad (3.35)
\end{aligned}
$$

Note that $Q(\varphi_\zeta = \infty) = 1$, and define the following first hitting time similar as (3.4) under Q

$$
\tau = \inf\left\{u \geqslant 0 : \int_0^u S_t dt > KT\right\}.
$$

Here τ is the first hitting time of the integrated stock price process to a fixed level $a = KT \in [0, \infty)$. Observe the equivalence between the two events under Q

$$
\left\{\int_0^t S_t dt \geqslant KT\right\} \Longleftrightarrow \{\tau \leqslant t\}, \text{ for } t \in [0, \infty)
$$

Rewrite equation (3.35) as

$$
\begin{aligned}
C_0 &= e^{-rT}\mathbb{E}^Q\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)\mathbb{1}_{\left\{\int_0^T S_t dt \geqslant KT\right\}}\right] \\
&= e^{-rT}\mathbb{E}^Q\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)\mathbb{1}_{\{\tau \leqslant T\}}\right] \\
&= e^{-rT}\mathbb{E}^Q\left[\left(\frac{1}{T}\left(KT + \int_\tau^T S_t dt\right) - K\right)\mathbb{1}_{\{\tau \leqslant T\}}\right] \\
&= e^{-rT}\mathbb{E}^Q\left[\left(\frac{1}{T}\int_\tau^T S_t dt\right)\mathbb{1}_{\{\tau \leqslant T\}}\right].
\end{aligned}
$$

By the law of iterated expectation, conditioning on (τ, S_τ)

$$\begin{aligned}
C_0 &= e^{-rT}\mathbb{E}^Q\left[\mathbb{E}^Q\left[\left(\frac{1}{T}\int_\tau^T S_t dt\right)\mathbb{1}_{\{\tau\leqslant T\}} \mid (\tau, S_\tau)\right]\right] \\
&= e^{-rT}\mathbb{E}^Q\left[\mathbb{E}^Q\left[\left(\frac{1}{T}\int_\tau^T S_t dt\right) \mid (\tau, S_\tau)\right]\mathbb{1}_{\{\tau\leqslant T\}}\right] \\
&= \mathbb{E}^Q\left[S_\tau \frac{e^{-r\tau}-e^{-rT}}{rT}\mathbb{1}_{\{\tau\leqslant T\}}\right]. \qquad (3.36)
\end{aligned}$$

Denote $g(x, y) = y\frac{e^{-rx}-e^{-rT}}{rT}$, then $g(\tau, S_\tau) = S_\tau\frac{e^{-r\tau}-e^{-rT}}{rT}$. Denote the joint density of (τ, S_τ) as $f(x, y) = Q(\tau \in dx, S_\tau \in dy)$, then the equation (3.36) can be rewritten as

$$\begin{aligned}
C_0 &= \mathbb{E}^Q\left[S_\tau \frac{e^{-r\tau}-e^{-rT}}{rT}\mathbb{1}_{\{\tau\leqslant T\}}\right] \\
&= \int_0^\infty\int_0^\infty g(x, y)f(x, y)\mathbb{1}_{\{x\leqslant T\}}dxdy \\
&= \int_0^\infty\int_0^T g(x, y)f(x, y)dxdy. \qquad (3.37)
\end{aligned}$$

Equation (3.37) is the analytical formula for the price of the arithmetic Asian option written as a double integral.

The next task is to determine the joint density function of (τ, S_τ). This is already given in Proposition 3.23 as

$$\begin{aligned}
f(x, y) &= P\left(\tau \in dx, S_\tau \in dy\right) \\
&= \frac{y^{\nu+1}}{2KTS_0^\nu}\exp\left\{-\frac{S_0^2+y^2}{2KT} - \frac{\nu^2\sigma^2}{8x}\right\} i_{\sigma^2 x/8}\left(\frac{S_0 y}{KT}\right)dxdy, \qquad (3.38)
\end{aligned}$$

where $i_y(z)$ is defined in (3.23). Combining equation (3.37) and equation (3.38), we obtain the analytical formula for the arithmetic Asian option in the Black-Scholes model as given in (3.34). This completes the proof. □

3.6 Conclusion of Chapter 3

This chapter studies properties of the first hitting time of an integral functional of a time-homogeneous diffusion to a fixed level. We provide a unified approach to compute the Laplace transform of this first hitting time. As an application, we provide an alternative proof to the main result in Metzler (2013) with a slightly stronger assumption. The links between the first hitting times and integral functionals of time-homogeneous diffusions are established, and is connected to relevant literature. We also show the link between the pricing of an arithmetic Asian option and this first hitting time. We derive an analytical formula for the price in the Black-Scholes model by linking it to the study of some functional of a standard Bessel process with no drift. Financial motivations behind the study of this hitting time are also discussed, with the newly introduced "timer option" as an example.

Chapter 4

Prices and asymptotics of some discrete volatility derivatives

This chapter is mainly based on the publication Bernard and Cui (2013) forthcoming in the *Applied Mathematical Finance*. Sections 4.8 and 4.9 are not part of the paper of Bernard and Cui (2013).

In this chapter, we study the fair strike of a discrete variance swap for a general time-homogeneous stochastic volatility model. In the special cases of Heston, Hull-White and Schöbel-Zhu stochastic volatility models we give simple explicit expressions (improving Broadie and Jain (2008a) in the case of the Heston model). We give conditions on parameters under which the fair strike of a discrete variance swap is higher or lower than that of the continuous variance swap. The interest rate and the correlation between the underlying price and its volatility are key elements in this analysis. We derive asymptotics for the discrete variance swaps and compare our results with those of Broadie and Jain (2008a), Jarrow et al. (2013) and Keller-Ressel and Griessler (2012).

4.1 Introduction

A variance swap is a derivative contract that pays at a fixed maturity T the difference between a given level (fixed leg) and a realized level of variance over the swap's life (floating leg). Nowadays, variance swaps on stock indices are broadly used and highly liquid. Less standardized variance swaps could be linked to other types of underlying assets such as currencies or commodities. They can be useful to hedging volatility risk exposure or to taking positions on future realized volatility. For example, Carr and Lee (2007) price options on realized variance and realized volatility by using variance swaps as pricing and hedging instruments. See Carr and Lee (2009) for a history of volatility derivatives. As noted by Jarrow et al. (2013), most academic studies[1] focus on continuously sampled variance and volatility swaps. However existing volatility derivatives tend to be based on the realized variance computed from the discretely sampled log stock

[1]See, for example, Howison, Rafailidis and Rasmussen (2004), Benth, Groth and Kufakunesu (2007) and Broadie and Jain (2008b).

price (see Windcliff, Forsyth and Vetzal (2006)), and continuously sampled derivatives prices may only be used as approximations. As pointed out in Sepp (2012), some care is needed to replace the discrete realized variance by the continuous quadratic variation. By standard probability arguments, the discretely sampled realized variance converges to the quadratic variation of the log stock process in probability. However, this does not guarantee that it converges in expectation. Jarrow et al. (2013) provide sufficient conditions such that the convergence in expectation happens when the stock is modeled by a general semi-martingale, and concrete examples where this convergence fails.

In this chapter, we study discretely sampled variance swaps in a general time-homogeneous model for stochastic volatility. For discretely sampled variance swaps, it is difficult to use the elegant and model-free approach of Dupire (1993) and Neuberger (1994), who independently proved that the fair strike for a continuously sampled variance swap on any underlying price process with continuous path is simply two units of the forward price of the log contract. Building on these results, Carr and Madan (1998) published an explicit expression to obtain this forward price from option prices (by synthesizing a forward contract with vanilla options). The Dupire-Neuberger theory was recently extended by Carr, Lee and Wu (2011) to the case when the underlying stock price is driven by a time-changed Lévy process (thus allowing jumps in the path of the underlying stock price). In this chapter, we adopt a parametric approach that allows us to derive explicit closed-form expressions and asymptotic behaviors with respect to key parameters such as the maturity of the contract, the risk-free rate, the sampling frequency, the volatility of the variance process (vol of vol), or the correlation between the underlying stock and its volatility. This is in line with the work of Broadie and Jain (2008a) in which the Heston model and the Merton jump diffusion model are considered. See also Itkin and Carr (2010), who study discretely sampled variance swaps in the 3/2 stochastic volatility model.

The main contributions of this chapter are as follows. We give an expression of the fair strike of the discretely sampled variance swap and derive its sensitivity to interest rates in a general time-homogeneous stochastic volatility model. In the case of the (correlated) Heston (1993) model, and the (correlated) Hull-White (1987) model, we obtain simple explicit closed-form formulas for the respective fair strikes of continuously and discretely sampled variance swaps. In the Heston model, our formula simplifies the results of Broadie and Jain (2008a) and is easy to analyze. Consequently, we are able to give asymptotic behaviors with respect to key parameters of the model and to the sampling frequency. In particular, we provide explicit conditions under which the discretely sampled variance swap is less valuable than the continuously sampled variance swap although the contrary is commonly observed in the literature (see Bühler (2006) for example). Thus the "convex-order conjecture" formulated by Keller-Ressel and Griessler (2012) may not hold for stochastic volatility models with correlation. We discuss practical implications and illustrate the risk to underestimate or overestimate prices of discretely sampled variance swaps when using a model for the corresponding continuously sampled ones with numerical examples. Based on the explicit closed-form formula for the discrete variance swap in the Heston model, utilizing symmetry properties of the Heston model under the change of numeraire, we manage to obtain closed-form formula for the fair strike of a special type of gamma swap in the Heston model. In Broadie and Jain (2008a), they study the fair strike of the discrete variance swap under the Merton's jump diffusion model, and in this chapter we provide an explicit formula for the fair strike of a discrete variance swap under the newly introduced Mixed Exponential Jump Diffusion (MEJD) model in Cai and Kou (2011). Since the mixed exponential distribution is dense with respect to the class of all distributions in the sense of weak convergence (see Botta and Harris (1986)), the MEJD can be used to approximate Merton's jump diffusion model, or essentially any jump diffusion model.

In this chapter, the new results, which contribute to the current literature, are as follows: Proposition 4.2.1, Proposition 4.2.2, Proposition 4.3.1, Proposition 4.3.2, Proposition 4.4.1, Proposition 4.5.1, Proposition 4.6.1, Proposition 4.6.2, Proposition 4.6.3, Proposition 4.6.4, Proposition 4.6.5, Proposition 4.6.6, Proposition 4.6.7, Proposition 4.6.8, Proposition 4.8.2, Proposition 4.8.3, Proposition 4.8.4, Proposition 4.8.5, and Proposition 4.9.1.

The chapter is organized as follows. Section 4.2 deals with the general time-homogeneous stochastic volatility model. Sections 4.3, 4.4 and 4.5 provide formulas for the fair strike of a discrete variance swap in the Heston, Hull-White and Schöbel-Zhu models. Section 4.6 contains asymptotic expansion formulas for the Heston, Hull-White and Schöbel-Zhu models, and discusses a counter-example to the "convex-order conjecture". A numerical analysis is given in Section 4.7. Section 4.8 discusses the pricing of a special type of discrete gamma swaps in the Heston model. Section 4.9 discusses the pricing of discrete variance swaps in the mixed exponential jump diffusion (MEJD) model. Sections 4.10, 4.11, 4.12, 4.13 and 4.14 give the proofs to the main results. Section 4.15 concludes the chapter.

4.2 Pricing of variance swaps in a time-homogeneous stochastic volatility model

In this section, we consider the problem of pricing a discrete variance swap under the following general time-homogeneous stochastic volatility model (M), where the stock price and its volatility can possibly be correlated. We assume a constant risk-free rate $r \geqslant 0$, and that under a risk-neutral probability measure $\mathbb{Q}$

$$(M) \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + m(V_t)dW_t^{(1)} \\ dV_t &= \mu(V_t)dt + \sigma(V_t)dW_t^{(2)} \end{cases} \tag{4.1}$$

where $\mathbb{E}[dW_t^{(1)}dW_t^{(2)}] = \rho dt$, with $W^{(1)}$, $W^{(2)}$ standard correlated Brownian motions. The state space of the stochastic process[2] V is $J = (l, r) =$

[2]When $m(x) = \sqrt{x}$, V means the variance process and $l \geqslant 0$. When $m(x) = x$, V means the volatility process, and there is no restriction on l.

$(0,\infty)$ if V is the variance process ($m(x) = \sqrt{x}$). If $m(x) = x$ and V is the volatility process, we may use $J = (l, r) = (-\infty, \infty)$. Assume that $\mu, \sigma : J \to \mathbb{R}$ are Borel functions satisfying the following Engelbert-Schmidt conditions, $\forall x \in J, \sigma(x) \neq 0$, $\frac{1}{\sigma^2(x)}, \frac{\mu(x)}{\sigma^2(x)}, \frac{m^2(x)}{\sigma^2(x)} \in L^1_{loc}(J)$. Here $L^1_{loc}(J)$ denotes the class of locally integrable functions, i.e. the functions $J \to \mathbb{R}$ that are integrable on compact subsets of J. Under the above conditions, the SDE (4.1) for V has a unique in law weak solution that possibly exits its state space J (see Theorem 5.5.15, p341, Karatzas and Shreve (1991)). Assume that $\frac{m(x)}{\sigma(x)}$ is differentiable at all $x \in J$.

In particular, this general model includes the Heston, the Hull-White, the Schöbel-Zhu, the 3/2 and the Stein-Stein models as special cases. In what follows, we study discretely and continuously sampled variance swaps with maturity T. In a variance swap, one counterparty agrees to pay at a fixed maturity T a notional amount times the difference between a fixed level and a realized level of variance over the swap's life. If it is continuously sampled, the realized variance corresponds to the quadratic variation of the underlying log price. When it is discretely sampled, it is the sum of the squared increments of the log price. Define their respective fair strikes as follows.

Definition 4.2.1. *The fair strike of the discrete variance swap associated with the partition* $0 = t_0 < t_1 < ... < t_n = T$ *of the time interval* $[0, T]$ *is defined as*

$$K_d^M(n) := \frac{1}{T} \sum_{i=0}^{n-1} \mathbb{E}\left[\left(\ln \frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right], \tag{4.2}$$

where the underlying stock price S *follows the time-homogeneous stochastic volatility model* (4.1) *and where the exponent* M *refers to the model* (M).

Definition 4.2.2. *The fair strike of the continuous variance swap is defined as*

$$K_c^M := \frac{1}{T}\mathbb{E}\left[\int_0^T m^2(V_s) ds\right], \tag{4.3}$$

where S follows the time-homogeneous stochastic volatility model (4.1).

In popular stochastic volatility models, $m(v) = \sqrt{v}$, so that $K_c^M = \frac{1}{T}\mathbb{E}\left[\int_0^T V_s ds\right]$. The derivation of the fair strike of a discrete variance swap in the time-homogeneous stochastic volatility model (4.1) is based on the following proposition.

Proposition 4.2.1. *Under the dynamics* (4.1) *for the stochastic volatility model* (M), *define*

$$f(v) = \int_0^v \frac{m(z)}{\sigma(z)} dz \quad \text{and} \quad h(v) = \mu(v) f'(v) + \frac{1}{2}\sigma^2(v) f''(v).$$

For all $t \leqslant s \leqslant t+\Delta$ and $t \leqslant u \leqslant t+\Delta$, assume that[3]

$$\begin{aligned} &\mathbb{E}\left[|h(V_s)h(V_u)|\right] < \infty, \quad \mathbb{E}\left[\left|h(V_s)m^2(V_u)\right|\right] < \infty, \\ &\mathbb{E}\left[\left|(f(V_{t+\Delta}) - f(V_t))(2\rho h(V_s) + m^2(V_s))\right|\right] < \infty. \end{aligned} \tag{4.4}$$

Define for $t \leqslant s \leqslant t+\Delta, \quad t \leqslant u \leqslant t+\Delta$,

$$\begin{aligned} &m_1(s) := \mathbb{E}\left[m^2(V_s)\right], && m_2(s,u) := \mathbb{E}\left[m^2(V_s)m^2(V_u)\right], \\ &m_3(s,u) := \mathbb{E}\left[h(V_s)h(V_u)\right], && m_4(s,u) := \mathbb{E}\left[h(V_s)m^2(V_u)\right], \\ &m_5(t,s) := \mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t))(2\rho h(V_s) + m^2(V_s))\right]. \end{aligned}$$

Then, we have

$$\begin{aligned} \mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] &= r^2\Delta^2 + (1-\rho^2 - r\Delta)\int_t^{t+\Delta} m_1(s)ds - \rho\int_t^{t+\Delta} m_5(t,s)ds \\ &+ \frac{1}{4}\int_t^{t+\Delta}\int_t^{t+\Delta} m_2(s,u)dsdu + \rho^2\mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t))^2\right] \\ &+ \rho^2\int_t^{t+\Delta}\int_t^{t+\Delta} m_3(s,u)dsdu + \rho\int_t^{t+\Delta}\int_t^{t+\Delta} m_4(s,u)dsdu. \end{aligned} \tag{4.5}$$

Proof. See Section 4.10. □

[3] These conditions ensure that we can apply Fubini's theorem to exchange the order of integration. They are easily verified in specific examples.

Proposition 4.2.1 gives the key equation in the analysis of discrete variance swaps. Observe that the final expression (4.5) only depends on covariances of functionals of V_t. Thus we can derive closed-form formulas for the fair strike of discrete variance swaps in those stochastic volatility models in which the terms m_i from Proposition 4.2.1 can be computed in closed-form. In the rest of the chapter, we provide three examples to apply this formula.

From now on, for simplicity, we consider the equi-distant sampling scheme in (4.2). Under this scheme, $t_i = iT/n$ and $\Delta = t_{i+1} - t_i = T/n$, for $i = 0, 1, ..., n-1$.

Remark 4.2.1. *From* (4.5) *it is clear that the fair strike of a discrete variance swap only depends on the risk-free rate r up to the second order, as there is no higher order terms of r. Interestingly, the second order coefficient of this expansion is model-independent whereas the first order coefficient is directly related to the strike of the corresponding continuously-sampled variance swap. Assume a constant sampling period $\frac{T}{n}$, the fair strike of the discrete variance swap can be expressed as*

$$K_d^M(n) = b^M(n) - \frac{T}{n}K_c^M r + \frac{T}{n}r^2, \tag{4.6}$$

where $b^M(n)$ does not depend on r. Its sensitivity[4] *to the risk-free rate r is equal to*

$$\frac{dK_d^M(n)}{dr} = \frac{T}{n}(2r - K_c^M), \tag{4.7}$$

so that the minimum of K_d^M as a function of r is attained when the risk-free rate takes the value r^* given by

$$r^* = \frac{K_c^M}{2}.$$

[4]The impact of stochastic interest rates on variance swaps is studied by Hörfelt and Torné (2010). Long-dated variance swaps will usually be sensitive to the interest rate volatility.

The next proposition deals with the special case when the risk-free rate r and the correlation coefficient ρ are both equal to 0.

Proposition 4.2.2. *(Fair strike when $r = 0\%$ and $\rho = 0$)*

In the special case when the constant risk-free rate is 0, and the underlying stock price is not correlated to its volatility, we observe that

$$K_d^M(n) \geqslant K_c^M.$$

Proof. Using Proposition 4.2.1 when $r = 0\%$ and $\rho = 0$, we obtain

$$\mathbb{E}\left[\left(\ln \frac{S_{t+\Delta}}{S_t}\right)^2\right] = \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta} m^2(V_s)ds\right)^2\right] + \int_t^{t+\Delta} \mathbb{E}\left[m^2(V_s)\right] ds.$$

We then add up the expectations of the squares of the log increments (as in (4.2)) and find that the fair strike of the discrete variance swap is always larger than the fair strike of the continuous variance swap given in (4.3). □

Proposition 4.2.2 has already appeared in the literature in specific models. See for example Corollary 6.2 of Carr, Lee and Wu (2011), where this result is proved in the more general setting of time-changed Lévy processes with independent time changes. However, we will see in the remainder of this chapter that Proposition 4.2.2 may not hold under more general assumptions, namely when the dynamic of the stock price is correlated to the one of the volatility.

4.3 Fair strike of the discrete variance swap in the Heston model

Assume that we work under the Heston stochastic volatility model with the following dynamics

$$(H) \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)}, \\ dV_t &= \kappa(\theta - V_t)dt + \gamma\sqrt{V_t}dW_t^{(2)}, \end{cases} \tag{4.8}$$

where $\mathbb{E}\left[dW_t^{(1)}dW_t^{(2)}\right] = \rho dt$. It is a special case of the general model (4.1), where we choose

$$m(x) = \sqrt{x},\ \mu(x) = \kappa(\theta - x),\ \sigma(x) = \gamma\sqrt{x}. \tag{4.9}$$

Use (4.48) in Lemma 4.10.1 in Section 4.10 with $f(v) = \frac{v}{\gamma}$ and $h(v) = \frac{\kappa}{\gamma}(\theta - v)$, the stock price is

$$S_t = S_0 e^{rt - \frac{1}{2}\xi_t + (V_t - V_0 - \kappa\theta t + \kappa\xi_t)\frac{\rho}{\gamma} + \sqrt{1-\rho^2}\int_0^t \sqrt{V_s}dW_s^{(3)}} \tag{4.10}$$

where $\xi_t = \int_0^t V_s ds$ and $W_t^{(3)}$ is such that $dW_t^{(1)} = \rho dW_t^{(2)} + \sqrt{1-\rho^2}dW_t^{(3)}$.

Using Proposition 4.2.1 for the time-homogeneous stochastic volatility model, we then derive a closed-form expression for the fair strike of a discrete variance swap and compare it with the fair strike of a continuous variance swap.

Proposition 4.3.1. *(Fair Strikes in the Heston Model)*

In the Heston stochastic volatility model (4.8)*, the fair strike* (4.2) *of the discrete variance swap is*

$$\begin{aligned} K_d^H(n) = \frac{1}{8n\kappa^3 T}\Bigg\{ & n\left(\gamma^2(\theta - 2V_0) + 2\kappa(V_0 - \theta)^2\right)\left(e^{-2\kappa T} - 1\right)\frac{1 - e^{\frac{\kappa T}{n}}}{1 + e^{\frac{\kappa T}{n}}} \\ & + 2\kappa T\left(\kappa^2 T(\theta - 2r)^2 + n\theta\left(4\kappa^2 - 4\rho\kappa\gamma + \gamma^2\right)\right) \\ & + 4(V_0 - \theta)\left(n\left(2\kappa^2 + \gamma^2 - 2\rho\kappa\gamma\right) + \kappa^2 T(\theta - 2r)\right)\left(1 - e^{-\kappa T}\right) \\ & - 2n^2\theta\gamma(\gamma - 4\rho\kappa)\left(1 - e^{-\frac{\kappa T}{n}}\right) + 4(V_0 - \theta)\kappa T\gamma(\gamma - 2\rho\kappa)\frac{1 - e^{-\kappa T}}{1 - e^{\frac{\kappa T}{n}}}\Bigg\} \end{aligned} \tag{4.11}$$

The fair strike of the continuous variance swap is

$$K_c^H = \frac{1}{T}\mathbb{E}\left[\int_0^T V_s ds\right] = \theta + (1 - e^{-\kappa T})\frac{V_0 - \theta}{\kappa T}. \tag{4.12}$$

Proof. See Section 4.11 for the proof of (4.11). The formula (4.12) for the fair strike of a continuous variance swap is already well-known and can be found for example in Broadie and Jain (2008a), formula (4.3), p772. □

Proposition 4.3.1 provides an explicit formula for the fair strike of a discrete variance swap as a function of model parameters. This formula simplifies the expressions obtained by Broadie and Jain (2008a) in equations (A-29) and (A-30), p793, where several sums from 0 to n are involved and can actually be computed explicitly as shown by the expression (4.11) above. We verified that our formula agrees with numerical examples presented in Table 5 (column 'SV') on p782 of Broadie and Jain (2008a).[5]

Contrary to what is stated in the introduction of the paper by Zhu and Lian (2011), the techniques of Broadie and Jain (2008a) can easily be extended to other types of payoffs. The following proposition gives explicit expressions for the volatility derivative considered by Zhu and Lian (2011).

Proposition 4.3.2. *For the following set of dates $t_i = \frac{iT}{n}$ with $i = 0, 1, ..., n$, denote $\Delta = T/n$, and assume $\alpha = 2\kappa\theta/\gamma^2 - 1 \geqslant 0$, and $\gamma^2 T < 1$. Then the fair price of a discrete variance swap with payoff $\frac{1}{T}\sum_{i=0}^{n-1}\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2$ is equal to*

$$K_d^{zl}(n) = \frac{1}{T}\sum_{i=0}^{n-1}\mathbb{E}\left[\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2\right] = \frac{1}{T}\left(a_0 + \sum_{i=1}^{n-1} a_i\right) + \frac{n - 2ne^{r\Delta}}{T},$$

where we define $a_i = \mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right]$, for $i = 0, 1, ..., n-1$. Then for $i = 0, 1, ..., n-1$, we have

$$a_i = \frac{e^{2r\Delta}}{S_0^2} M(2, \Delta) e^{q(2)V_0\left(\frac{\eta(t_i)e^{-\kappa t_i}}{\eta(t_i) - q(2)} - 1\right)} \left(\frac{\eta(t_i)}{\eta(t_i) - q(2)}\right)^{\alpha+1},$$

where

$$M(u,t) = \mathbb{E}[e^{uX_t}] = S_0^u e^{\frac{\kappa\theta}{\gamma^2}\left((\kappa - \gamma\rho u - d(u))t - 2\ln\left(\frac{1 - g(u)e^{-d(u)t}}{1 - g(u)}\right)\right)} e^{V_0 \frac{\kappa - \gamma\rho u - d(u)}{\gamma^2}\frac{1 - e^{-d(u)t}}{1 - g(u)e^{-d(u)t}}},$$

[5] The formula has been implemented in Matlab and is available online at `http://www.runmycode.org/CompanionSite/Site135` as well as the other formulas and asymptotic expansions, which appear in this chapter.

with the following auxiliary functions

$$d(u) = \sqrt{(\kappa - \gamma\rho u)^2 + \gamma^2(u - u^2)}, \qquad g(u) = \frac{\kappa - \gamma\rho u - d(u)}{\kappa - \gamma\rho u + d(u)},$$
$$q(u) = \frac{\kappa - \gamma\rho u - d(u)}{\gamma^2}\frac{1 - e^{-d(u)\Delta}}{1 - g(u)e^{-d(u)\Delta}}, \quad \eta(u) = \frac{2\kappa}{\gamma^2}\left(1 - e^{-\kappa u}\right)^{-1}.$$

Proof. See Section 4.12. □

Remark 4.3.1. *The formula in the above Proposition 4.3.2 is consistent with the one obtained in equation (2.34) by Zhu and Lian (2011). In particular, we are able to reproduce all numerical results but one presented in Table* 3.1*, p246 of Zhu and Lian (2011) using their set of parameters:* $\kappa = 11.35$*,* $\theta = 0.022$*,* $\gamma = 0.618$*,* $\rho = -0.64$*,* $V_0 = 0.04$*,* $r = 0.1$*,* $T = 1$ *and* $S_0 = 1$ *(all numbers match except the case when* $n = 4$ *we get 263.2 instead of 267.6).*

Proposition 4.3.2 gives a formula for pricing the variance swap with payoff $\frac{1}{T}\sum_{i=0}^{n-1}\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2$*, but it is straightforward to extend its proof to the following payoff* $\frac{1}{T}\sum_{i=0}^{n-1}\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^k$*, with an arbitrary integer power* k*.*

4.4 Fair strike of the discrete variance swap in the Hull-White model

The correlated Hull-White stochastic volatility model is as follows

$$(HW) \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)}, \\ dV_t &= \mu V_t dt + \sigma V_t dW_t^{(2)}, \end{cases} \tag{4.13}$$

where $\mathbb{E}[dW_t^{(1)}dW_t^{(2)}] = \rho dt$. Referring to equation (4.1), we have $m(x) = \sqrt{x}$, $\mu(x) = \mu x$, $\sigma(x) = \sigma x$, so it is straightforward to determine $f(v) = \frac{2}{\sigma}\sqrt{v}$, $h(v) = \left(\frac{\mu}{\sigma} - \frac{\sigma}{4}\right)\sqrt{v}$, and apply (4.48) in Lemma 4.10.1 in the Section

4.10 to obtain

$$S_T = S_0 \exp\left\{ rT - \frac{1}{2}\int_0^T V_t dt + \frac{2\rho}{\sigma}(\sqrt{V_T} - \sqrt{V_0}) \right.$$
$$\left. -\rho\left(\frac{\mu}{\sigma} - \frac{\sigma}{4}\right)\int_0^T \sqrt{V_t}dt + \sqrt{1-\rho^2}\int_0^T \sqrt{V_t}dW_t^{(3)} \right\}.$$

Proposition 4.4.1. *(Fair Strikes in the Hull-White Model)*

In the Hull-White stochastic volatility model (4.13), *the fair strike* (4.2) *of the discrete variance swap is*

$$K_d^{HW}(n) = \frac{r^2T}{n} + \left(1 - \frac{rT}{n}\right)K_c^{HW} - \frac{V_0^2\left(e^{\left(2\mu+\sigma^2\right)T} - 1\right)\left(e^{\frac{\mu T}{n}} - 1\right)}{2T\mu(\mu+\sigma^2)\left(e^{\frac{(2\mu+\sigma^2)T}{n}} - 1\right)}$$
$$+ \frac{V_0^2\left(e^{\left(2\mu+\sigma^2\right)T} - 1\right)}{2T(2\mu+\sigma^2)(\mu+\sigma^2)} + \frac{8\rho\left(e^{\frac{3(4\mu+\sigma^2)T}{8}} - 1\right)V_0^{3/2}\sigma(e^{\frac{\mu T}{n}} - 1)}{\mu T\,(4\,\mu + 3\,\sigma^2)\left(e^{\frac{3(4\mu+\sigma^2)T}{8n}} - 1\right)}$$
$$- \frac{64\rho\left(e^{\frac{3(4\mu+\sigma^2)T}{8}} - 1\right)V_0^{3/2}\sigma}{3T(4\mu+\sigma^2)\,(4\,\mu+3\,\sigma^2)}. \quad (4.14)$$

The fair strike of the continuous variance swap is

$$K_c^{HW} = \frac{1}{T}\mathbb{E}\left[\int_0^T V_s ds\right] = \frac{V_0}{T\mu}(e^{\mu T} - 1). \quad (4.15)$$

Proof. The proof can be found in Section 4.13. ☐

4.5 Fair strike of the discrete variance swap in the Schöbel-Zhu model

The correlated Schöbel-Zhu stochastic volatility model (see Schöbel and Zhu (1999)) can be described by the following dynamics[6]

$$(SZ) \quad \begin{cases} \frac{dS_t}{S_t} &= rdt + V_t dW_t^{(1)}, \\ dV_t &= \kappa(\theta - V_t)dt + \gamma dW_t^{(2)}, \end{cases} \tag{4.16}$$

where $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$. Referring to equation (4.1), we have $m(x) = x, \mu(x) = -\kappa(x-\theta), \sigma(x) = \gamma$, so it is straightforward to apply (4.48) in Lemma 4.10.1 given in Section 4.10 with $f(v) = \frac{v^2}{2\gamma}$ and $h(v) = \frac{\kappa\theta}{\gamma}v - \frac{\kappa}{\gamma}v^2 + \frac{\gamma}{2}$ to obtain

$$S_T = S_0 \exp\left\{(r - \frac{\gamma\rho}{2})T - \frac{\kappa\theta\rho}{\gamma}\int_0^T V_t dt - \left(\frac{1}{2} - \frac{\rho\kappa}{\gamma}\right)\int_0^T V_t^2 dt + \frac{\rho}{2\gamma}(V_T^2 - V_0^2) + \sqrt{1-\rho^2}\int_0^T V_t dW_t^{(3)}\right\}.$$

Proposition 4.5.1. *(Fair Strikes in the Schöbel-Zhu Model)*

In the Schöbel-Zhu stochastic volatility model (4.16)*, the fair strike* (4.2) *of the discrete variance swap is computed from* (4.5) *but does not have a simple expression.*[7] *The fair strike of the continuous variance swap is*

$$K_c^{SZ} = \frac{\gamma^2}{2\kappa} + \theta^2 + \left(\frac{(V_0-\theta)^2}{2\kappa T} - \frac{\gamma^2}{4\kappa^2 T}\right)(1 - e^{-2\kappa T}) + \frac{2\theta(V_0 - \theta)}{\kappa T}(1 - e^{-\kappa T}). \tag{4.17}$$

Proof. The proof can be found in Section 4.14. □

Remark 4.5.1. *In the literature, there is an alternative method to derive*

[6]We shall note that here $m(V_t) = V_t$ (where $m(\cdot)$ is defined in (4.1)) instead of $\sqrt{V_t}$, thus the process V_t models the volatility and not the variance. In particular in the Schöbel-Zhu model, the variance process $Y_t = V_t^2$ follows $dY_t = (\gamma^2 + 2\kappa\theta\sqrt{Y_t} - 2\kappa Y_t)dt + 2\gamma\sqrt{Y_t}dW_t^{(2)}$.

[7]See Proposition 4.6.7 for an explicit expansion.

the fair strikes of discrete variance swaps. Hong (2004) first proposed to use the forward characteristic functions of the log stock returns to calculate the fair strikes. This method applies to all stock price models where we have a closed-form forward characteristic function for the log stock price. The method can be applied to affine processes (e.g. Heston, Hull-White models). Note that the Schöbel-Zhu model can be transformed to an affine model by rewriting the model in terms of the variance process $Y_t = V_t^2$*, and treat* $(S_t, Y_t, \sqrt{Y_t})$ *as state variables. Along this strand of literature, Itkin and Carr (2010) considered using it to price discrete variance swaps under time-changed Lévy processes. Crosby and Davis (2012) consider the pricing of generalized discrete variance swaps under time-changed Lévy processes.*

4.6 Asymptotics

In the time-homogeneous stochastic volatility model, this section presents asymptotics for the fair strikes of discrete variance swaps in the Heston, the Hull-White and the Schöbel-Zhu models based on the explicit expressions derived in the previous sections 4.3, 4.4 and 4.5.

The expansions as functions of the number of sampling periods n are given in Propositions 4.6.1, 4.6.4 and 4.6.7 (respectively for the Heston, Hull-White and Schöbel-Zhu models). In the Heston model, our results are consistent with Proposition 4.2 of Broadie and Jain (2008a), in which it is proved that $K_d^H(n) = K_c^H + \mathcal{O}\left(\frac{1}{n}\right)$. The expansion below is more precise in that at least the first leading term in the expansion is given explicitly. See also Theorem 3.8 of Jarrow et al. (2013) in a more general context. In particular, Jarrow et al. (2013) give a sufficient condition for the convergence of the fair strike of a discrete variance swap to that of a continuously monitored variance swap. In our setting, which is in the absence of jumps, their sufficient condition reduces to $E[\int_0^T m^4(V_s)ds] < \infty$. This latter condition is obviously satisfied in the three examples considered in this chapter (the Heston, the Hull-White and the Schöbel-Zhu models).

Expansions as a function of the maturity T (for small maturities) are also given in order to complement results of Keller-Ressel and Muhle-Karbe (2012) (see for example Corollary 2.7 which gives qualitative properties of the discretization gap[8] as the maturity $T \to 0$).

4.6.1 Heston Model

We first expand the fair strike of the discrete variance swap with respect to the number of sampling periods n.

Proposition 4.6.1. *(Expansion of the fair strike $K_d^H(n)$ w.r.t. n)*

In the Heston model, the expansion of the fair strike of a discrete variance swap, $K_d^H(n)$, is given by

$$K_d^H(n) = K_c^H + \frac{a_1^H}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{4.18}$$

where

$$a_1^H = r^2 T - rTK_c^H + \left(\frac{\gamma(\theta - V_0)}{2\kappa}(1 - e^{-\kappa T}) - \frac{\theta\gamma T}{2}\right)\rho + \left(\frac{\theta^2}{4} + \frac{\theta\gamma^2}{8\kappa}\right)T + c_1,$$

with

$$c_1^H = \frac{\left[\gamma^2\theta - 2\kappa(V_0 - \theta)^2\right]\left(e^{-2T\kappa} - 1\right) + 2(V_0 - \theta)(e^{-T\kappa} - 1)\left[\gamma^2(e^{-T\kappa} - 1) - 4\kappa\theta\right]}{16\kappa^2}.$$

Proof. This proposition is a straightforward expansion from (4.11) in Proposition 4.3.1. ☐

We know that $K_d^H(n) = b^H(n) + \frac{T}{n} r(r - K_c^H)$ from (4.6) in Remark 4.2.1. It is thus clear that a_1^H contains all the terms in the risk-free rate r and thus that all the higher terms in the expansion (4.18) with respect to n are independent of the risk-free rate.

Remark 4.6.1. *The first term in the expansion* (4.18)*, a_1^H, is a linear function of ρ. Observe that the coefficient in front of ρ, $\frac{\gamma(\theta - V_0)}{2\kappa}(1 - e^{-\kappa T}) -$*

[8] See Definition 2.6 on p112 of Keller-Ressel and Muhle-Karbe (2012).

$\frac{\theta\gamma T}{2}$ *is negative,*[9] *so that* a_1^H *is always a decreasing function of* ρ*. We have that*

$$a_1^H \geqslant 0 \quad \Longleftrightarrow \quad \rho \leqslant \rho_0^H$$

where $\rho_0^H = \frac{r^2T - rTK_c^H + \left(\frac{\theta^2}{4} + \frac{\theta\gamma^2}{8\kappa}\right)T + c_1^H}{-\left(\frac{\gamma(\theta - V_0)}{2\kappa}(1 - e^{-\kappa T}) - \frac{\theta\gamma T}{2}\right)}$.

Proposition 4.6.2. *(Expansion of the fair strike for small maturity)*

In the Heston model, $K_d^H(n)$ *can be expanded when* $T \to 0$ *as*

$$K_d^H(n) = V_0 + b_1^H T + b_2^H T^2 + \mathcal{O}\left(T^3\right) \tag{4.19}$$

where

$$\begin{aligned}
b_1^H &= \frac{\kappa(\theta - V_0)}{2} + \frac{1}{4n}\left((V_0 - 2r)^2 - 2\rho V_0\gamma\right) \\
b_2^H &= \frac{\kappa^2(V_0 - \theta)}{6} + \frac{(V_0 - \theta)\kappa(\gamma\rho + 2r - V_0) + \frac{\gamma^2 V_0}{2}}{4n} + \frac{\gamma\rho\kappa(V_0 + \theta) - \frac{\gamma^2 V_0}{2}}{12n^2}.
\end{aligned}$$

Note also that $K_c^H = V_0 + \frac{\kappa}{2}\left(\theta - V_0\right)T + \frac{\kappa}{6}^2\left(V_0 - \theta\right)T^2 + \mathcal{O}\left(T^3\right)$ *and thus*

$$K_d^H(n) - K_c^H = \frac{1}{4n}\left((V_0 - 2r)^2 - 2\rho V_0\gamma\right)T + \mathcal{O}(T^2).$$

Proof. This proposition is a straightforward expansion from (4.11) in Proposition 4.3.1. □

Proposition 4.6.2 is consistent with Corollary 2.7 [b] on p113 of Keller-Ressel and Muhle-Karbe (2012), where it is clear that the limit of $K_d(n) - K_c$ is 0 when $T \to 0$.

Notice that in the case $\rho \leqslant 0$, in the Heston model, $K_d^H(n)$ is non-negative and decreasing in n as the maturity T goes to 0. However, this property cannot be generalized to all correlation levels as it depends on the sign of $(V_0 - 2r)^2 - 2\gamma V_0\rho$.

Proposition 4.6.3. *(Expression of the fair strike w.r.t.* γ*)*

[9] This can be easily seen from the fact that for all $x > 0$, $(\theta - V_0)(1 - e^{-x}) - \theta x \leqslant \theta(1 - e^{-x} - x) < 0$, and note that here $x = \kappa T > 0$.

In the Heston model, $K_d^H(n)$ is a quadratic function of γ:

$$K_d^H(n) = \frac{1}{8n\kappa^3 T}\left(h_0^H + h_1^H\gamma + h_2^H\gamma^2\right), \tag{4.20}$$

where

$$\begin{aligned} h_0^H &= 2\,n\kappa\,(V_0-\theta)^2\left(e^{-2\,\kappa\,T}-1\right)\frac{1-e^{\frac{\kappa\,T}{n}}}{1+e^{\frac{\kappa\,T}{n}}} + 2\,\kappa\,T\left(\kappa^2 T\,(\theta-2\,r)^2 + 4\,\kappa^2 n\theta\right) \\ &\quad + 4\,(V_0-\theta)\left(2\,\kappa^2 n + \kappa^2 T\,(\theta-2\,r)\right)\left(1-e^{-\kappa\,T}\right), \end{aligned}$$

$$h_1^H = 8\rho\kappa\left(n\theta(n-ne^{-\frac{\kappa T}{n}}-\kappa T) - (V_0-\theta)\left(n\left(1-e^{-\kappa\,T}\right) + \kappa T\frac{1-e^{-\kappa\,T}}{1-e^{\frac{\kappa\,T}{n}}}\right)\right),$$

$$\begin{aligned} h_2^H &= n\,(\theta-2\,V_0)\left(e^{-2\,\kappa\,T}-1\right)\frac{1-e^{\frac{\kappa\,T}{n}}}{1+e^{\frac{\kappa\,T}{n}}} - 2\,n^2\theta\left(1-e^{-\frac{\kappa\,T}{n}}\right) \\ &\quad + 4\,(V_0-\theta)\left(n-ne^{-\kappa\,T} + \kappa\,T\frac{1-e^{-\kappa\,T}}{1-e^{\frac{\kappa\,T}{n}}}\right) + 2\,\kappa\,Tn\theta. \end{aligned}$$

Proposition 4.6.3 shows that the discrete fair strike in the Heston model is a quadratic function of the volatility of variance γ. From Figure 4.6, we observe that the discrete fair strikes evolve in a parabolic shape as γ varies.

4.6.2 Hull-White Model

Proposition 4.6.4. *(Expansion of $K_d^{HW}(n)$ w.r.t. n)*

In the Hull-White model, the expansion of the fair strike of the discrete variance swap, $K_d^{HW}(n)$, is given by

$$K_d^{HW}(n) = K_c^{HW} + \frac{a_1^{HW}}{n} + \frac{a_2^{HW}}{n^2} + \frac{a_3^{HW}}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right) \tag{4.21}$$

where

$$a_1^{HW} = r^2T - rTK_c^{HW} + \frac{V_0^2}{4}\frac{e^{(2\mu+\sigma^2)T}-1}{2\mu+\sigma^2} - \frac{4\rho\sigma V_0^{\frac{3}{2}}}{3}\frac{e^{\frac{3}{8}(4\mu+\sigma^2)T}-1}{4\mu+\sigma^2},$$

$$a_2^{HW} = -\frac{V_0^2\sigma^2 T}{24}\frac{e^{(2\mu+\sigma^2)T}-1}{2\mu+\sigma^2} - \frac{\rho V_0^{\frac{3}{2}}\sigma T(4\mu-3\sigma^2)}{36}\frac{e^{\frac{3}{8}(4\mu+\sigma^2)T}-1}{4\mu+\sigma^2},$$

$$a_3^{HW} = -\frac{\mu T^2 V_0^2(\mu+\sigma^2)}{48}\frac{e^{(2\mu+\sigma^2)T}-1}{2\mu+\sigma^2} + \frac{\mu T^2\rho\sigma V_0^{\frac{3}{2}}(4\mu+3\sigma^2)}{72}\frac{e^{\frac{3}{8}(4\mu+\sigma^2)T}-1}{4\mu+\sigma^2}.$$

Proof. This proposition is a straightforward expansion from (4.14) in Proposition 4.4.1. □

Observe that $K_d^{HW}(n) = b^{HW}(n) - \frac{K_c^{HW}T}{n}r + \frac{T}{n}r^2$ where $b^{HW}(n) = K_d^{HW}(r=0) > K_c^{HW}$ is independent of r.

If we neglect higher order terms in the expansion (4.21), we observe that the position of the fair strike of the discrete variance swap with respect to the fair strike of the continuous variance swap is driven by the sign of a_1^{HW} and we have the following observation.

Remark 4.6.2. *The first term in the expansion* (4.21)*,* a_1^{HW}*, is a linear function of* ρ*.*

$$a_1^{HW} \geqslant 0 \quad \Longleftrightarrow \quad \rho \leqslant \rho_0^{HW}$$

where $\rho_0^{HW} = \frac{3(4\mu+\sigma^2)\left(r^2T - rTK_c^{HW} + \frac{V_0^2}{4}\frac{e^{(2\mu+\sigma^2)T}-1}{2\mu+\sigma^2}\right)}{4\sigma V_0^{\frac{3}{2}}(e^{\frac{3}{8}(4\mu+\sigma^2)T}-1)} > 0.$

ρ_0^{HW} can take values strictly larger than 1 as it appears clearly in the right panel of Figure 4.4. In this latter case, the fair strike of the discrete variance swap is larger than the fair strike of the continuous variance swap for all levels of correlation and for sufficiently high values of n. The minimum value of $K_d^{HW}(n)$ as a function of r is obtained when $r = r^* = \frac{K_c^{HW}}{2}$. After replacing r by r^* in the expression of ρ_0^{HW}, ρ_0^{HW} can easily be shown to be positive[10].

[10] It reduces to studying the sign of $\frac{e^{(2\mu+\sigma^2)T}-1}{(2\mu+\sigma^2)T} - \frac{(e^{\mu T}-1)^2}{\mu^2T^2}$. It is an increasing function of σ, so it is larger than $\frac{e^{2\mu T}-1}{2\mu T} - \frac{(e^{\mu T}-1)^2}{\mu^2T^2}$, which is always positive because its minimum is 0 obtained when $\mu T = 0$.

Proposition 4.6.5. *(Expansion of $K_d^{HW}(n)$ for small maturity)*

In the Hull-White model, $K_d^{HW}(n)$ can be expanded when $T \to 0$ as

$$K_d^{HW}(n) = V_0 + b_1^{HW} T + b_2^{HW} T^2 + \mathcal{O}\left(T^3\right), \tag{4.22}$$

where

$$\begin{aligned}
b_1^{HW} &= \frac{V_0\,\mu}{2} + \frac{1}{4n}\left((V_0 - 2r)^2 - 2\rho {V_0}^{3/2}\sigma\right),\\
b_2^{HW} &= \frac{V_0\mu^2}{6} + \frac{V_0}{4n}\left(\frac{\sigma^2 V_0}{2} - \frac{3\rho\, {V_0}^{1/2}\sigma(\sigma^2+4\mu)}{8} + \mu(V_0 - 2r)\right)\\
&\quad + \frac{{V_0}^{3/2}\sigma\left(\rho(3\sigma^2 - 4\mu) - 4\sigma\sqrt{V_0}\right)}{96n^2}.
\end{aligned}$$

Note also that $K_c^{HW} = V_0 + \frac{V_0\mu}{2}T + \frac{V_0\mu^2}{6}T^2 + \mathcal{O}\left(T^3\right)$, and thus

$$K_d^{HW}(n) - K_c^{HW} = \frac{1}{4n}\left((V_0 - 2r)^2 - 2\rho {V_0}^{3/2}\sigma\right) T + \mathcal{O}(T^2).$$

Proof. This proposition is a straightforward expansion from (4.14) in Proposition 4.4.1. □

Note that the expansion for small maturities in the Hull White model is similar to the one in the Heston model given in Proposition 4.6.2.

Proposition 4.6.6. *(Expansion of $K_d^{HW}(n)$ w.r.t. σ)*

In the Hull-White model, the fair strike of a discrete variance swap, $K_d^{HW}(n)$, verifies

$$K_d^{HW}(n) = h_0^{HW} + h_1^{HW}\sigma + \mathcal{O}(\sigma^2), \tag{4.23}$$

where

$$\begin{aligned}
h_0^{HW} &= \frac{r^2 T}{n} + \left(1 - \frac{rT}{n}\right) V_0\,\frac{e^{T\mu} - 1}{T\mu} - \frac{{V_0}^2}{2}\,\frac{e^{2\,T\mu} - 1}{e^{2\,\frac{T\mu}{n}} - 1}\,\frac{e^{\frac{T\mu}{n}} - 1}{T\mu^2} + \frac{{V_0}^2\left(e^{2\,T\mu} - 1\right)}{4T\mu^2},\\
h_1^{HW} &= 2\rho\frac{e^{3/2\,T\mu} - 1}{e^{3/2\frac{T\mu}{n}} - 1}{V_0}^{3/2}\,\frac{e^{\frac{T\mu}{n}} - 1}{T\mu^2} - \frac{4\rho\left(e^{3/2T\mu} - 1\right){V_0}^{3/2}}{3T\mu^2}.
\end{aligned}$$

The expansion of the fair strike in the Hull-White model with respect to the volatility of volatility is very different from the one in the Heston model as it is not a quadratic function of σ, and it also involves higher order terms of σ.

4.6.3 Schöbel-Zhu Model

We first expand the fair strike of the discrete variance swap with respect to the number of sampling periods n. The following result is similar to Proposition 4.6.1 and 4.6.4. In particular we find that the first term in the expansion is also linear in ρ and has a similar behaviour as in the Heston and Hull-White model.

Proposition 4.6.7. *(Expansion of $K_d^{SZ}(n)$ w.r.t. n)*

In the Schöbel-Zhu model, the expansion of the fair strike of the discrete variance swap, $K_d^{SZ}(n)$, is given by

$$K_d^{SZ}(n) = K_c^{SZ} + \frac{a_1^{SZ}}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{4.24}$$

where

$$a_1^{SZ} = r^2 T - rTK_c^{SZ} + d_1 - d_2 \frac{\gamma}{2\kappa}\rho, \tag{4.25}$$

with

$$\begin{aligned} d_1 := {} & \frac{T{V_0}^4}{4} - \frac{E(T+D)}{16\kappa^2} + \left(\frac{3{V_0}^2\gamma^2}{4} + \frac{E}{32\kappa} + \frac{\kappa {V_0}^3(\theta - V_0)}{2}\right) D^2 \\ & + \left(\frac{2\theta\,\kappa^2 {V_0}^3}{3} - \frac{{V_0}^4\kappa^2}{6} - \frac{E}{48} - \frac{{V_0}^2\theta^2\kappa^2}{2} - \gamma^2\kappa\, V_0\theta + \frac{3{V_0}^2\kappa\,\gamma^2}{4} - \frac{\gamma^4}{4}\right) D^3 \\ & + \left(\frac{E}{8\kappa} + 3\gamma^2(\theta - V_0)\theta + \frac{3{V_0}^2\gamma^2}{2} + V_0\kappa(\theta - V_0)\left(2\theta^2 - \theta V_0 + V_0^2\right)\right) \frac{\kappa^2 D^4}{8}, \end{aligned}$$

and

$$d_2 = T\left(\gamma^2 + 2\,\kappa\,\theta^2\right) + \left(2\kappa(\theta^2 - V_0^2) + \gamma^2\right) D + \frac{\kappa}{2}\,\left(\gamma^2 - 2\,\kappa\,(\theta - V_0)^2\right) D^2,$$

where

$$E := 4\,{V_0}^4\kappa^2 - 4\,\theta^4\kappa^2 - 3\,\gamma^4 - 12\,\gamma^2\theta^2\kappa, \qquad D := \frac{e^{-\kappa T} - 1}{\kappa}.$$

Proof. This proposition is a straightforward expansion from the formula of $K_d^{SZ}(n)$ in Proposition 4.5.1. Note that although the formula of $K_d^{SZ}(n)$ does not have a simple form, its asymptotic expansion can be easily computed with Maple for instance. □

Remark 4.6.3. *Similarly as in the Heston and the Hull-White models, the first term in the expansion* (4.24), a_1^{SZ}, *is a linear function of* ρ, *but the sign of its slope is not clear in general.*

Proposition 4.6.8. *(Expansion of the fair strike for small maturity)*

In the Schöbel-Zhu model, $K_d^{SZ}(n)$ can be expanded when $T \to 0$ as

$$K_d^{SZ}(n) = V_0^2 + b_1^{SZ} T + \mathcal{O}(T^2) \tag{4.26}$$

where

$$b_1^{SZ} = \kappa V_0(\theta - V_0) + \frac{\gamma^2}{2} + \frac{1}{n}\left(r^2 - rV_0^2 + \frac{V_0^2(V_0^2 - 4\rho\gamma)}{4}\right).$$

Note also that $K_c^{SZ} = {V_0}^2 + \left(V_0\kappa(\theta - V_0) + \frac{\gamma^2}{2}\right) T + \mathcal{O}\left(T^2\right)$ *and thus,*

$$K_d^{SZ}(n) - K_c^{SZ} = \frac{1}{4n}\left((V_0^2 - 2r)^2 - 4\rho V_0^2\gamma\right) T + \mathcal{O}(T^2).$$

Proof. This proposition is a straightforward expansion from the formula of $K_d^{SZ}(n)$ in Proposition 4.5.1. □

Note that the form of the expansion is similar for the three models under study (compare Propositions 4.6.2, 4.6.5 and 4.6.8). We find that the difference between the discrete and the continuous strikes has a first term involving the product of 2ρ by a function of the initial variance value and the volatility of the variance process, and respectively γ in the Heston, σ in the Hull-White and 2γ in the Schöbel-Zhu model. See for example footnote 6 where the dynamics of the variance is derived in the Schöbel-Zhu model.

4.6.4 Discussion on the convex-order conjecture

As motivated in Keller-Ressel and Griessler (2012), it is of interest to study the *systematic bias* for fixed n and T when using the quadratic variation to approximate the realized variance. Bühler (2006) and Keller-Ressel and Muhle-Karbe (2012) show numerical evidence of this bias (see also Section 4.7 for further evidence in the Heston and the Hull-White models). Keller-Ressel and Griessler (2012) propose the following "**convex-order**

conjecture":

$$\mathbb{E}[f(RV(X,\mathcal{P}))] \geqslant \mathbb{E}[f([X,X]_T)]$$

where f is convex, $\mathcal{P}$ refers to the partition of $[0,T]$ in $n+1$ division points and $X = \log(S_T/S_0)$. $RV(X,\mathcal{P})$ is the discrete realized variance $(\sum_{i=1}^n (\log(S_{t_i}/S_{t_{i-1}}))^2)$ and $[X,X]_T$ is the continuous quadratic variation $(\int_0^T m^2(V_s)ds$ in our setting).

When $f(x) = x/T$ and the correlation can be positive, the conjecture is violated, see for example Figure 4.1 to 4.4 where $K_d^M(n)$ can be below K_c^M. When $\rho = 0$, the process has conditionally independent increments and satisfies other assumptions in Keller-Ressel and Griessler (2012). Proposition 4.2.2 ensures that $K_d^M(n) \geqslant K_c^M$, which is consistent with their results.

4.7 Numerics

This section illustrates with numerical examples in the Heston, the Hull-White and the Schöbel-Zhu models.

4.7.1 Heston and Hull-White models

Given parameters for the Heston model, we then choose the parameters in the Hull-White model so that the continuous strikes match. Precisely, we obtain μ by solving numerically $K_c^H = K_c^{HW}$, and find σ such that the variances of V_T in the respective Heston and the Hull-White models match. From (4.54) and (4.55), the variance for V_T for the Heston model is given by

$$Var^H(V_T) = \frac{\gamma^2}{2\kappa}(\theta + 2e^{-\kappa T}(V_0 - \theta) + e^{-2\kappa T}(\theta - 2V_0)).$$

The variance for V_T for the Hull-White model can be computed using (4.62)

$$Var^{HW}(V_T) = V_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1).$$

The parameters for the Heston model are taken from reasonable parameter sets in the literature. Precisely the first set of parameters is similar to the one used by Broadie and Jain (2008a). The second set corresponds to Table 2 in Broadie and Kaya (2006). The values for the parameters of the Hull-White model are obtained consistently using the procedure described above[11].

					Heston			(matched) Hull-White	
	T	r	V_0	ρ	γ	θ	κ	μ	σ
Set 1	1	3.19%	0.010201	-0.7	0.31	0.019	6.21	1.003	0.42
Set 2	5	5%	0.09	-0.3	1	0.09	2	2.9×10^{-9}	0.52

Table 4.1: Parameter sets

Figure 4.1 displays cases when the fair strike of the discrete variance swap $K_d^M(n)$ may be smaller than the fair strike of the continuous variance swap K_c^M. The first graph obtained in the Heston model (the model M is denoted by the exponent H for Heston) shows that K_d^H is first higher than K_c^H, crosses this level and stays below K_c^H until it converges to the value K_c^H as $n \to \infty$. It means that options on discrete realized variance may be overvalued when the continuous quadratic variation is used to approximate the discrete realized variance. Note that this unusual pattern happens when $\rho = 0.7$, which may happen for example in foreign exchange markets.

[11]For the two sets of parameters above, we compute the critical interest rate r^* as defined in Remark 4.2.1. Set 1: $r^* = 0.88\%$; Set 2: $r^* = 0.605\%$, and we can see that the interest rates are both larger than r^*.

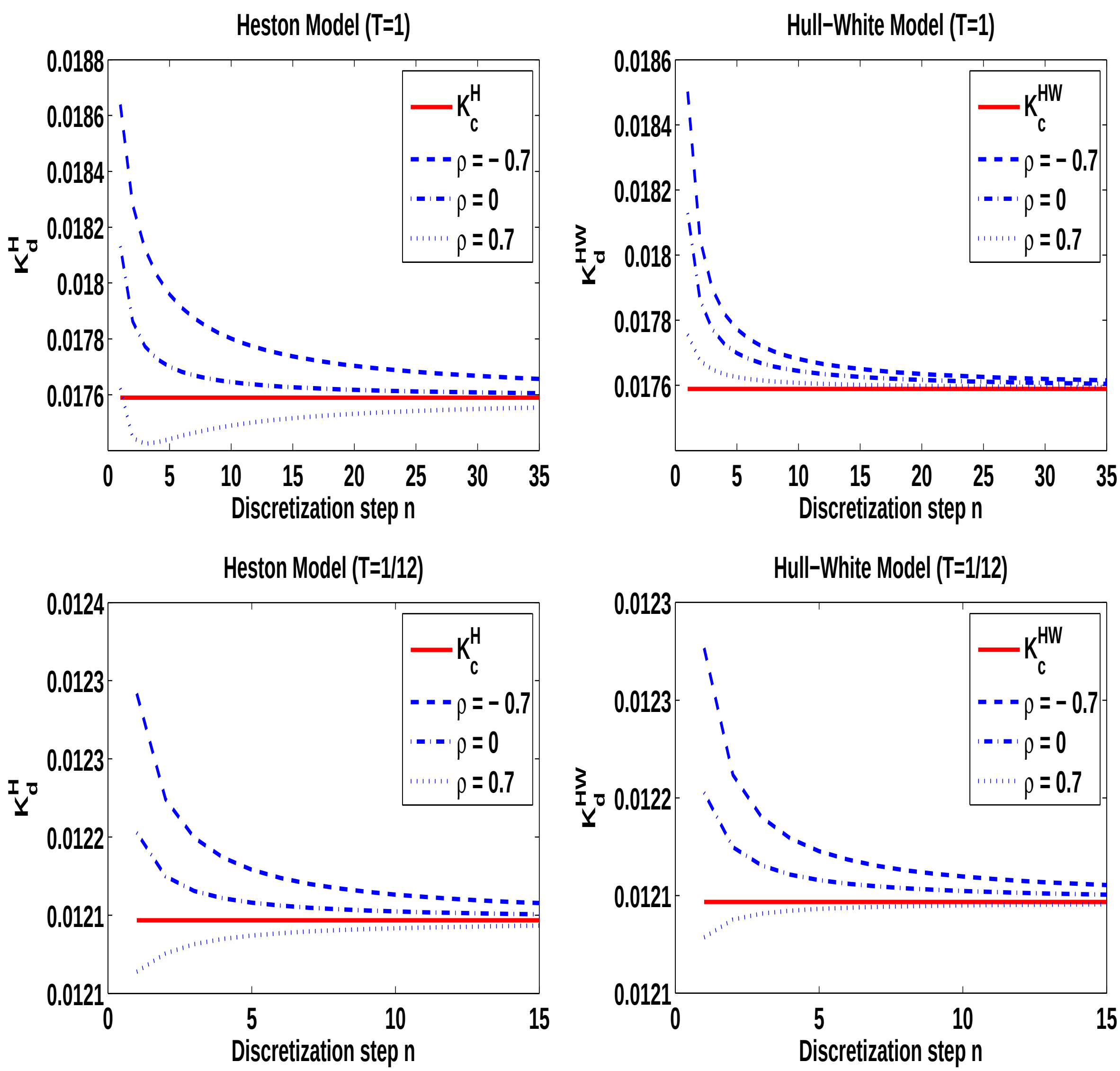


Figure 4.1: Sensitivity to the number of sampling periods n and to ρ

Parameters correspond to Set 1 in Table 4.1 except for ρ that can take three possible values $\rho = -0.7$, $\rho = 0$ or $\rho = 0.7$ and for T that is equal to $T = 1$ for the two upper graphs and $T = 1/12$ for the two lower graphs. When $T = 1/12$, the parameters for the Hull-White model are adjusted according to the procedure described in Section 4.7.1. In the case when $T = 1/12$, one has $\mu = 4.03$ and $\sigma = 1.78$.

Figure 4.2 highlights another type of convergence showing the complexity of the behaviour of the fair strike of the discrete variance swap with respect to that of the continuous variance swap.

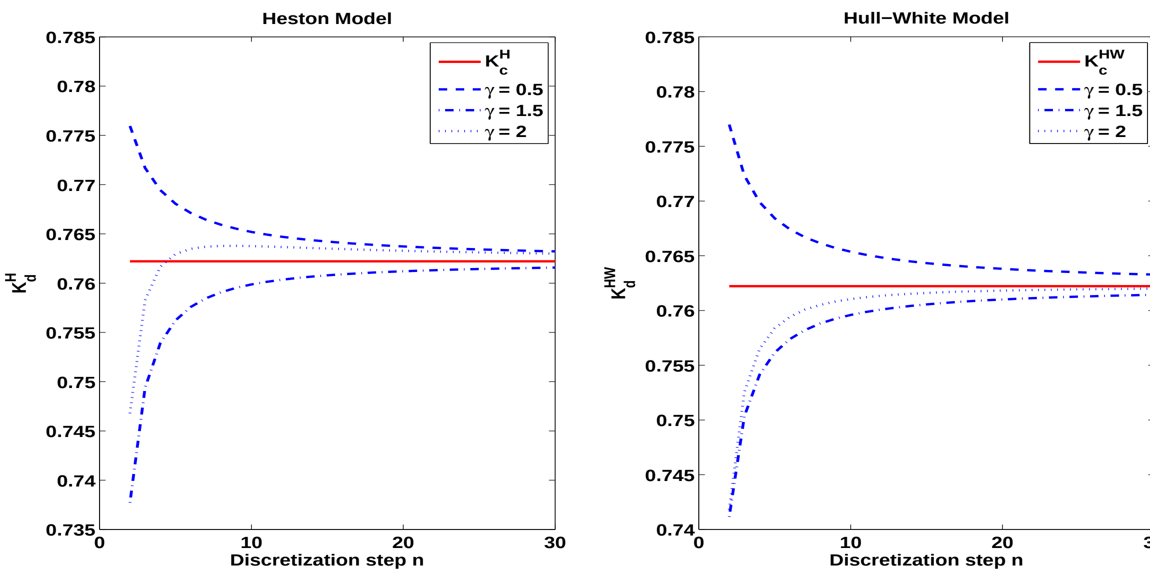


Figure 4.2: Sensitivity to the number of sampling periods n and to γ

Parameters are set to unusual values to show that any types of behaviors can be expected. $\rho = 0.6$, $r = 3.19\%$, $\theta = 0.019$, $\kappa = .1$, $V_0 = 0.8$ and γ takes three possible values: 0.5, 1.5 and 2.

Figure 4.3 displays on the same graphs the discrete fair strike $K_d(n)$ and the first two terms of the expansion formula $K_c^H + \frac{a_1^H}{n}$ for the Heston model and $K_c^{HW} + \frac{a_1^{HW}}{n}$ for the Hull-White model (see Propositions 4.6.1 and 4.6.4 for the exact expressions of a_1^H and a_1^{HW}). It shows that the first term of this expansion is already highly informative as it clearly appears to fit very well for small values of n in both models.

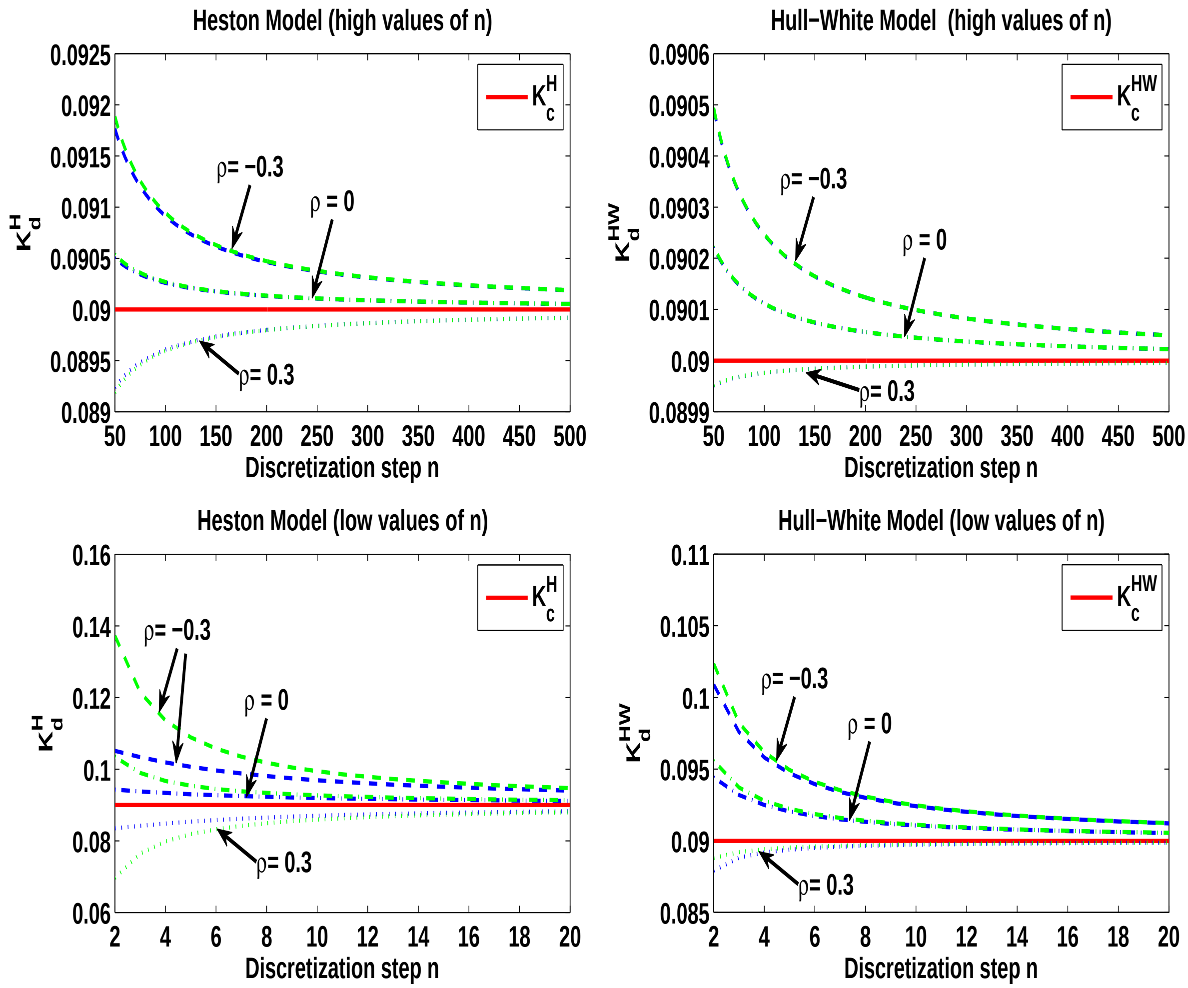


Figure 4.3: Asymptotic expansion $K_c^M + a/n$ with respect to the number of sampling periods n and to ρ

Parameters correspond to Set 2 in Table 4.1 except for ρ that can take three possible values $\rho = -0.3$, $\rho = 0$ or $\rho = 0.3$. The upper graphs correspond to large number of discretization steps whereas lower graphs have relatively small values of n.

Figure 4.4 further illustrates that the discrete fair strike (for a daily monitoring) can be lower than the continuous fair strike as $K_d^M - K_c^M$ may be negative for high values of the correlation coefficient both in the Heston and the Hull-White models. In Remark 4.6.1 and 4.6.2, it is noted that the first term in the asymptotic expansion with respect to n is linear in ρ.

From Figure 4.3 it is clear that the first term has an important explanatory power. This justifies the linear behavior observed in Figure 4.4 of the difference between discrete and continuous fair strikes with respect to ρ. Computations of ρ_0^H and ρ_0^{HW} for each of the risk-free rate levels $r = 0\%$, $r = 3.2\%$ and $r = 6\%$ confirm that it is always positive when $r = 0\%$ (which is consistent with Proposition 4.2.2) and that it can be higher than 1, which ensures that for n sufficiently high, the discrete fair strike is always higher than the continuous fair strike.

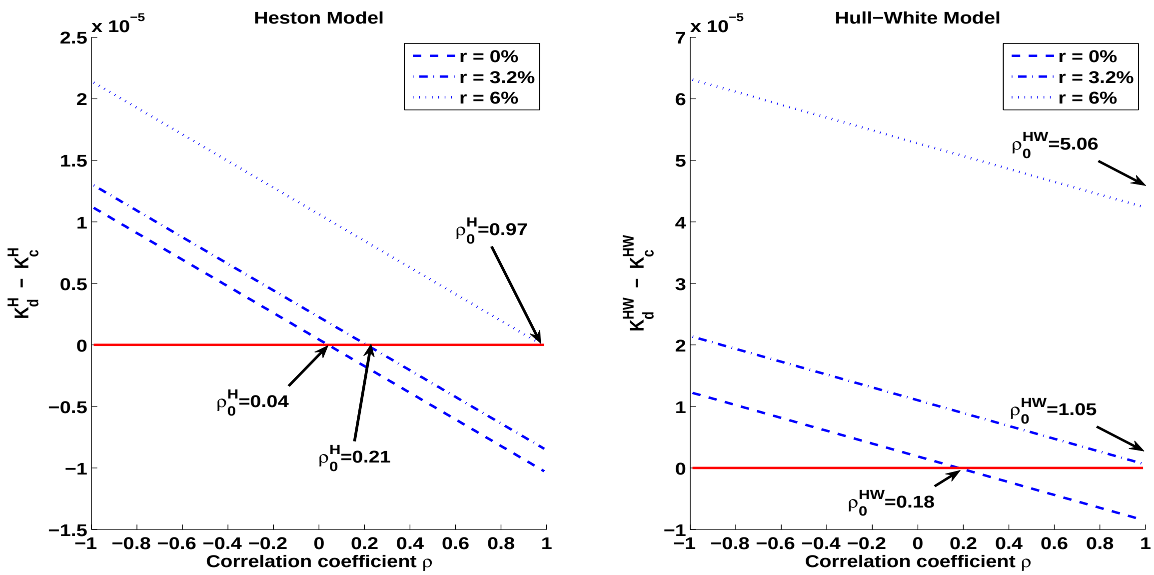


Figure 4.4: Asymptotic expansion with respect to the correlation coefficient ρ and the risk-free rate r

Parameters correspond to Set 1 in Table 4.1 except for r that can take three possible values $r = 0\%$, $r = 3.2\%$ or $r = 6\%$. Here $n = 250$, which corresponds to a daily monitoring as $T = 1$.

Figure 4.5 shows that as the time to maturity T goes to 0, the discrete fair strike is converging to the continuous fair strike at approximately a quadratic rate. This is consistent with Proposition 4.6.2 and Proposition 4.6.5.

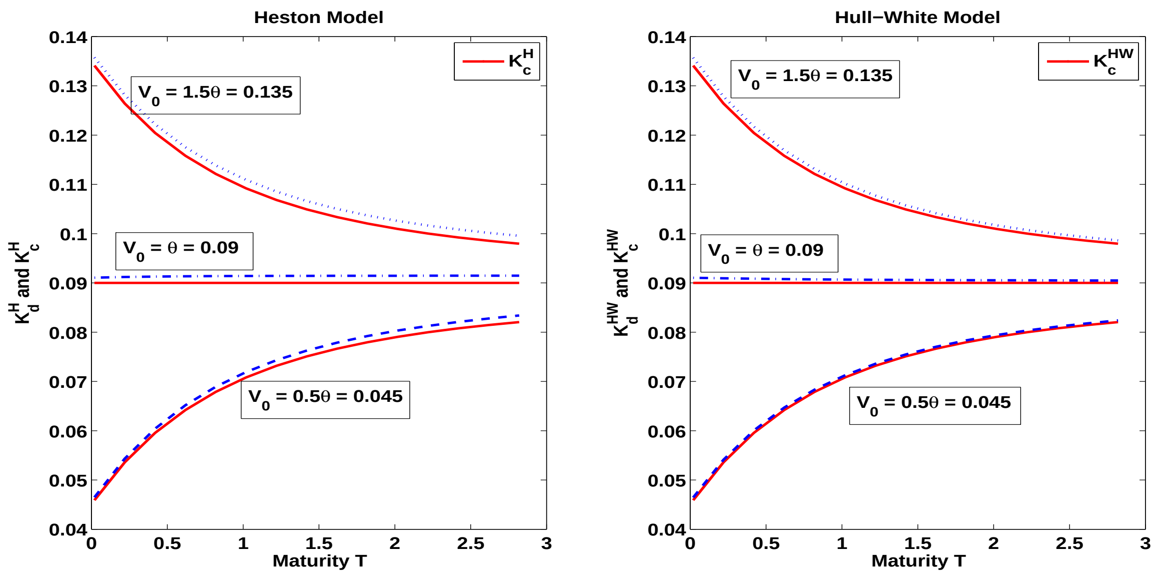


Figure 4.5: Discrete and continuous fair strikes with respect to the maturity T and to V_0

Parameters correspond to Set 2 in Table 4.1 except for T and V_0. Also we choose a monthly monitoring to compute the discrete fair strike. When $\theta = V_0$, K_c^H is independent of the maturity T.

Figure 4.6 shows that the discrepancy between the discrete fair strike and the continuous fair strike is exacerbated by the volatility of the underlying variance process. We observe that the gap between the discrete fair strike and the continuous fair strike, with respect to γ, is wider in the Heston model than in the Hull-White model. This illustrates, from a numerical viewpoint, that the discrete fair strike in the Heston model is more sensitive to the volatility of variance parameter than that of the Hull-White model. In particular, the continuous fair strike K_c^H is independent of γ. For each γ we compute the corresponding σ for the Hull-White model such that the variances match as described in Section 4.7.1. We then observe similar patterns in the Heston and the Hull-White models. From the left panel of Figure 4.6, we can see that the shape of the discrete fair strike in the Heston model with respect to γ evolves similar to a parabola, and this is consistent with Proposition 4.6.3. From the right panel of Figure 4.6, we can see that the discrete fair strike in the Hull-White model does not exhibit a parabolic shape with respect to γ, and this is explained by Proposition 4.6.6, which states that it is a higher order polynomial of σ.

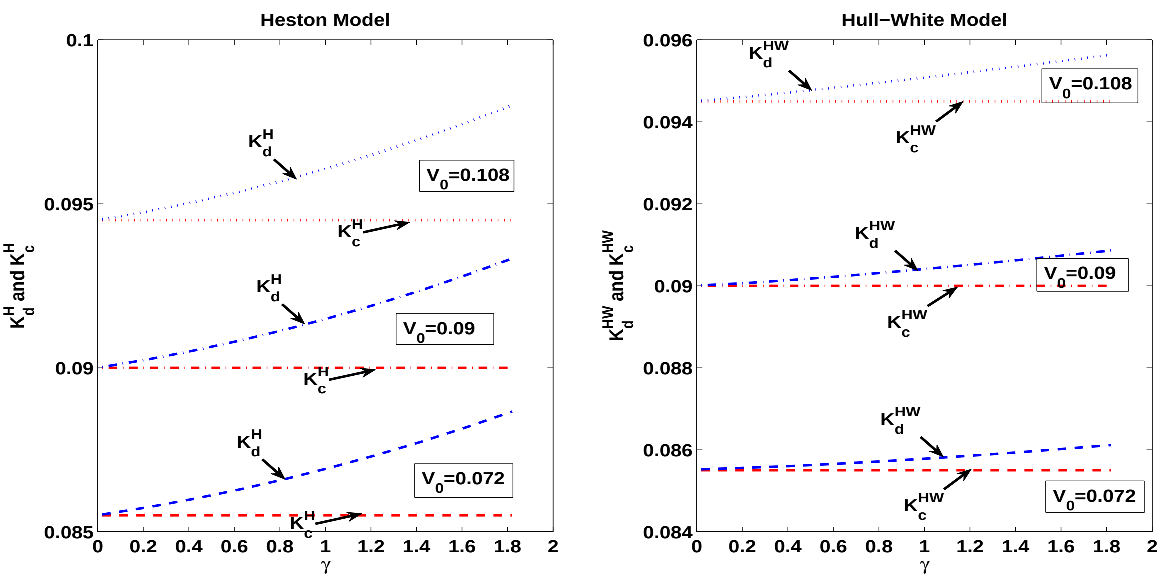


Figure 4.6: Discrete and continuous fair strikes with respect to the parameter γ and to V_0

Parameters correspond to Set 2 in Table 4.1 except for γ and V_0 that are indicated on the graphs. A monthly monitoring is used to compute the discrete fair strike. The continuous fair strike K_c^H is independent of γ, so that it is easy to identify the different curves on the graph.

4.7.2 Schöbel-Zhu model

For the Schöbel-Zhu model, we reproduce a similar numerical analysis and take parameters consistent with the Heston model. Note that the V process in the Schöbel-Zhu model corresponds to the volatility process instead of the variance process[12]. Then we choose $\theta = \sqrt{0.019}$ and $V_0 = \sqrt{0.010201}$. Other parameters are taken from set 1 of Table 4.1.

Both the left and right panels of Figure 4.7 show that K_d^{SZ} can be below K_c^{SZ} until it converges to the value K_c^{SZ} as $n \to \infty$. This unusual pattern

[12]The notation V_t in the Schöbel-Zhu model corresponds to the square root of what is denoted by V_t in the Heston model.

happens when the correlation is positive similarly in the Heston and the Hull-White models.

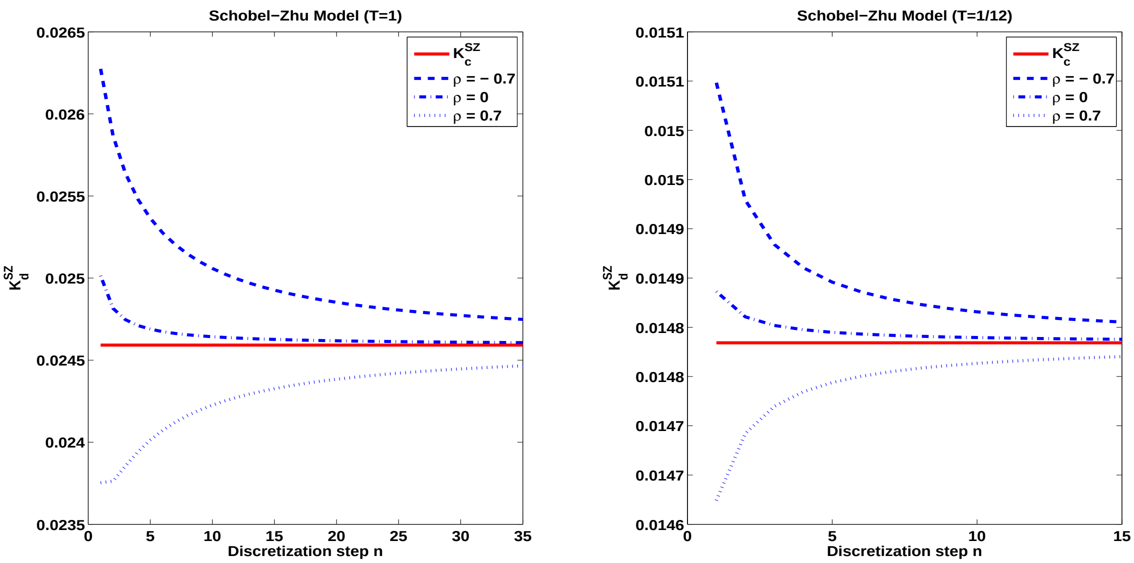


Figure 4.7: Sensitivity to the number of sampling periods n and to ρ

Parameters are similar to Set 1 in Table 4.1 for the Heston model except for ρ that can take three possible values $\rho = -0.7$, $\rho = 0$ or $\rho = 0.7$ and for T that is equal to $T = 1$ for the left panel and $T = 1/12$ for the right panel. Precisely, we use the following parameters for the Schöbel-Zhu model. $\kappa = 6.21$, $\theta = \sqrt{0.019}$, $\gamma = 0.31$, $r = 0.0319$, $V_0 = \sqrt{0.010201}$.

Figure 4.8 illustrates that the discrete fair strike (for a daily monitoring) can be lower than the continuous fair strike as $K_d^{SZ} - K_c^{SZ}$ may be negative for high values of the correlation coefficient. From Figure 4.8 it is clear that the first term also has an important explanatory power. This justifies the linear behavior observed in Figure 4.8 of the difference between discrete and continuous fair strikes with respect to ρ. Computations of ρ_0^{SZ} (defined as the zero of a_1^{SZ} computed in Proposition 4.6.7) for each of the risk-free rate levels $r = 0\%$, $r = 3.2\%$ and $r = 6\%$ confirm that it is always positive when $r = 0\%$ (which is consistent with Proposition 4.2.2).

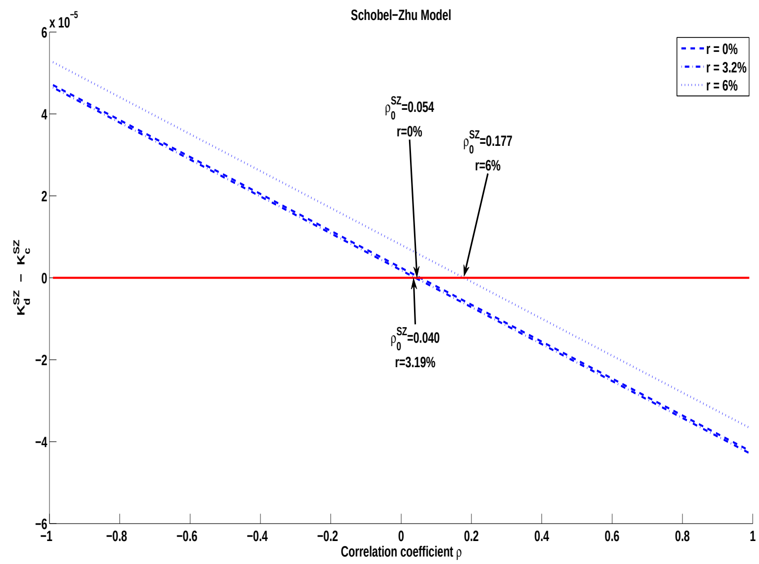


Figure 4.8: Asymptotic expansion with respect to the correlation coefficient ρ and the risk-free rate r.

Parameters are similar to Set 1 in Table 4.1 for the Heston model except for r that can take three possible values $r = 0\%$, $r = 3.2\%$ or $r = 6\%$. Precisely, we use the following parameters for the Schöbel-Zhu model: $\kappa = 6.21$, $\theta = \sqrt{0.019}$, $\gamma = 0.31$, $\rho = -0.7$, $T = 1$, $V_0 = \sqrt{0.010201}$. Here $n = 250$, which corresponds to a daily monitoring as $T = 1$.

4.8 Fair strike of the special discrete gamma swap in the Heston model

In this section we give a closed-form formula for the fair strike of a special discrete gamma swap in the Heston model.

Let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of the time interval $[0, T]$ into n equal segments: $t_i = iT/n$, for $i = 0, 1, ..., n$. The discrete gamma swap pays at a fixed maturity T the difference between a given level (fixed leg) and a weighted realized level of variance over the swap's life (floating leg). From Lee (2010), the floating leg of a standard discrete gamma swap (without dividend) is

$$Notional \times \frac{1}{T} \times \sum_{i=0}^{n-1} \frac{S_{t_{i+1}}}{S_0} \left(\ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2. \tag{4.27}$$

For the ease of exposition, we consider a special discrete gamma swap (without dividend), and its floating leg is

$$Notional \times \frac{1}{T} \times \sum_{i=0}^{n-1} \frac{S_{t_{i+1}}}{S_0 e^{rt_{i+1}}} \left(\ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2. \tag{4.28}$$

Note that the difference of the contract described by (4.28) and the one described by (4.27) is that there is an additional term $e^{rt_{i+1}}$ in the denominator of each of the weighting terms in (4.28). This additional term will be canceled out later in the derivation and makes the derivation easier by utilizing some symmetry properties of the stock price in the Heston model under the change of numeraire. For the standard gamma swap payoff (4.27), we can still obtain a closed-form formula for the fair strike by similar derivations as in Section 4.11. Since the purpose here is to illustrate the applications of the symmetry ideas in reducing the problem of calculating the fair strike of a discrete special discounted gamma swap to the problem of calculating the fair strike of a discrete variance swap, in the following we shall stick to the payoff in equation (4.28).

Rewrite (4.28) as $Notional \times V_g(0, n, T)$, where we define

$$V_g(0, n, T) = \frac{1}{T} \sum_{i=0}^{n-1} \frac{S_{t_{i+1}}}{S_0 e^{rt_{i+1}}} \left(\ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2.$$

Then the fair strike of this special discrete gamma swap is $\Gamma_d^H = \mathbb{E}^Q[V_g(0, n, T)]$. We now illustrate the relationship of the fair strike of this special discrete gamma swap with that of a discrete variance swap. Under the risk-neutral measure Q, in the Heston model, from Proposition 2.5.1 in Chapter 2, the underlying (discounted) stock price $(e^{-rt}S_t)_{t\in[0,T]}, T \in [0, \infty)$ is a true martingale. Define the numeraire measure Q_S as $\frac{dQ_S}{dQ} |_{\mathcal{F}_t} = \frac{S_t}{S_0 e^{rt}}, t > 0$. To the best of our knowledge, the following proposition first appeared in Theorem 1, p5 of Del Baño Rollin (2008), and also see Proposition 2.2 and equation (8) on p2042 in Del Baño Rollin et al. (2010). We state the results using our notation, and for completeness, we provide a proof.

Proposition 4.8.1. *(Theorem 1, Del Baño Rollin (2008))*

Under Q, assume that the stock price follows the Heston model with the SDE (4.8)*, and denote $S_t \sim Hes(\kappa, \theta, \gamma, \rho, r)$. Also assume $\kappa > \rho\gamma$*[13]*. Under the numeraire measure Q_S, define $S_t^{'} = 1/S_t$. Then $S_t^{'} \sim Hes(\kappa - \rho\gamma, \frac{\kappa\theta}{\kappa - \rho\gamma}, \gamma, -\rho, -r)$. This means that under Q_S, $S_t^{'}$ follows the Heston model dynamic, but with different parameters.*

Proof. By the Girsanov theorem, under the numeraire measure Q_S

$$\widetilde{W}_t^{(1)} = W_t^{(1)} - \langle W_t^{(1)}, \int_0^t \sqrt{V_u} dW_u^{(1)} \rangle = W_t^{(1)} - \int_0^t \sqrt{V_u} du.$$

and

$$\begin{aligned}
\widetilde{W}_t^{(2)} &= W_t^{(2)} - \langle W_t^{(2)}, \int_0^t \sqrt{V_u} dW_u^{(1)} \rangle \\
&= W_t^{(2)} - \langle W_t^{(2)}, \rho \int_0^t \sqrt{V_u} dW_u^{(2)} \rangle - \sqrt{1-\rho^2} \langle W_t^{(2)}, \rho \int_0^t \sqrt{V_u} dW_u^{(3)} \rangle \\
&= W_t^{(2)} - \rho \int_0^t \sqrt{V_u} du,
\end{aligned}$$

where $\widetilde{W}_t^{(1)}$ and $\widetilde{W}_t^{(2)}$ are standard Brownian motions under Q_S. Then

$$\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^{(1)} \\
&= rS_t dt + \sqrt{V_t} S_t (d\widetilde{W}_t^{(1)} + \sqrt{V_t} dt) \\
&= (r + V_t) S_t dt + \sqrt{V_t} S_t d\widetilde{W}_t^{(1)},
\end{aligned} \tag{4.29}$$

[13] We assume this for the ease of exposition. This condition is termed *good correlation regime* on p12 of Jacquier and Martini (2011). This is not an overly restrictive assumption given that in the equity stock market, the correlation between the stock price and the volatility is usually negative due to the leverage effect.

and

$$
\begin{aligned}
dV_t &= \kappa(\theta - V_t)dt + \gamma\sqrt{V_t}(d\widetilde{W}_t^{(2)} + \rho\sqrt{V_t}dt) \\
&= (\kappa - \rho\gamma)\left(\frac{\kappa\theta}{\kappa - \rho\gamma} - V_t\right)dt + \gamma\sqrt{V_t}d\widetilde{W}_t^{(2)}.
\end{aligned} \tag{4.30}
$$

Observe that under Q_S, the dynamics of V_t in equation (4.30) still follows a CIR process, but with different parameters. If the original CIR process is denoted by $CIR(\kappa, \theta, \gamma)$, then the new CIR process is $CIR(\kappa - \rho\gamma, \frac{\kappa\theta}{\kappa-\rho\gamma}, \gamma)$. Denote $S_t' = 1/S_t$ as the reciprocal process of the stock price under Q_S and apply Itō's lemma

$$
\begin{aligned}
d\left(\frac{1}{S_t}\right) &= -\frac{1}{S_t^2}dS_t + \frac{1}{S_t^3}(dS_t)^2 \\
&= -\frac{1}{S_t^2}\left((r + V_t)S_t dt + \sqrt{V_t}S_t d\widetilde{W}_t^{(1)}\right) + \frac{1}{S_t^3}V_t S_t^2 dt \\
&= \frac{1}{S_t}\left((-r)dt - \sqrt{V_t}d\widetilde{W}_t^{(1)}\right).
\end{aligned} \tag{4.31}
$$

Notice that the two Brownian motions $\widetilde{W}_t^{(1)}$ and $\widetilde{W}_t^{(2)}$ still have correlation ρ, thus $-\widetilde{W}_t^{(1)}$ and $\widetilde{W}_t^{(2)}$ shall have correlation $-\rho$. Substitute $1/S_t$ in (4.31) by S_t', denote $r' = -r$ and $\rho' = -\rho$, $\kappa' = \kappa - \rho\gamma$, and $\theta' = \frac{\kappa\theta}{\kappa-\rho\gamma}$. Then

$$
\begin{aligned}
dS_t' &= S_t'\left(r'dt + \sqrt{V_t}d(-\widetilde{W}_t^{(1)})\right), \\
dV_t &= \kappa'\left(\theta' - V_t\right)dt + \gamma\sqrt{V_t}d\widetilde{W}_t^{(2)},
\end{aligned} \tag{4.32}
$$

where $E[d(-\widetilde{W}_t^{(1)}), d\widetilde{W}_t^{(2)}] = \rho' dt$. Comparing the SDE (4.32) with the SDE (4.8), we can see that they have exactly the same form except with different parameters. This completes the proof. □

As a first application, we can compute the value of a continuous entropy contract in the Heston model.

Proposition 4.8.2. *In the Heston model, assume $\kappa > \rho\gamma$. The price of a continuous entropy contract is*

$$\mathbb{E}^Q[S_T \ln S_T] = \frac{S_0 e^{rT}\left(\kappa\theta T + (1 - e^{-(\kappa-\rho\gamma)T})(V_0 - \frac{\kappa\theta}{\kappa-\rho\gamma})\right)}{2(\kappa-\rho\gamma)}.$$

Proof. Apply the change of numeraire, we have

$$\begin{aligned}
\mathbb{E}^Q[S_T \ln S_T] &= \mathbb{E}^{Q_S}\left[\frac{S_0 e^{rT}}{S_T} S_T \ln S_T\right] \\
&= S_0 e^{rT} \mathbb{E}^{Q_S}[\ln S_T] \\
&= -S_0 e^{rT} \mathbb{E}^{Q_S}[\ln S'_T] \\
&= \frac{S_0 e^{rT}}{2} \mathbb{E}^{Q_S}\left[\int_0^T V'_s ds\right] \\
&= \frac{S_0 e^{rT}}{2} T K_c^H \left(\kappa - \rho\gamma, \frac{\kappa\theta}{\kappa-\rho\gamma}, -r, -\rho\right) \\
&= \frac{S_0 e^{rT}\left(\kappa\theta T + (1 - e^{-(\kappa-\rho\gamma)T})(V_0 - \frac{\kappa\theta}{\kappa-\rho\gamma})\right)}{2(\kappa-\rho\gamma)}.
\end{aligned}$$

In the above, S'_T denotes the reciprocal of the stock price under the numeraire measure Q_S, and from Proposition 4.8.1, it follows the Heston model with a different set of parameters. This completes the proof. □

In the second application, we give a closed-form formula for the fair strike of the special discrete gamma swap defined in (4.28) in the Heston model.

Proposition 4.8.3. *In the Heston model, assume $\kappa > \rho\gamma$. The fair strike of the special discrete gamma swap is*

$$\Gamma_d^H(\kappa,\theta,r,\rho,n) = \mathbb{E}^Q[V_g(0,n,T)]$$
$$= \frac{1}{8n(\kappa-\rho\gamma)^3 T}\Bigg\{ n\left(\gamma^2\left(\frac{\kappa\theta}{\kappa-\rho\gamma} - 2V_0\right) + 2(\kappa-\rho\gamma)\left(V_0 - \frac{\kappa\theta}{\kappa-\rho\gamma}\right)^2\right)$$
$$\times\left(e^{-2(\kappa-\rho\gamma)T} - 1\right)\frac{1-e^{\frac{(\kappa-\rho\gamma)T}{n}}}{1+e^{\frac{(\kappa-\rho\gamma)T}{n}}}$$
$$+2(\kappa-\rho\gamma)T\left((\kappa-\rho\gamma)^2 T\left(\frac{\kappa\theta}{\kappa-\rho\gamma}+2r\right)^2 + n\frac{\kappa\theta}{\kappa-\rho\gamma}\left(4(\kappa-\rho\gamma)^2 + 4\rho(\kappa-\rho\gamma)\gamma + \gamma^2\right)\right)$$
$$+4\left(V_0 - \frac{\kappa\theta}{\kappa-\rho\gamma}\right)\left(n\left(2(\kappa-\rho\gamma)^2 + \gamma^2 + 2\rho(\kappa-\rho\gamma)\gamma\right) + (\kappa-\rho\gamma)^2 T\left(\frac{\kappa\theta}{\kappa-\rho\gamma}+2r\right)\right)$$
$$\times\left(1 - e^{-(\kappa-\rho\gamma)T}\right)$$
$$-2n^2\frac{\kappa\theta}{\kappa-\rho\gamma}\gamma\left(\gamma + 4\rho(\kappa-\rho\gamma)\right)\left(1 - e^{-\frac{(\kappa-\rho\gamma)T}{n}}\right)$$
$$+4\left(V_0 - \frac{\kappa\theta}{\kappa-\rho\gamma}\right)(\kappa-\rho\gamma)T\gamma\left(\gamma + 2\rho(\kappa-\rho\gamma)\right)\frac{1-e^{-(\kappa-\rho\gamma)T}}{1-e^{\frac{(\kappa-\rho\gamma)T}{n}}}\Bigg\}. \quad (4.33)$$

The fair strike of the continuous special gamma swap is

$$\Gamma_c^H(\kappa,\theta,\rho) = \frac{\kappa\theta T + V_0 - \frac{\kappa\theta}{\kappa-\rho\gamma}}{(\kappa-\rho\gamma)T} + 1 - e^{-(\kappa-\rho\gamma)T}. \quad (4.34)$$

Proof. From Proposition 4.8.1

$$\Gamma_d^H = \mathbb{E}^Q[V_g(0,n,T)] = \frac{1}{T}\sum_{i=0}^{n-1}\mathbb{E}^Q\left[\frac{S_{t_{i+1}}}{S_0 e^{rt_{i+1}}}\left(\ln\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right]$$
$$= \frac{1}{T}\sum_{i=0}^{n-1}\mathbb{E}^{Q_S}\left[\frac{S_0 e^{rt_{i+1}}}{S_{t_{i+1}}}\frac{S_{t_{i+1}}}{S_0 e^{rt_{i+1}}}\left(\ln\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right]$$
$$= \frac{1}{T}\sum_{i=0}^{n-1}\mathbb{E}^{Q_S}\left[\left(\ln\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right]$$

$$= \frac{1}{T} \sum_{i=0}^{n-1} \mathbb{E}^{Q_S} \left[\left(\ln \frac{S'_{t_{i+1}}}{S'_{t_i}} \right)^2 \right].$$

Thus the fair strike of a special discrete gamma swap in the Heston model is equal to the fair strike of another discrete variance swap associated to the stock S', which follows the Heston dynamics with different parameters. Denote its fair strike as $K_d^H(\kappa, \theta, r, \rho, n)$. Similarly denote the fair strike of the gamma swap as $\Gamma_d^H(\kappa, \theta, r, \rho, n)$. Then

$$\Gamma_d^H(\kappa, \theta, r, \rho, n) = K_d^H \left(\kappa - \rho\gamma, \frac{\kappa\theta}{\kappa - \rho\gamma}, -r, -\rho, n \right). \tag{4.35}$$

Thus the closed-form formula (4.33) of the special discrete gamma swap is a consequence of the explicit closed-form formula for K_d^H in Proposition 4.3.1. Similarly from (4.35), as $n \to \infty$

$$\begin{aligned}
\Gamma_c^H(\kappa, \theta, \rho) &= \lim_{n\to\infty} \Gamma_d^H(\kappa, \theta, r, \rho, n) \\
&= \lim_{n\to\infty} K_d^H \left(\kappa - \rho\gamma, \frac{\kappa\theta}{\kappa - \rho\gamma}, -r, -\rho, n \right) \\
&= K_c^H \left(\kappa - \rho\gamma, \frac{\kappa\theta}{\kappa - \rho\gamma}, -r, -\rho \right) \\
&= \frac{\kappa\theta T + V_0 - \frac{\kappa\theta}{\kappa - \rho\gamma}}{(\kappa - \rho\gamma)T} + 1 - e^{-(\kappa - \rho\gamma)T}.
\end{aligned}$$

This completes the proof. □

Remark 4.8.1. *When $r = 0\%$, the payoff of the special gamma swap agrees with the payoff of a standard gamma swap. In this case, the continuous strike of the standard gamma swap is still given by* (4.34) *since it does not depend on r. Zheng and Kwok (2013) give a closed-form explicit formula for the fair strike of the continuous standard gamma swap in their equation* (3.5) *in the stochastic volatility with simultaneous jumps (SVSJ) model. Take $J^S = 0$, $J^V = 0$, $r = 0$, $\lambda = 0$ and replace ε with our parameter γ, it can be verified that their formula* (3.5) *agrees with our formula* (4.34) *here.*

When $r = 0\%$, our formula (4.33) is more explicit than their formula (3.1) for the discrete fair strike of a standard gamma swap. Zheng and Kwok (2013) use the forward characteristic function and obtain their formula by solving a system of Ricatti ODEs. Our approach here explore the nice symmetry property of the Heston model and the derivation is simpler.

Remark 4.8.2. *In the above calculation, we see that the term $e^{rt_{i+1}}$ is canceled out, and it is possible to directly link the fair strike of this special discrete gamma swap to the fair strike of a discrete variance swap. More generally, in a model where the reciprocal of the stock price $1/S$ under the numeraire measure Q_S has the same dynamics as the stock price S under the original risk-neutral measure Q except with some differences in the parameters, we shall have similar relationship between the fair strike of a special discrete gamma swap and that of a discrete variance swap.*

4.8.1 Asymptotics of special discrete gamma swaps in the Heston model

We work in the Heston stochastic volatility model. First expand the fair strike of the special discrete gamma swap with respect to the number of sampling periods n.

Proposition 4.8.4. *(Expansion of the fair strike of the special discrete gamma swap w.r.t. n)*

Assume $\kappa > \rho\gamma$, the asymptotic behavior of the fair strike of a special discrete gamma swap in the Heston model is

$$\Gamma_d^H(n) = \Gamma_c^H + \frac{a_1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{4.36}$$

where

$$a_1 = r^2T + rT\Gamma_c^H - \left(\frac{\gamma(\frac{\kappa\theta}{\kappa-\rho\gamma} - V_0)}{2(\kappa-\rho\gamma)}(1 - e^{-(\kappa-\rho\gamma)T}) - \frac{\kappa\theta\gamma T}{2(\kappa-\rho\gamma)} \right) \rho$$
$$+ \left(\frac{\kappa^2\theta^2}{4(\kappa-\rho\gamma)^2} + \frac{\kappa\theta\gamma^2}{8(\kappa-\rho\gamma)^2} \right) T + c_1, \tag{4.37}$$

with

$$c_1 = \frac{1}{16(\kappa-\rho\gamma)^2} \left(\left[\gamma^2 \frac{\kappa\theta}{\kappa-\rho\gamma} - 2(\kappa-\rho\gamma)\left(V_0 - \frac{\kappa\theta}{\kappa-\rho\gamma} \right)^2 \right] \left(e^{-2T(\kappa-\rho\gamma)} - 1\right) \right.$$
$$\left. +2\left(V_0 - \frac{\kappa\theta}{\kappa-\rho\gamma} \right) \left(e^{-T(\kappa-\rho\gamma)} - 1\right) \left[\gamma^2 (e^{-T(\kappa-\rho\gamma)} - 1) - 4\kappa\theta \right] \right).$$

Proof. This proposition is a straightforward expansion from (4.33) in Proposition 4.8.3. This completes the proof. □

Remark 4.8.3. *The first term in the expansion* (4.36)*,* a_1*, is a linear function of* ρ*. Observe that the coefficient in front of* ρ*,* $-\left(\frac{\gamma(\frac{\kappa\theta}{\kappa-\rho\gamma} - V_0)}{2(\kappa-\rho\gamma)}(1 - e^{-(\kappa-\rho\gamma)T}) - \frac{\kappa\theta\gamma T}{2(\kappa-\rho\gamma)} \right)$ *is positive*[14]*, so that* a_1 *is always an increasing function of* ρ*. Then*

$$a_1 \geqslant 0 \quad \Longleftrightarrow \quad \rho \geqslant \rho_0,$$

where

$$\rho_0 = \frac{r^2T + rT\Gamma_c^H + \left(\frac{\kappa^2\theta^2}{4(\kappa-\rho\gamma)^2} + \frac{\kappa\theta\gamma^2}{8(\kappa-\rho\gamma)^2} \right) T + c_1}{-\left(\frac{\gamma(\frac{\kappa\theta}{\kappa-\rho\gamma} - V_0)}{2(\kappa-\rho\gamma)}(1 - e^{-(\kappa-\rho\gamma)T}) - \frac{\kappa\theta\gamma T}{2(\kappa-\rho\gamma)} \right)}.$$

Proposition 4.8.5. *(Expansion of the fair strike for small maturity)*

In the Heston model, an expansion of $\Gamma_d^H(n)$ *when* $T \to 0$ *is calculated*

[14]This can be easily seen from the fact that for all $x > 0$, $(\frac{\kappa\theta}{\kappa-\rho\gamma} - V_0)(1 - e^{-x}) - \frac{\kappa\theta}{\kappa-\rho\gamma}x \leqslant \frac{\kappa\theta}{\kappa-\rho\gamma}(1 - e^{-x} - x) < 0$, and note that here $x = (\kappa - \rho\gamma)T > 0$.

as

$$\Gamma_d^H(n) = V_0 + b_1 T + b_2 T^2 + \mathcal{O}\left(T^3\right), \tag{4.38}$$

where

$$b_1 = \frac{(\kappa\theta - V_0(\kappa - \rho\gamma))}{2} + \frac{1}{4n}\left((V_0 + 2r)^2 + 2\gamma V_0 \rho\right),$$

$$b_2 = \frac{(\kappa - \rho\gamma)^2 (V_0 - \frac{\kappa\theta}{\kappa - \rho\gamma})}{6} + \frac{(V_0 - \frac{\kappa\theta}{\kappa - \rho\gamma})(\kappa - \rho\gamma)(-\gamma\rho - 2r - V_0) + \frac{\gamma^2 V_0}{2}}{4n} + \frac{-\gamma\rho(\kappa - \rho\gamma)(V_0 + \frac{\kappa\theta}{\kappa - \rho\gamma}) - \frac{\gamma^2 V_0}{2}}{12n^2}.$$

Note that

$$\Gamma_c^H = V_0 + \frac{\kappa - \rho\gamma}{2}\left(\frac{\kappa\theta}{\kappa - \rho\gamma} - V_0\right) T + \frac{\kappa - \rho\gamma^2}{6}\left(V_0 - \frac{\kappa\theta}{\kappa - \rho\gamma}\right) T^2 + \mathcal{O}\left(T^3\right),$$

then

$$\Gamma_d^H(n) - \Gamma_c^H = \frac{1}{4n}\left((V_0 + 2r)^2 + 2\gamma V_0 \rho\right) T + \mathcal{O}(T^2).$$

Proof. This proposition is a straightforward expansion from (4.33) in Proposition 4.8.3. This completes the proof. □

4.9 Discrete variance swap in the mixed exponential jump diffusion model

Broadie and Kaya (2006) give a closed-form formula of the fair strike of the discrete variance swap in the Merton's jump diffusion model. The mixed exponential distribution is dense with respect to the class of all distributions in the sense of weak convergence (see Botta and Harris (1986)). Cai and Kou (2011) propose a new class of jump diffusions named "mixed exponential jump diffusions"(MEJD). In particular, the MEJD can be used to approximate Merton's jump diffusion. For the literature on fitting mixed exponential distributions to a given distribution, refer to the papers of

Botta and Harris (1986) and Dufresne (2007). The MEJD model can also be used to approximate Lévy processes and for its applications in option pricing, please refer to Crosby, Le Saux and Mijatović (2010), and Pistorius and Stolte (2012), and the references therein.

The underlying stock price in the MEJD model is given as follows

$$\frac{dS_t}{S_{t-}} = rdt + \sigma dW_t + d\sum_{i=1}^{N_t}(V_i - 1),$$

where V_i is the jump size, S_{t-} is the stock price immediately before the jump time at t. The return process $X_t = \ln(S_t/S_0)$ follows the MEJD process. $N_t, t \geqslant 0$ is a Poisson process with rate λ counting the number of jumps up to time t. $W_t,\ t \geqslant 0$ is a standard Brownian motion, and $Y_i = \ln(V_i), i = 1, 2, ...$ is a sequence of independent and identically distributed mixed exponential random variables with probability density function given as below

$$f_Y(x) = p_u \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x} \mathbb{1}_{x \geqslant 0} + q_d \sum_{j=1}^{n} q_j \theta_j e^{\theta_j x} \mathbb{1}_{x<0}, \tag{4.39}$$

with

$$\begin{aligned}
&p_u \geqslant 0, q_d = 1 - p_u \geqslant 0,\\
&p_i \in (-\infty, \infty), i = 1, ..., m; \sum_{i=1}^{m} p_i = 1,\\
&q_j \in (-\infty, \infty), j = 1, ..., n; \sum_{j=1}^{n} q_j = 1,\\
&\quad \eta_i > 1, i = 1, ..., m, \theta_j > 0, j = 1, ..., n.
\end{aligned}$$

In addition, the parameters p_i and q_j need to satisfy some conditions to guarantee that $f_Y(x)$ is always non-negative and is a true probability density function. From p5 of Cai and Kou (2011), a simple sufficient condition is $\sum_{i=1}^{k} p_i \eta_i \geqslant 0$, for all $k = 1, ..., m$, and $\sum_{j=1}^{l} q_j \theta_j \geqslant 0$, for all $l = 1, ..., n$. For alternative conditions, see Bartholomew (1969).

Under the risk-neutral measure Q, the MEJD process is

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, X_0 = 0,$$

where $\mu = r - \frac{\sigma^2}{2} - \lambda\xi$ and

$$\xi = \mathbb{E}[e^{Y_1}] - 1 = p_u \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - 1} + q_d \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + 1} - 1.$$

Similarly, the moment generating function of X_t is

$$\mathbb{E}[e^{xX_t}] = e^{G(x)t}, t \geqslant 0, x \in (-\theta_1, \eta_1), \tag{4.40}$$

where

$$G(x) = \frac{\sigma^2}{2}x^2 + \mu x + \lambda \left(p_u \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + q_d \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1 \right). \tag{4.41}$$

Note that $(-\theta_1, \eta_1)$ contains a neighborhood of 0, and all moments of X_t exist. Thus we can calculate the moments of the process X_t by differentiating the above moment generating function given in (4.40).

Now we derive the explicit formula for the fair strike of the discrete variance swap in the MEJD model.

Proposition 4.9.1. *Consider equi-distant sampling and denote* $\Delta = t_{i+1} - t_i = T/N$*, for* $i = 0, 1, ..., N-1$*. The fair strike of the discrete variance swap in the MEJD model is*

$$\begin{aligned} K_d &= \frac{1}{T} \sum_{i=0}^{N-1} \mathbb{E}\left[\left(\ln(S_{t_{i+1}}/S_{t_i}) \right)^2 \right] \\ &= \left(\sigma^2 + \lambda \left(2p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i^2} + 2q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j^2} \right) \right) + \frac{T}{N} \left(\mu + \lambda \left(p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i} - q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j} \right) \right)^2 . \end{aligned}$$

The fair strike of the continuous variance swap is

$$K_c = \sigma^2 + \lambda \left(2p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i^2} + 2q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j^2} \right). \tag{4.42}$$

Proof. First calculate

$$\begin{aligned}
\left(\ln(S_{t_{i+1}}/S_{t_i})\right)^2 &= \left(\mu\Delta + \sigma(W_{t_i+\Delta} - W_{t_i}) + \sum_{j=N_{t_i}}^{N_{t_{i+1}}} Y_j \right)^2 \\
&= \left(\mu\Delta + \sigma\sqrt{\Delta} Z_{i+1} + \sum_{j=N_{t_i}}^{N_{t_{i+1}}} Y_j \right)^2 \\
&= \mu^2\Delta^2 + \sigma^2\Delta Z_{i+1}^2 + 2\mu\sigma\Delta^{\frac{3}{2}} Z_{i+1} + \left(\sum_{j=N_{t_i}}^{N_{t_{i+1}}} Y_j \right)^2 \\
&\quad + 2\mu\Delta \sum_{j=N_{t_i}}^{N_{t_{i+1}}} Y_j + 2\sigma\sqrt{\Delta} Z_{i+1} \sum_{j=N_{t_i}}^{N_{t_{i+1}}} Y_j,
\end{aligned} \tag{4.43}$$

where Z_{i+1} are independent and identically distributed standard Normal random variables with mean 0 and variance 1, for $i = 0, 1, ..., N-1$. Here N_{t_i} is the number of jumps in the stock price during $[0, t_i]$, $i = 0, 1, ..., N-1$. Taking expectations on both sides of (4.43)

$$\mathbb{E}\left[\left(\ln(S_{t_{i+1}}/S_{t_i})\right)^2 \right] = \mu^2\Delta^2 + \sigma^2\Delta + \mathbb{E}\left[\left(\sum_{j=N_{t_i}}^{N_{t_{i+1}}} Y_j \right)^2 \right] + 2\mu\Delta\mathbb{E}\left[\sum_{j=1}^{N_{t_{i+1}}} Y_j \right]$$

$$\begin{aligned}
&= \mu^2\Delta^2 + \sigma^2\Delta + \mathbb{E}\left[\left(\sum_{j=1}^{N_{t_{i+1}}} Y_j \right)^2 \right] - \mathbb{E}\left[\left(\sum_{j=1}^{N_{t_i}} Y_j \right)^2 \right] \\
&\quad - 2\mathbb{E}\left[\sum_{j=1}^{N_{t_i}} Y_j \right] \left(\mathbb{E}\left[\sum_{j=1}^{N_{t_{i+1}}} Y_j \right] - \mathbb{E}\left[\sum_{j=1}^{N_{t_i}} Y_j \right] \right) + 2\mu\Delta \left(\mathbb{E}\left[\sum_{j=1}^{N_{t_{i+1}}} Y_j \right] - \mathbb{E}\left[\sum_{j=1}^{N_{t_i}} Y_j \right] \right).
\end{aligned} \tag{4.44}$$

From (4.40) and (4.41)

$$\mathbb{E}\left[e^{x\sum_{j=1}^{N_{t_i}} Y_j}\right] = \exp\left\{\lambda t_i \left(p_u \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + q_d \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1\right)\right\}. \quad (4.45)$$

Then by differentiating the above moment generating function (4.45) and substituting $x = 0$

$$\mathbb{E}\left[\sum_{j=1}^{N_{t_i}} Y_j\right] = \lambda t_i \left(p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i} - q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j}\right), \quad (4.46)$$

and

$$\mathbb{E}\left[\left(\sum_{j=1}^{N_{t_i}} Y_j\right)^2\right] = \lambda^2 t_i^2 \left(p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i} - q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j}\right)^2 + \lambda t_i \left(2p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i^2} + 2q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j^2}\right). \quad (4.47)$$

Substitute (4.46) and (4.47) by their corresponding expressions into (4.44)

$$\begin{aligned}
&\mathbb{E}\left[\left(\ln(S_{t_{i+1}}/S_{t_i})\right)^2\right] \\
&= \mu^2\Delta^2 + \sigma^2\Delta + \mathbb{E}\left[\left(\sum_{j=1}^{N_{t_{i+1}}} Y_j\right)^2\right] - \mathbb{E}\left[\left(\sum_{j=1}^{N_{t_i}} Y_j\right)^2\right] \\
&\quad - 2\mathbb{E}\left[\sum_{j=1}^{N_{t_i}} Y_j\right]\left(\mathbb{E}\left[\sum_{j=1}^{N_{t_{i+1}}} Y_j\right] - \mathbb{E}\left[\sum_{j=1}^{N_{t_i}} Y_j\right]\right) + 2\mu\Delta\left(\mathbb{E}\left[\sum_{j=1}^{N_{t_{i+1}}} Y_j\right] - \mathbb{E}\left[\sum_{j=1}^{N_{t_i}} Y_j\right]\right) \\
&= \mu^2\Delta^2 + \sigma^2\Delta + \lambda^2(t_{i+1}^2 - t_i^2)\left(p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i} - q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j}\right)^2 \\
&\quad + \lambda\Delta\left(2p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i^2} + 2q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j^2}\right) + 2\mu\lambda\Delta^2\left(p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i} - q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j}\right) \\
&\quad - 2\lambda^2\Delta t_i\left(p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i} - q_d \sum_{j=1}^{n} \frac{q_j}{\theta_j}\right)^2.
\end{aligned}$$

By summing the individual terms up

$$
\begin{aligned}
K_d &= \frac{1}{T}\sum_{i=0}^{N-1}\mathbb{E}\left[\left(\ln(S_{t_{i+1}}/S_{t_i})\right)^2\right] \\
&= \left(\sigma^2+\lambda\left(2p_u\sum_{i=1}^{m}\frac{p_i}{\eta_i^2}+2q_d\sum_{j=1}^{n}\frac{q_j}{\theta_j^2}\right)\right) \\
&\quad +\frac{T}{N}\left(\mu+\lambda\left(p_u\sum_{i=1}^{m}\frac{p_i}{\eta_i}-q_d\sum_{j=1}^{n}\frac{q_j}{\theta_j}\right)\right)^2.
\end{aligned}
$$

Letting $N\to\infty$, the fair strike of the continuous variance swap is

$$
K_c=\sigma^2+\lambda\left(2p_u\sum_{i=1}^{m}\frac{p_i}{\eta_i^2}+2q_d\sum_{j=1}^{n}\frac{q_j}{\theta_j^2}\right),
$$

and the convergence of K_d to K_c is of the order $\mathcal{O}(\frac{1}{N})$. This completes the proof. □

4.10 Proof of Proposition 4.2.1

Using Itō's lemma and Cholesky decomposition, (4.1) becomes

$$
\begin{aligned}
d\left(\ln\left(S_t\right)\right) &= \left(r-\frac{1}{2}m^2(V_t)\right)dt+\rho m(V_t)dW_t^{(2)}+\sqrt{1-\rho^2}m(V_t)dW_t^{(3)}, \\
dV_t &= \mu(V_t)dt+\sigma(V_t)dW_t^{(2)},
\end{aligned}
$$

where $W_t^{(2)}$ and $W_t^{(3)}$ are two standard independent Brownian motions.

Proposition 4.2.1 is then a direct application of the following lemma (see Lemma 3.1 of Bernard and Cui (2011) for its proof).

Lemma 4.10.1. *Under the model given in* (4.1)*, we have*

$$S_T = S_0 \exp\left\{rT - \frac{1}{2}\int_0^T m^2(V_t)dt + \rho(f(V_T) - f(V_0)) \right. \\ \left. -\rho\int_0^T h(V_t)dt + \sqrt{1-\rho^2}\int_0^T m(V_t)dW_t^{(3)}\right\}, \quad (4.48)$$

where $f(v) = \int_0^v \frac{m(z)}{\sigma(z)}dz$ *and* $h(v) = \mu(v)f'(v) + \frac{1}{2}\sigma^2(v)f''(v)$.

Now from equation (4.48) in Lemma 4.10.1, we compute the following key elements in the fair strike of the discrete variance swap. Assume that the time interval is $[t, t+\Delta]$, then

$$\ln\left(\frac{S_{t+\Delta}}{S_t}\right) = r\Delta - \frac{1}{2}\int_t^{t+\Delta} m^2(V_s)ds + \rho\left(f(V_{t+\Delta}) - f(V_t) - \int_t^{t+\Delta} h(V_s)ds\right) \\ + \sqrt{1-\rho^2}\int_t^{t+\Delta} m(V_s)dW_s^{(3)}.$$

Then we can compute

$$\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] = r^2\Delta^2 + \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta} m^2(V_s)ds\right)^2\right] - r\Delta\mathbb{E}\left[\int_t^{t+\Delta} m^2(V_s)ds\right] + \mathbb{E}\left[A^2\right] \\ + \mathbb{E}\left[\left(2r\Delta - \int_t^{t+\Delta} m^2(V_s)ds\right)A\right] + (1-\rho^2)\mathbb{E}\left[\int_t^{t+\Delta} m^2(V_s)ds\right], \quad (4.49)$$

where $A = \rho\left(f(V_{t+\Delta}) - f(V_t) - \int_t^{t+\Delta} h(V_s)ds\right)$, and

$$A^2 = \rho^2\left((f(V_{t+\Delta}) - f(V_t))^2 + \left(\int_t^{t+\Delta} h(V_s)ds\right)^2 - 2(f(V_{t+\Delta}) - f(V_t))\int_t^{t+\Delta} h(V_s)ds\right).$$

Using the above expressions for A and A^2 in (4.49), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] \\
&= r^2\Delta^2 + \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta} m^2(V_s)ds\right)^2\right] + (1-\rho^2 - r\Delta)\mathbb{E}\left[\int_t^{t+\Delta} m^2(V_s)ds\right] \\
&\quad + \rho^2\mathbb{E}[((f(V_{t+\Delta}) - f(V_t))^2] + \rho^2\mathbb{E}\left[\left(\int_t^{t+\Delta} h(V_s)ds\right)^2\right] + 2r\rho\Delta\mathbb{E}[(f(V_{t+\Delta}) - f(V_t))] \\
&\quad - \mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t))\int_t^{t+\Delta}(2\rho^2 h(V_s) + \rho m^2(V_s))ds\right] - 2r\rho\Delta\mathbb{E}\left[\int_t^{t+\Delta} h(V_s)ds\right] \\
&\quad + \rho\mathbb{E}\left[\left(\int_t^{t+\Delta} h(V_s)ds\right)\left(\int_t^{t+\Delta} m^2(V_s)ds\right)\right]. \qquad (4.50)
\end{aligned}$$

By Itō's lemma, f defined in Lemma 4.10.1 verifies $df(V_t) = h(V_t)dt + m(V_t)dW_t^{(2)}$. Integrating the above SDE from t to $t+\Delta$, we have

$$\begin{aligned}
&f(V_{t+\Delta}) - f(V_t) = \int_t^{t+\Delta} h(V_s)ds + \int_t^{t+\Delta} m(V_s)dW_s^{(2)}, \\
&\mathbb{E}\left[f(V_{t+\Delta}) - f(V_t)\right] - \mathbb{E}\left[\int_t^{t+\Delta} h(V_s)ds\right] = \mathbb{E}\left[\int_t^{t+\Delta} m(V_s)dW_s^{(2)}\right] = 0. \qquad (4.51)
\end{aligned}$$

Rearrange (4.50) and use (4.51) to simplify the terms, and we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(\ln\frac{S_{t+\Delta}}{S_t}\right)^2\right] &= r^2\Delta^2 - r\Delta\mathbb{E}\left[\int_t^{t+\Delta} m^2(V_s)ds\right] + \frac{1}{4}\mathbb{E}\left[\left(\int_t^{t+\Delta} m^2(V_s)ds\right)^2\right] \\
&\quad + (1-\rho^2)\mathbb{E}\left[\int_t^{t+\Delta} m^2(V_s)ds\right] + \rho^2\mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t))^2\right] \\
&\quad + \rho^2\mathbb{E}\left[\left(\int_t^{t+\Delta} h(V_s)ds\right)^2\right] + \rho\mathbb{E}\left[\int_t^{t+\Delta} h(V_s)ds \int_t^{t+\Delta} m^2(V_s)ds\right] \\
&\quad - \rho\mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t))\int_t^{t+\Delta}(2\rho h(V_s) + m^2(V_s))ds\right]. \qquad (4.52)
\end{aligned}$$

Now we apply Fubini's theorem and partial integration to further simplify

(4.52). Note that $m^2(V_s) \geqslant 0$, Q-a.s., then by Fubini's theorem for non-negative measurable functions, $\mathbb{E}\left[\int_t^{t+\Delta} m^2(V_s)ds\right] = \int_t^{t+\Delta} \mathbb{E}\left[m^2(V_s)\right] ds$. Similarly we have $\mathbb{E}\left[\left(\int_t^{t+\Delta} m^2(V_s)ds\right)^2\right] = \int_t^{t+\Delta}\int_t^{t+\Delta} \mathbb{E}\left[m^2(V_s)m^2(V_u)\right] dsdu$ for any $t \leqslant s \leqslant t+\Delta$ and any $t \leqslant u \leqslant t+\Delta$,

If $\mathbb{E}\left[\mid h(V_s)h(V_u) \mid\right] < \infty$ for any $t \leqslant s \leqslant t+\Delta$ and any $t \leqslant u \leqslant t+\Delta$, then we have $\mathbb{E}\left[\left(\int_t^{t+\Delta} h(V_s)ds\right)^2\right] = \int_t^{t+\Delta}\int_t^{t+\Delta} \mathbb{E}\left[h(V_s)h(V_u)\right] dsdu$.

If $\mathbb{E}\left[\mid h(V_s)m^2(V_u) \mid\right] < \infty$ for any $t \leqslant s \leqslant t+\Delta$ and any $t \leqslant u \leqslant t+\Delta$, then we have

$$\mathbb{E}\left[\int_t^{t+\Delta} h(V_s)ds \int_t^{t+\Delta} m^2(V_s)ds\right] = \int_t^{t+\Delta}\int_t^{t+\Delta} \mathbb{E}\left[h(V_s)m^2(V_u)\right] dsdu.$$

If $\mathbb{E}\left[\mid (f(V_{t+\Delta}) - f(V_t))(2\rho h(V_s) + m^2(V_s)) \mid\right] < \infty$ for all $t \leqslant s \leqslant t+\Delta$, then we have

$$\begin{aligned}&\mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t))\int_t^{t+\Delta}(2\rho h(V_s) + m^2(V_s))ds\right]\\&= \int_t^{t+\Delta} \mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t))(2\rho h(V_s) + m^2(V_s))\right] ds.\end{aligned}$$

Thus we finally have proved (4.5) from Proposition 4.2.1. This completes the proof. □

4.11 Proof of Proposition 4.3.1

Proof. We apply Proposition 4.2.1 to the Heston stochastic volatility model. We first compute $f(x) = \frac{x}{\gamma}$ and $h(x) = \frac{\kappa\theta - \kappa x}{\gamma}$, then we have

$$
\begin{aligned}
K_d^H &= \frac{1}{T} \sum_{i=0}^{n-1} \mathbb{E}\left[\left(\ln \frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] \\
&= \frac{1}{T}\left(\frac{a^2T^2}{n} + b^2 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathbb{E}[V_s V_u] ds du + \left(\frac{2abT}{n} + 1 - \rho^2\right) \int_0^T \mathbb{E}[V_s] ds\right. \\
&\quad + \frac{\rho^2}{\gamma^2} \sum_{i=0}^{n-1} \mathbb{E}[(V_{t_{i+1}} - V_{t_i})^2] + \frac{2\rho a T}{n\gamma}(\mathbb{E}[V_T] - \mathbb{E}[V_0]) \\
&\quad \left. + \frac{2\rho b}{\gamma} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \mathbb{E}\left[V_{t_{i+1}} V_s\right] ds - \int_{t_i}^{t_{i+1}} \mathbb{E}\left[V_{t_i} V_s\right] ds\right)\right). \qquad (4.53)
\end{aligned}
$$

Furthermore, for all $t \geqslant 0$

$$
\mathbb{E}[V_t] = \theta + e^{-\kappa t}(V_0 - \theta), \qquad (4.54)
$$

and for all $0 < s \leqslant t$

$$
\begin{aligned}
\mathbb{E}[V_t V_s] &= \theta^2 + e^{-\kappa t}(V_0 - \theta)\left(\theta + \frac{\gamma^2}{\kappa}\right) + e^{-\kappa s}\theta(V_0 - \theta) \\
&\quad + e^{-\kappa(t+s)}\left((\theta - V_0)^2 + \frac{\gamma^2}{2\kappa}(\theta - 2V_0)\right) + \frac{\gamma^2}{2\kappa}\theta e^{-\kappa(t-s)}. \qquad (4.55)
\end{aligned}
$$

In particular, this formula holds for $t = s$ and gives $\mathbb{E}[V_t^2]$. These formulas already appear in Broadie and Jain (2008a) (formula (A-15)). To compute K_d^H, (4.54) and (4.55) are the only expressions needed, and they should then be integrated and summed.

We have computed all terms in (4.53) with the help of Maple and also have simplified the final expression given by Maple. It turns out that in the case of the Heston model, all terms can be computed explicitly and the final simplified expression for (4.53) does not require any sums or integrals. We finally obtain an explicit formula for K_d^H as a function of the parameters of the model. This completes the proof. □

4.12 Proof of Proposition 4.3.2

Proof. Denote the log stock price without drift as $X_t = \ln S_t - rt$, and $X_0 = x_0$. Denote $V_0 = v_0$, $\Delta = T/n$. We have that $\mathbb{E}\left[\left(\frac{S_{t_{i+1}}-S_{t_i}}{S_{t_i}}\right)^2\right] = \mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] + 1 - 2e^{r\Delta}$. Thus the goal is to calculate the second moment $\mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right]$, and note that it is closely linked to the moment generating function of the log stock price X. Recall the following formulation of the moment generating function $M(u,t) = \mathbb{E}[e^{uX_t}]$ from Albrecher et al. (2007)

$$M(u,t) = S_0^u \exp\left\{\frac{\kappa\theta}{\gamma^2}\left((\kappa - \gamma\rho u - d(u))t - 2\ln\left(\frac{1-g(u)e^{-d(u)t}}{1-g(u)}\right)\right)\right\}$$
$$\times \exp\left\{V_0 \frac{\kappa - \gamma\rho u - d(u)}{\gamma^2} \frac{1-e^{-d(u)t}}{1-g(u)e^{-d(u)t}}\right\}, \tag{4.56}$$

where the auxiliary functions are given by

$$d(u) = \sqrt{(\kappa - \gamma\rho u)^2 + \gamma^2(u - u^2)}, \quad g(u) = \frac{\kappa - \gamma\rho u - d(u)}{\kappa - \gamma\rho u + d(u)}.$$

We first separate out the case of $i = 0$ and $i = 1, ..., n-1$. For the first case, we have

$$\mathbb{E}\left[\left(\frac{S_{t_1}}{S_0}\right)^2\right] = \frac{1}{S_0^2}\mathbb{E}\left[e^{2\ln S_{t_1}}\right] = \frac{e^{2rt_1}}{S_0^2}M(2,t_1) = \frac{e^{2r\Delta}}{S_0^2}M(2,\Delta). \tag{4.57}$$

For the second case, with $i = 1, 2, ..., n-1$, we have

$$\mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = \mathbb{E}\left[e^{2\ln\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)}\right] = e^{2r\Delta}\mathbb{E}\left[\mathbb{E}\left[e^{2(X_{t_{i+1}} - X_{t_i})} \mid \mathcal{F}_{t_i}\right]\right]$$
$$= \exp\left\{2r\Delta + \frac{\kappa\theta}{\gamma^2}\left((\kappa - 2\gamma\rho - d(2))\Delta - 2\ln\frac{1-g(2)e^{-d(2)\Delta}}{1-g(2)}\right)\right\}$$
$$\times \mathbb{E}\left[\exp\left\{V_{t_i}\frac{\kappa - 2\gamma\rho - d(2)}{\gamma^2}\frac{1-e^{-d(2)\Delta}}{1-g(2)e^{-d(2)\Delta}}\right\}\right]. \tag{4.58}$$

We first define $\alpha = 2\kappa\theta/\gamma^2 - 1 \geqslant 0$, and $\eta(t) = \frac{2\kappa}{\gamma^2}(1 - e^{-\kappa t})^{-1}$. Then from Theorem 3.1[15] in Hurd and Kuznetsov (2008), we have

$$\mathbb{E}[e^{uV_T}] = \left(\frac{\eta(T)}{\eta(T) - u}\right)^{\alpha+1} e^{V_0 \frac{\eta(T)u}{\eta(T)-u} e^{-\kappa T}}. \tag{4.59}$$

Combine equations (4.58) and (4.59), for $i = 1, ..., n-1$, we finally have

$$\mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = e^{2r\Delta + \frac{\kappa\theta}{\gamma^2}\left((\kappa - 2\gamma\rho - d(2))\Delta - 2\ln\frac{1-g(2)e^{-d(2)\Delta}}{1-g(2)}\right)} e^{V_0 \frac{\eta(t_i)q(2)}{\eta(t_i)-q(2)} e^{-\kappa t_i}} \left(\frac{\eta(t_i)}{\eta(t_i) - q(2)}\right)^{\alpha+1}, \tag{4.60}$$

where $q(u) = \frac{\kappa - \gamma\rho u - d(u)}{\gamma^2} \frac{1-e^{-d(u)\Delta}}{1-g(u)e^{-d(u)\Delta}}$. Using the definition of $M(u,t)$, we can factor out $M(2,\Delta)$ from (4.60) and finally we have

$$\mathbb{E}\left[\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = \frac{e^{2r\Delta}}{S_0^2} M(2,\Delta) e^{q(2)V_0\left(\frac{\eta(t_i)e^{-\kappa t_i}}{\eta(t_i)-q(2)} - 1\right)} \left(\frac{\eta(t_i)}{\eta(t_i) - q(2)}\right)^{\alpha+1}. \tag{4.61}$$

When $i = 0$, we have $t_i = 0$ and since $\eta_u \to \infty$ as $u \to 0$, we use L'Hôpital's rule

$$\frac{\eta_{t_0}}{\eta_{t_0} - q(2)} = \lim_{u\to 0} \frac{\eta_u}{\eta_u - q(2)} = \lim_{u \to 0} \frac{\eta'_u}{\eta'_u} = 1.$$

Thus a_0 is a special case of the formula in (4.61) when $i = 0$. From Theorem 3.1 in Hurd and Kuznetsov (2008), equation (4.56) and consequently the above (4.58), (4.59) are well-defined if $u < \eta(T)$[16]. Note that the formula (4.61) involves the $u = 2$ case. A sufficient condition for $u = 2 < \eta(T)$ to hold is $\gamma^2 T < 1$ (since $2 < \eta(T)$ is equivalent to $1 - \frac{\kappa}{\gamma^2} < e^{-\kappa T}$).

Then the final formula for the discrete fair strike follows by summing the above terms $a_i, i = 0, 1, ..., n-1$. This completes the proof. □

[15] Note that in terms of our notation, the parameters in Hurd and Kuznetsov (2008) and our parameters have the correspondence $a = \kappa\theta, b = \kappa, c = \gamma$.

[16] Note that $\eta(t)$ is a decreasing function in t, thus $u < \eta(T)$ is sufficient for $u < \eta(t_i)$ for all $i = 0, 1, ..., n$.

4.13 Proof of Proposition 4.4.1

Proof. For the Hull-White model, from Proposition 4.2.1, we first compute $f(x) = \frac{2}{\sigma}\sqrt{x}$ and $h(x) = \left(\frac{\mu}{\sigma} - \frac{\sigma}{4}\right)\sqrt{x}$, then we have

$$\begin{aligned}
\mathbb{E}\left[\left(\ln\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] &= (1-\rho^2-\frac{rT}{n})\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}}\mathbb{E}\left[V_s\right]ds + r^2\frac{T^2}{n^2} \\
&- \frac{2\rho}{\sigma}\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}}\mathbb{E}\left[\left(\sqrt{V_{\frac{(i+1)T}{n}}} - \sqrt{V_{\frac{iT}{n}}}\right)V_s\right]ds + 2\rho^2q^2\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}}\int_{\frac{iT}{n}}^{u}\mathbb{E}\left[\sqrt{V_s}\sqrt{V_u}\right]dsdu \\
&+ \frac{4\rho^2}{\sigma^2}\mathbb{E}\left[\left(\sqrt{V_{\frac{(i+1)T}{n}}} - \sqrt{V_{\frac{iT}{n}}}\right)^2\right] - \frac{4\rho^2 q}{\sigma}\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}}\mathbb{E}\left[\left(\sqrt{V_{\frac{(i+1)T}{n}}} - \sqrt{V_{\frac{iT}{n}}}\right)\sqrt{V_s}\right]ds \\
&+ \frac{1}{2}\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}}\int_{\frac{iT}{n}}^{u}\mathbb{E}\left[V_sV_u\right]dsdu + \rho q\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}}\int_{\frac{iT}{n}}^{u}\mathbb{E}\left[\sqrt{V_s}V_u\right]dsdu \\
&+ \rho q\int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}}\int_{u}^{\frac{(i+1)T}{n}}\mathbb{E}\left[\sqrt{V_s}V_u\right]dsdu,
\end{aligned}$$

with $q = \frac{\mu}{\sigma} - \frac{\sigma}{4}$.

We now compute the following covariance terms that are useful in the simplification of the fair strike $K_d^{HW}(n)$. In the Hull-White model, the stochastic variance process V_t follows a geometric Brownian motion. Thus we have $V_t = V_0\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t^{(2)}\right)$. Note that

$$\mathbb{E}\left[V_s^a\right] = V_0^a e^{a\mu s}e^{\frac{a^2-a}{2}\sigma^2 s},$$

which will be useful below for $a = 1/2$, $a = 1$ and $a = 2$.

$$\mathbb{E}\left[V_s\right] = V_0e^{\mu s}, \quad \mathbb{E}\left[\sqrt{V_s}\right] = \sqrt{V_0}e^{\frac{\mu}{2}s - \frac{1}{8}\sigma^2 s} = \sqrt{V_0}e^{\frac{\sigma}{2}qs}, \quad \mathbb{E}\left[V_s^2\right] = V_0^2e^{2\mu s + \sigma^2 s}.$$

The fair strike for the continuous variance swap is straightforward and is equal to $\mathbb{E}\left[\int_0^T V_s ds\right] = \frac{V_0}{\mu}(e^{\mu T} - 1)$. Similarly

$$\mathbb{E}\left[\int_0^T \sqrt{V_s} ds\right] = \int_0^T \sqrt{V_0} e^{\frac{\sigma}{2} q s} ds = \sqrt{V_0} \frac{2}{\sigma q}\left(e^{\frac{\sigma q T}{2}} - 1\right),$$

and for $s < u$, we have the following results

$$\begin{aligned}
\mathbb{E}\left[V_s V_u\right] &= V_0^2 \exp\left(\mu(u+s) + \sigma^2 s\right), \\
\mathbb{E}\left[\sqrt{V_s}\sqrt{V_u}\right] &= V_0 \exp\left(\frac{\mu}{2}(u+s) - \frac{\sigma^2}{8}(u-s)\right), \\
\mathbb{E}\left[\sqrt{V_s}\, V_u\right] &= V_0^{\frac{3}{2}} \exp\left(\mu\left(\frac{s}{2} + u\right) + \frac{3\sigma^2}{8} s\right), \\
\mathbb{E}\left[V_s \sqrt{V_u}\right] &= V_0^{\frac{3}{2}} \exp\left(\mu\left(s + \frac{u}{2}\right) - \frac{\sigma^2}{8} u + \frac{\sigma^2}{2} s\right).
\end{aligned} \tag{4.62}$$

After some tedious calculations with the help of Maple, we can obtain an explicit formula as the one appearing in Proposition 4.4.1. This completes the proof. □

4.14 Proof of Proposition 4.5.1

Proof. For the Schöbel-Zhu model, from the key equation in Proposition 4.2.1, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\ln \frac{S_{t+\Delta}}{S_t}\right)^2\right] &= r^2\Delta^2 + (1-\rho^2 - r\Delta)\int_t^{t+\Delta} m_1(s) ds - \rho \int_t^{t+\Delta} m_5(t,s) ds \\
&\quad + \frac{1}{4}\int_t^{t+\Delta}\int_t^{t+\Delta} m_2(s,u) ds du + \frac{\rho^2}{4\gamma^2}\mathbb{E}\left[\left(V_{t+\Delta}^2 - V_t^2\right)^2\right] \quad (4.63) \\
&\quad + \rho^2 \int_t^{t+\Delta}\int_t^{t+\Delta} m_3(s,u) ds du + \rho \int_t^{t+\Delta}\int_t^{t+\Delta} m_4(s,u) ds du,
\end{aligned}$$

where

$$\begin{aligned}
m_1(s) &:= \mathbb{E}\left[m^2(V_s)\right] = \mathbb{E}\left[V_s^2\right], t \leqslant s \leqslant t+\Delta, \\
m_2(s,u) &:= \mathbb{E}\left[m^2(V_s)m^2(V_u)\right] = \mathbb{E}\left[V_s^2 V_u^2\right], t \leqslant s \leqslant t+\Delta, \quad t \leqslant u \leqslant t+\Delta, \\
m_3(s,u) &:= \mathbb{E}\left[h(V_s)h(V_u)\right], t \leqslant s \leqslant t+\Delta, \quad t \leqslant u \leqslant t+\Delta, \qquad (4.64) \\
m_4(s,u) &:= \mathbb{E}\left[h(V_s)m^2(V_u)\right], t \leqslant s \leqslant t+\Delta, \quad t \leqslant u \leqslant t+\Delta, \\
m_5(t,s) &:= \mathbb{E}\left[(f(V_{t+\Delta}) - f(V_t))(2\rho h(V_s) + m^2(V_s))\right], t \leqslant s \leqslant t+\Delta,
\end{aligned}$$

and $\mathbb{E}\left[\left(V_{t+\Delta}^2 - V_t^2\right)^2\right] = \mathbb{E}\left[V_{t+\Delta}^4\right] + \mathbb{E}\left[V_t^4\right] - 2\mathbb{E}\left[V_{t+\Delta}^2 V_t^2\right]$. We compute the following two terms in (4.63) by expanding the products out. For $s \leqslant u$

$$\begin{aligned}
m_3(s,u) =& \mathbb{E}\left[\left(\frac{\kappa\theta}{\gamma}V_s - \frac{\kappa}{\gamma}V_s^2 + \frac{\gamma}{2}\right)\left(\frac{\kappa\theta}{\gamma}V_u - \frac{\kappa}{\gamma}V_u^2 + \frac{\gamma}{2}\right)\right] \\
=& \mathbb{E}\left[\frac{\kappa^2\theta^2}{\gamma^2}V_sV_u - \frac{\kappa^2\theta}{\gamma^2}(V_sV_u^2 + V_s^2V_u) + \frac{\kappa\theta}{2}(V_s + V_u)\right. \\
&\left. - \frac{\kappa}{2}(V_s^2 + V_u^2) + \frac{\kappa^2}{\gamma^2}V_s^2V_u^2 + \frac{\gamma^2}{4}\right],
\end{aligned}$$

and for $t \leqslant s \leqslant t+\Delta$

$$\begin{aligned}
m_5(t,s) =& \frac{1}{2\gamma}\mathbb{E}\left[(V_{t+\Delta}^2 - V_t^2)\left(2\rho\left(\frac{\kappa\theta}{\gamma}V_s - \frac{\kappa}{\gamma}V_s^2 + \frac{\gamma}{2}\right) + V_s^2\right)\right] \\
=& \mathbb{E}\left[\frac{\rho\kappa\theta}{\gamma^2}(V_{t+\Delta}^2 V_s - V_t^2 V_s) + \frac{\gamma - 2\rho\kappa}{2\gamma^2}(V_{t+\Delta}^2 V_s^2 - V_t^2 V_s^2)\right. \\
&\left. + \frac{\rho}{2}(V_{t+\Delta}^2 - V_t^2)\right].
\end{aligned}$$

It is clear from the above expressions of m_i for $i = 1, 2, ..., 5$ that they are all functions of $\mathbb{E}[V_s]$, $\mathbb{E}[V_s^2]$, $\mathbb{E}[V_s^4]$, $\mathbb{E}[V_sV_u]$, $\mathbb{E}[V_s^2V_u]$, $\mathbb{E}[V_sV_u^2]$ and $\mathbb{E}[V_s^2V_u^2]$. We now compute these seven expressions.

Lemma 4.14.1. *For the Ornstein-Uhlenbeck process V, introduce the auxiliary deterministic functions $e_s := (V_0 - \theta)e^{-\kappa s} + \theta$, and $v(s) := \frac{\gamma^2}{2\kappa}(1 -$*

$e^{-2\kappa s})$, *then*

$$\mathbb{E}\left[V_s\right] = e_s, \tag{4.65}$$
$$\mathbb{E}\left[V_s^2\right] = e_s^2 + v(s), \tag{4.66}$$
$$\mathbb{E}\left[V_s^3\right] = e_s^3 + 3e_s v(s), \tag{4.67}$$
$$\mathbb{E}\left[V_s^4\right] = e_s^4 + 6e_s^2 v(s) + 3v^2(s). \tag{4.68}$$

For $t \leqslant s \leqslant u \leqslant t + \Delta$

$$\begin{aligned}\mathbb{E}\left[V_s V_u\right] &= e^{-\kappa(u-s)}\mathbb{E}\left[V_s^2\right] + \theta(1 - e^{-\kappa(u-s)})\mathbb{E}\left[V_s\right],\\ \mathbb{E}\left[V_s^2 V_u^2\right] &= e^{-2\kappa(u-s)}\mathbb{E}\left[V_s^4\right] + 2\theta e^{-\kappa(u-s)}(1 - e^{-\kappa(u-s)})\mathbb{E}\left[V_s^3\right]\\ &\quad + \left(\theta^2(1 - e^{-\kappa(u-s)})^2 + \frac{\gamma^2}{2\kappa}(1 - e^{-2\kappa(u-s)})\right)\mathbb{E}\left[V_s^2\right].\end{aligned} \tag{4.69}$$

For $t \leqslant s \leqslant u \leqslant t + \Delta$

$$\begin{aligned}\mathbb{E}\left[V_s V_u^2\right] &= e^{-2\kappa(u-s)}\mathbb{E}\left[V_s^3\right] + 2\theta e^{-\kappa(u-s)}(1 - e^{-\kappa(u-s)})\mathbb{E}\left[V_s^2\right]\\ &\quad + \left(\theta^2(1 - e^{-\kappa(u-s)})^2 + \frac{\gamma^2}{2\kappa}(1 - e^{-2\kappa(u-s)})\right)\mathbb{E}\left[V_s\right].\end{aligned} \tag{4.70}$$

For $t \leqslant s \leqslant u \leqslant t + \Delta$

$$\mathbb{E}\left[V_s^2 V_u\right] = e^{-\kappa(u-s)}\mathbb{E}\left[V_s^3\right] + \theta(1 - e^{-\kappa(u-s)})\mathbb{E}\left[V_s^2\right]. \tag{4.71}$$

Proof. The stochastic variance process V_s follows

$$dV_s = -\kappa(V_s - \theta)ds + \gamma dW_s^{(2)}.$$

On p120 of Jeanblanc, Yor and Chesney (2009), one finds that the exact solution of the above SDE is

$$V_s = (V_0 - \theta)e^{-\kappa s} + \theta + \gamma \int_0^s e^{-\kappa(s-t)} dW_t^{(2)}.$$

We can compute

$$e_s := \mathbb{E}[V_s] = (V_0 - \theta)e^{-\kappa s} + \theta, \tag{4.72}$$

$$v(s) := Var[V_s] = \frac{\gamma^2}{2\kappa}(1 - e^{-2\kappa s}), \tag{4.73}$$

and the higher moments can also be computed

$$\mathbb{E}\left[V_s^2\right] = e_s^2 + v(s). \tag{4.74}$$

$$\mathbb{E}\left[V_s^3\right] = e_s^3 + 3e_s v(s). \tag{4.75}$$

$$\mathbb{E}\left[V_s^4\right] = e_s^4 + 6e_s^2 v(s) + 3v^2(s). \tag{4.76}$$

For $s \leqslant u$, $\mathbb{E}[V_u \mid V_s] = \mathbb{E}\left[(V_s - \theta)e^{-\kappa(u-s)} + \theta\right]$, and

$$\begin{aligned}\mathbb{E}[V_s V_u] &= \mathbb{E}[V_s \mathbb{E}[V_u \mid V_s]] \\ &= \mathbb{E}\left[V_s((V_s - \theta)e^{-\kappa(u-s)} + \theta)\right] \\ &= e^{-\kappa(u-s)}\mathbb{E}\left[V_s^2\right] + \theta(1 - e^{-\kappa(u-s)})\mathbb{E}[V_s].\end{aligned}$$

Now we can compute the continuous fair strike as

$$\begin{aligned}K_c = \frac{1}{T}\mathbb{E}\left[\int_0^T V_s^2 ds\right] &= \frac{1}{T}\int_0^T \left[((V_0 - \theta)e^{-\kappa s} + \theta)^2 + \frac{\gamma^2}{2\kappa}(1 - e^{-2\kappa s})\right] ds \\ = \left((V_0 - \theta)^2 - \frac{\gamma^2}{2\kappa}\right)\frac{1 - e^{-2\kappa T}}{2\kappa T} &+ 2\theta(V_0 - \theta)\frac{1 - e^{-\kappa T}}{\kappa T} + \theta^2 + \frac{\gamma^2}{2\kappa}.\end{aligned}$$

For $s \leqslant u$

$$\begin{aligned}\mathbb{E}\left[V_s^2 V_u^2\right] &= \mathbb{E}\left[V_s^2 \mathbb{E}\left[V_u^2 \mid V_s\right]\right] \\ &= \mathbb{E}\left[V_s^2\left(((V_s - \theta)e^{-\kappa(u-s)} + \theta)^2 + \frac{\gamma^2}{2\kappa}(1 - e^{-2\kappa(u-s)})\right)\right] \\ &= e^{-2\kappa(u-s)}\mathbb{E}\left[V_s^4\right] + 2\theta e^{-\kappa(u-s)}(1 - e^{-\kappa(u-s)})\mathbb{E}\left[V_s^3\right] \\ &+ \left(\theta^2(1 - e^{-\kappa(u-s)})^2 + \frac{\gamma^2}{2\kappa}(1 - e^{-2\kappa(u-s)})\right)\mathbb{E}\left[V_s^2\right],\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[V_s V_u^2\right] &= \mathbb{E}\left[V_s \mathbb{E}\left[V_u^2 \mid V_s\right]\right] \\
&= \mathbb{E}\left[V_s((V_s-\theta)e^{-\kappa(u-s)}+\theta)^2 + \frac{\gamma^2}{2\kappa}(1-e^{-2\kappa(u-s)})\right] \\
&= e^{-2\kappa(u-s)}\mathbb{E}\left[V_s^3\right] + 2\theta e^{-\kappa(u-s)}(1-e^{-\kappa(u-s)})\mathbb{E}\left[V_s^2\right] \\
&+ \left(\theta^2(1-e^{-\kappa(u-s)})^2 + \frac{\gamma^2}{2\kappa}(1-e^{-2\kappa(u-s)})\right)\mathbb{E}\left[V_s\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[V_s^2 V_u\right] &= \mathbb{E}\left[V_s^2 \mathbb{E}\left[V_u \mid V_s\right]\right] \\
&= \mathbb{E}\left[V_s^2((V_s-\theta)e^{-\kappa(u-s)}+\theta)\right] \\
&= e^{-\kappa(u-s)}\mathbb{E}\left[V_s^3\right] + \theta(1-e^{-\kappa(u-s)})\mathbb{E}\left[V_s^2\right].
\end{aligned}
$$

In the above expressions, the moments $\mathbb{E}[V_s]$, $\mathbb{E}[V_s^2]$, $\mathbb{E}[V_s^3]$ and $\mathbb{E}[V_s^4]$ are already calculated in (4.72), (4.74), (4.75), and (4.76). Then we can substitute the corresponding inputs into equation (4.63), sum up the terms, and obtain $K_d^{SZ}(n)$. This completes the proof. □

4.15 Conclusion of Chapter 4

This chapter provides explicit expressions of the fair strike of discretely sampled variance swaps in the Heston, the Hull-White, and the Schöbel-Zhu models. For the Heston model, the explicit closed-form formulae simplifies the expression obtained by Broadie and Jain (2008a) in equations (A-29) and (A-30) on p793, where several sums from 0 to n are involved. Our formulae are more explicit (as there is no sums involved in the discrete fair strikes), and easier to use. The explicit closed-form formulae for the Hull-White model and the Schöbel-Zhu model are new. Asymptotics of the fair strikes with respect to key parameters such as $n \to \infty$, $T \to 0$, $\kappa \to \infty$, $\gamma \to 0$ are new and consistent with theoretical results obtained in Keller-Ressel and Muhle-Karbe (2012).

Part III

Nearly unbiased Monte Carlo simulation

Chapter 5

Nearly exact option price simulation using characteristic functions

5.1 Introduction

This chapter is based on the publication Bernard, Cui and McLeish (2012) in the *International Journal of Theoretical and Applied Finance*. In this chapter, we propose a new and nearly unbiased Fourier inversion technique using Monte Carlo simulations. Our approach allows us to simulate directly from the characteristic function without any discretization or biased approximation. We then show that it can be useful to solve multidimensional complex problems in finance and illustrate the study with the pricing of some exotic options and with the simulation of first passage times.

It is well-known that Monte Carlo simulations can outperform numerical integration techniques when the problem involves high dimensions. However, the exact number of dimensions at which Monte Carlo techniques start to outperform deterministic methods (Fourier expansion methods) is generally unknown and depends on the problem at hand[1].

One application of our approach is to allow us to simulate from the characteristic function directly. Existing methods involve a discretization and/or a truncation and therefore a bias. Standard inversion techniques usually require the discretization of an integral (Abate and Whitt (1995), Weeks (1966)). Some simulation techniques have made use of the saddlepoint approximation (Carr and Madan (2009), Lewis (2000), McLeish (2013)) but they are biased estimates because of truncation. Our Monte Carlo approach is unbiased when the support of the distribution is finite and is nearly unbiased otherwise. It can be applied to problems involving the inverse of a characteristic function in which the characteristic function can be efficiently evaluated. In many financial market models, the characteristic function of the log stock price is known, for example in affine stochastic volatility models (Duffie, Pan and Singleton (2000)), with time-changed Lévy processes (Carr and Wu (2004)), or in affine stochastic volatility combined with affine stochastic interest rate models (van Haastrecht and Pelsser (2011a, 2011b)). We will illustrate our technique with

[1]For example, Genz and Malik (1980) show that 8 dimensions is the turning point after which the deterministic Genz-Malik rules are beaten by the Monte Carlo method.

the pricing of standard call options and forward-starting options in the Heston stochastic volatility model.

Another application of our method is to simulate first passage times directly without simulating the trajectories of the underlying processes. This is especially useful when the characteristic function of the first passage time has a simple expression but its probability density function is complicated. For example, the characteristic function of the first hitting time of an Ornstein-Uhlenbeck process to a given level is known (Alili, Patie and Pedersen (2005)). Similarly the characteristic function of the "Parisian time"[2] has a simple closed-form expression in the Black-Scholes framework. We are then able to get an unbiased simulation of this first passage time and also give the price of a Parisian option. Similar as barrier option prices (Broadie, Glasserman and Kou (1997)), Parisian option prices obtained by Monte Carlo simulations are very sensitive to the discretization step used in the simulations of the trajectories of the underlying (Bernard and Boyle (2011)). In this chapter we illustrate our study with the pricing of continuously monitored Parisian options. We also show that it can easily be extended to Parisian options with multiple levels, which requires multidimensional integrations and can be handled through Monte Carlo simulations easily. These multi-level Parisian options have recently appeared in CEO compensation packages.

In the option pricing literature, especially for multi-dimensional option pricing problems, several authors have proposed the use of (deterministic) Fourier approaches, see Dempster and Hong (2000), Fang and Oosterlee (2008), Jackson, Jaimungal and Surkov (2008), Leentvaar and Oosterlee (2008), and Ruijter and Oosterlee (2012). The above papers all use efficient numerical techniques (by deterministic Fourier expansion methods) to find option prices. Alternatively, simulation is a general approach often requiring less programming efforts than deterministic numerical techniques. However, numerical methods, carefully adapted to the problem at hand, are usually faster than simulation when we want to estimate a single quantity.

[2]The *Parisian time* is the first time that the underlying process spends more than a given amount of time above (resp. below) a given barrier.

For example, in Table 6.1 of Ruijter and Oosterlee (2012), the convergence of the numerical scheme is reached in milliseconds. The simulation method can not achieve similar speed due to the additional noise in the random numbers. However, the information obtained through numerical methods is often one-dimensional (e.g. option prices or Greeks), whereas the simulation allows the estimation of a number of parameters with a single run, including error estimates. Thus it is hard to make a "direct comparison" between numerical methods and simulation methods. They both have their pros and cons.

Our inversion technique relies on the Fourier inversion formula used in Fang and Oosterlee (2008). However unlike that paper, we do not truncate the Fourier series at an arbitrary fixed number of terms but add an unbiased estimator of the truncation error and are thus able to obtain an unbiased estimate of the inverse of the characteristic function using a very small number of terms of the Fourier series.

In this chapter, the new result, which contributes to the current literature, is as follows: Theorem 5.3.1. It presents a novel randomization idea applied to the unbiased estimation of the density function, and is later applied to the construction of unbiased importance sampling weights in our importance sampling Monte Carlo algorithm for estimating option prices.

The chapter is organized as follows. In Section 5.2, we first describe two financial problems that require the inversion of characteristic functions and involve more than one-dimensional integration. The first problem is the pricing of forward-starting options in the Heston model. The second problem is the pricing of Parisian-type options in the Black-Scholes setting (standard Parisian options and multi-level Parisian options). We first show how the no-arbitrage pricing of a series of multi-level Parisian options can be reduced to a problem with similar complexity to a standard Parisian option. In this case it can still be argued that deterministic methods (Fourier expansion methods) are more appropriate in that they would give a faster answer (although slightly biased). These multi-level Parisian options are indeed of practical importance as they appear in the design of recent executive stock options (see Bernard and Boyle (2011) and Bernard and Le Courtois (2012)). The pricing and risk management of executive stock options are different from the no-arbitrage pricing of traded options. Next, we present the Fourier inversion technique by Monte Carlo in Section 5.3. In Section 5.4, we illustrate this approach by solving the two original problems presented in Section 5.2. Section 5.5 concludes the chapter.

5.2 Option pricing

The purpose of this chapter is to propose a nearly unbiased inversion of characteristic functions by Monte Carlo simulations. This section presents two pricing problems for which it is difficult to obtain unbiased estimates. First we look at pricing in the Heston stochastic volatility model. We start by a standard call option and then price a forward-starting option which is a mildly path-dependent option that depends on two dates. We then solve the problem of pricing continuously monitored Parisian options in the Black-Scholes setting. Standard Parisian options and multi-level Parisian options are considered. These two applications are here for the purpose of illustration. Our technique can be applied to problems involving the inversion of a characteristic function that can be easily evaluated and when the corresponding cumulative distribution function is unknown or difficult to invert.

5.2.1 Option pricing in the Heston stochastic volatility model

In the Heston model, the dynamics of the stock price S_t and its variance V_t can be written under a risk neutral probability as follows

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t(\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)), \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_1(t), \end{aligned} \tag{5.1}$$

where $V_0 > 0$, $\kappa > 0$, $\theta > 0$, $\sigma > 0$, where W_1 and W_2 are independent Brownian motion processes, and $-1 \leqslant \rho \leqslant 1$ is the correlation between the Brownian motion processes driving the stock price process and its stochastic variance process. In this model, the process V_t is a Feller or Cox-Ingersoll-Ross process (Cox, Ingersoll and Ross (1985)).

Forward-starting options

A forward-starting option is the advance purchase of a put or a call option with a strike price that will be determined at a later date. A forward starting option becomes active at a specified date in the future. Its premium is paid in advance. The time to expiration and the factor K in the strike are established at the time the forward-starting option is purchased. Typically a forward-starting call option written on an underlying S has a terminal payoff $(S_{T_2} - KS_{T_1})^+$, where $0 < T_1 < T_2$ and where the contract is issued at time 0.

Relevant characteristic function

The stochastic differential system (5.1) can be integrated in the following form

$$S_T = S_0 e^{\left(r - \frac{\rho\kappa\theta}{\sigma}\right)T + \frac{\rho}{\sigma}(V_T - V_0) + \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right)\int_0^T V_t dt + \sqrt{\int_0^T V_t dt}\sqrt{1-\rho^2}Z}, \qquad (5.2)$$

where $Z \sim N(0,1)$ is independent of $(V_T, \int_0^T V_s ds)$. This formula can be found in Broadie and Kaya (2006) or in Lemma 2.1 of Bernard and Cui (2011).

Given the expression (5.2), S_T has a lognormal distribution conditional on $(V_T, \int_0^T V_s ds)$. Therefore it is of particular interest to simulate jointly the spot variance V_T at time T and the accumulated variance $\int_0^T V_s ds$ over the period $[0, T]$ to simulate the underlying stock prices and get option prices.

The marginal distribution of V_T is well-known (see Glasserman ((2004), Section 3.4.1)). Broadie and Kaya (2006) provide an expression for the conditional characteristic function of $\int_0^T V_s ds$ given the values V_0 and V_T.

$$\begin{aligned}
\phi(u) =& \mathbb{E}\left[\exp\left(iu\int_0^T V_s ds\right) \,\middle|\, V_0, V_T\right] \\
=& \frac{D(u)e^{-\frac{1}{2}(D(u)-\kappa)T}(1-e^{-\kappa T})}{\kappa(1-e^{-D(u)T})} \cdot \frac{I_\nu\left(\sqrt{V_0 V_T}\frac{4D(u)e^{-\frac{1}{2}D(u)T}}{\sigma^2(1-e^{-D(u)T})}\right)}{I_\nu\left(\sqrt{V_0 V_T}\frac{4\kappa e^{-\frac{1}{2}\kappa T}}{\sigma^2(1-e^{-\kappa T})}\right)} \\
&. \exp\left(\frac{V_0+V_T}{\sigma^2}\left[\frac{\kappa(1+e^{-\kappa T})}{1-e^{-\kappa T}} - \frac{D(u)(1+e^{-D(u)T})}{1-e^{-D(u)T}}\right]\right), \qquad (5.3)
\end{aligned}$$

where $D(u) = \sqrt{\kappa^2 - 2\sigma^2 iu}$, and $\nu = 2\kappa\theta/\sigma^2 - 1$, and $I_\nu(x)$ is the modified Bessel function of the first kind with degrees of freedom ν.

Notice that the characteristic function (5.3) contains the complex logarithm problem because of the presence of $D(u)$. In particular one observes a complex-valued modified Bessel function. If we restrict the logarithm to its principal branch, as it is done in most software packages, the characteristic function can become discontinuous and results in inaccurate option prices. This issue is rigorously analyzed in Lord and Kahl (2010). They propose the following formulation for the characteristic function. Define

$$z(u) = \frac{D(u)e^{-\frac{1}{2}D(u)T}}{1-e^{-D(u)T}},$$

and

$$f(u) = \frac{D(u)}{1-e^{-D(u)T}}.$$

Then to avoid the complex logarithm problem, from Lemma 4.2 and Theorem 4.3 in Lord and Kahl (2010), the characteristic function should be evaluated as

$$\phi(u) \times \frac{\exp(\nu \ln(z(u)))}{z(u)^\nu}, \qquad (5.4)$$

where $\phi(u)$ is given in (5.3) and evaluated using the principal branch for the modified Bessel function. The term $\ln(z(u))$ is evaluated based on the expression $\ln(z(u)) = -\frac{1}{2}D(u)T + \ln(f(u))$, with $\ln(f(u))$ restricted to its principal branch. The denominator of (5.4) is evaluated using the principal branch of the complex power function.

5.2.2 First two moments of $\int_0^T V_s ds|V_0, V_T$

For the first two moments of the aggregate volatility process, we cite the following result from Tse and Wan (2013) (with appropriate modifications to our notation).

Lemma 5.2.1. *(Proposition 3.1, Tse and Wan (2013))*

Let $\delta = 4\kappa\theta/\sigma^2$, $\nu = \sigma/2 - 1$, $C_1 = coth(\kappa T/2)$, $C_2 = csch^2(\kappa T/2)$, $C_z = 2\kappa(\sigma^2 sinh(\kappa T/2))^{-1}$ and $z = C_z\sqrt{V_0 V_T}$. The mean and the variance of $I_c = \int_0^T V_s ds|V_0, V_T$ are

$$k'(0) = \mathbb{E}[I_c] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[\eta]\mathbb{E}[Z]$$
$$k''(0) = Var[I_c] = \sigma^2_{X_1} + \sigma^2_{X_2} + \mathbb{E}[\eta]\sigma^2_Z + (\mathbb{E}[\eta^2] - \mathbb{E}[\eta]^2)\mathbb{E}[Z^2],$$

where

$$\begin{aligned}
\mathbb{E}[X_1] &= (V_0 + V_T)(C_1/\kappa - TC_2/2),\\
\sigma^2_{X_1} &= (V_0 + V_T)(\sigma^2 C_1/\kappa^3 + \sigma^2 T C_2/(2\kappa^2) - \sigma^2 T^2 C_1 C_2/(2\kappa)),\\
\mathbb{E}[X_2] &= \delta\sigma^2(-2 + \kappa T C_1)/(4\kappa^2),\\
\sigma^2_{X_2} &= \delta\sigma^4(-8 + 2\kappa T C_1 + \kappa^2 T^2 C_2)/(8\kappa^4),\\
\mathbb{E}[Z] &= 4\mathbb{E}[X_2]/\delta,\\
\sigma^2_Z &= 4\sigma^2_{X_2}/\delta,\\
\mathbb{E}[\eta] &= zI_{\nu+1}(z)/(2I_\nu(z)),\\
\mathbb{E}[\eta^2] &= z^2 I_{\nu+2}(z)/(4I_\nu(z)) + \mathbb{E}[\eta].
\end{aligned}$$

Pricing options by inversion and conditioning

To obtain an unbiased estimator of the price of a standard call option, we simulate from the couple $(V_T, \int_0^T V_s ds)$ using the formulation of the characteristic function in (5.4) and the method presented in Section 5.3. Moreover using the expression (5.2), conditional on $(V_T, \int_0^T V_s ds)$, the stock price S_T is lognormal and therefore the Black-Scholes option price can be used (see Theorem 2.1 in Bernard and Cui (2011)). This is an important step in reducing the variance of the estimate. The price of a standard call option with maturity T can be expressed as

$$\mathbb{E}\left[C_{BS}\left(\widehat{S}_0, K, r, \widehat{\sigma}, T\right)\right],$$

where C_{BS} is the Black-Scholes formula with initial underlying price

$$\begin{aligned}\widehat{S}_0 :=& \widehat{S}_0\left(V_T, \int_0^T V_s ds\right)\\ =& S_0 \exp\left(rT - \frac{\rho\kappa\theta T}{\sigma} + \frac{\rho}{\sigma}(V_T - V_0) + \left(\frac{\rho\kappa}{\sigma} - \frac{\rho^2}{2}\right)\int_0^T V_t dt\right),\end{aligned}$$

and volatility level $\widehat{\sigma} := \widehat{\sigma}(\int_0^T V_s ds/T) = \sqrt{(1-\rho^2)\int_0^T V_t dt/T}$.

Our method provides an alternative to that presented in Fang and Oosterlee (2008), who simply truncate the Fourier series with sufficiently many terms that a high degree of precision is possible. However the advantages of obtaining an unbiased Monte Carlo simulation are not evident, unless we have a high dimensional problem. It is well-known that the complexity of Monte Carlo techniques, unlike other numerical methods, does not increase with the number of dimensions. For example, an integral over the inverse of the characteristic function, essentially a two-dimensional problem, would theoretically require an infinite number of characteristic functions inversions.

Consider now for example pricing a forward-starting option. The idea is to simulate $(V_{T_1}, \int_0^{T_1} V_s ds)$ and then simulate $(V_{T_2}, \int_{T_1}^{T_2} V_s ds)$ conditional on the first simulation. It obviously involves a two-dimensional integral. More generally, this technique can be extended to more than two dimensions and an iterative unbiased simulation of $(S_{T_1}, S_{T_2}, ..., S_{T_n})$ is possible. Such approach allows to price path-dependent derivatives in the Heston stochastic volatility model, such as discrete Asian options, discrete Lookback options, discrete barrier or Parisian options.

Moreover the inversion proposed in this chapter (Section 5.3) requires only a very limited number of terms in the Fourier series (as small as 3 to 10 terms) instead of comparatively larger terms (16 terms) to get a precise unbiased estimate by the Fourier-cosine series of Fang and Oosterlee (2008).

5.2.3 Parisian options

We now develop a second example where the technique proposed in this chapter is very powerful. It consists of the pricing of Parisian type options and more generally of the unbiased simulation of first passage times for which the characteristic functions are known. A Parisian option is similar to a barrier option but the activation (resp. deactivation) condition is more complex. The underlying process needs not only to reach some given threshold but to stay beyond it for some period of time. As explained by Labart and Lelong (2009), "[a]s for standard barrier options, using simulations leads to a biased problem, due to the choice of the discretization time step in the Monte Carlo algorithm". Using our approach we can obtain unbiased estimates of Parisian option prices without discretizing the entire path of the underlying process.

The simulation technique presented in this chapter requires an explicit expression of the characteristic function of the random variable to be simulated. Here we are interested in Parisian times which are defined below. There are very few models for which their characteristic functions have been derived. In this chapter we choose to work in the Black-Scholes model where expressions for the characteristic functions of Parisian times are available. These characteristic functions are also obtained by Dassios and Wu (2011) when the underlying is a standard compound Poisson process with negative jumps, which is not a very good model for stock prices.

Albrecher et al. (2012) provide a procedure for pricing Parisian options under a jump diffusion model with two sided jumps but they do not provide the characteristic function of the Parisian time directly.

Financial market

To evaluate this option, we assume a Black-Scholes financial market, thus a complete, frictionless, arbitrage-free financial market. Let Q denote the (unique) risk-neutral measure. The underlying stock price S is modeled by the following diffusion:

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dZ_t^Q, \tag{5.5}$$

where Z^Q is a Q-Brownian motion, r is the constant continuously compounded risk-free rate, q the continuous dividend rate and σ the volatility. The solution of (5.5) is $S_t = xe^{\sigma(mt+Z_t^Q)}$ where $m = \frac{1}{\sigma}\left(r-q-\frac{\sigma^2}{2}\right)$ and $x = S_0$. Denote by $\bar{Q}$ the probability measure defined on $\mathcal{F}_T$ by the Radon-Nikodym density :

$$\left.\frac{d\bar{Q}}{dQ}\right|_{\mathcal{F}_T} = \exp\left(-mZ_T^{\bar{Q}} + \frac{m^2T}{2}\right),$$

then $Z_t^{\bar{Q}} = Z_t^Q + mt$ is a $\bar{Q}$-Brownian motion (using Girsanov's theorem). Under $\bar{Q}$, S_t is of the following form and has no drift

$$S_t = xe^{\sigma Z_t^{\bar{Q}}}. \tag{5.6}$$

Up and in Parisian option

To specify a Parisian option, we introduce some additional variables. Let T be the maturity of the option and K its strike price. Let $L > S_0$ be the barrier level and D the sojourn time. The option is activated if the underlying spends more than a time interval D (continuously) above the barrier, L before the maturity T. Since this is an *up* option, we monitor the time spent *above* the barrier. To formulate this, we consider the functional $g_t^L(S)$ which is the last time before t the process S reaches the barrier L:

$$g_t^L(S) = \sup\{s \leqslant t \mid S_s = L\},$$

where we use the usual convention that $\sup\{\emptyset\} = 0$. Note that $g_t^L(S)$ is not a stopping time. We denote by τ, the Parisian time: that is the first time the price remains longer than D units of time above the barrier L

$$\tau = \inf\{t > 0 \mid (t - g_t^L(S))\mathbb{1}_{S_t \geqslant L} \geqslant D\}.$$

These formal definitions are illustrated in Figure 5.1. We show two possible trajectories of the underlying S. To activate the option, the process $(S_t)_{t\in[0,T]}$ starting at $S_0 = 100$ has to stay continuously more than 9 months above the level $L = 180$ in the next three years. In case the Parisian condition is satisfied, g_τ^L is the last time the underlying hits the barrier level L before τ. Note that the dotted trajectory in Figure 5.1 would have activated a standard up and in barrier option with level L but the path does not stay above L long enough to activate the Parisian option.

Mathematical properties

The derivation of the price of a Parisian option requires a few mathematical properties that were originally given by Chesney, Jeanblanc and Yor (1997). Given the expression (5.6) for S_t under $\bar{Q}$, the barrier level for the Brownian motion $Z^{\bar{Q}}$ is given by

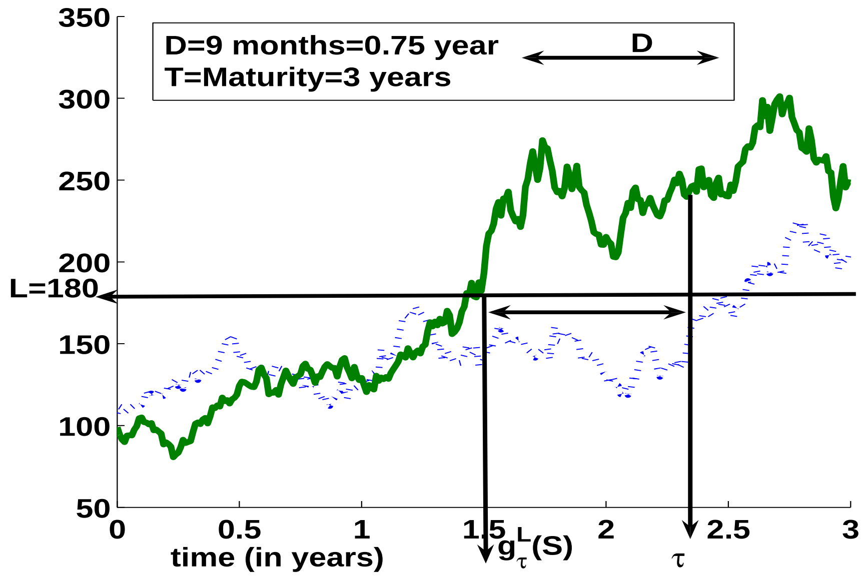
$$\ell = \frac{\ln(L/S_0)}{\sigma}. \tag{5.7}$$

We now recall properties of τ and $Z_\tau^{\bar{Q}}$ under $\bar{Q}$. First τ is a stopping time and τ and $Z_\tau^{\bar{Q}}$ are independent. The distribution of Z_τ under $\bar{Q}$ is given by

$$\bar{Q}\left(Z_\tau^{\bar{Q}} \in dy\right) = \left(\frac{y-\ell}{D}\right)\exp\left(-\frac{(y-\ell)^2}{2D}\right)1_{y>\ell}dy. \tag{5.8}$$

Figure 5.1: Illustration of the Parisian condition

Two possible trajectories of the underlying S. The barrier level $L = 180$. We show the first time the Parisian condition is met: it is indicated by τ.



The standardized density $f(x) = xe^{-x^2/2}$, for $x > 0$ has moment generating function $E(e^{pX}) = 1 + p\sqrt{2\pi}e^{\frac{p^2}{2}}\Phi(p)$ where Φ is the standard cumulative normal distribution function. Therefore $Z_\tau^{\bar{Q}}$ has the distribution of $\ell + \sqrt{D}X$, then $Z_\tau^{\bar{Q}}$ has characteristic function

$$\mathbb{E}\left[e^{iwZ_\tau^{\bar{Q}}}\right] = \left(\frac{L}{S_0}\right)^{\frac{iw}{\sigma}} \Psi(i\sqrt{D}w), \tag{5.9}$$

where $\Psi(z) := 1 + z\sqrt{2\pi}e^{\frac{z^2}{2}}\Phi(z)$.

The characteristic function of τ is given by Labart and Lelong (2009) using the original expression for the Laplace transform of Chesney et al. (1997). Let θ denote $\sqrt{-2iu}$. Labart and Lelong (2009) prove[3] that the

[3]Furthermore, in the Appendix of their chapter, after Lemma B.2 Labart and Lelong (2009) prove that the density of τ exists, is C^∞, that all derivatives ($k \geqslant 0$) of the density f verify $f^{(k)}(t) \to 0$ when t goes to $+\infty$ and that $\mathbb{E}\left[e^{iu\tau}\right] = \mathcal{O}\left(e^{-|\ell|\sqrt{|u|}}\right)$ (see Lemma

characteristic function of the up Parisian time is given by

$$\mathbb{E}\left[e^{iu\tau}\right] = \begin{cases} \frac{e^{-\ell\theta}}{\Psi(\theta\sqrt{D})} = \frac{\left(\frac{S_0}{L}\right)^{\frac{\theta}{\sigma}}}{\Psi(\theta\sqrt{D})} & \text{if } S_0 \leqslant L \\ \\ M(u,\sigma,\theta,S_0,L) & \text{otherwise,} \end{cases} \tag{5.10}$$

where $M(u,\sigma,\theta,S_0,L) = e^{iuD}\left(1 - 2\Phi\left(\frac{\ell}{\sqrt{D}}\right)\right) + \frac{e^{\theta\ell}\Phi\left(\theta\sqrt{D}+\frac{\ell}{\sqrt{D}}\right)+e^{-\theta\ell}\Phi\left(-\theta\sqrt{D}+\frac{\ell}{\sqrt{D}}\right)}{\Psi(\theta\sqrt{D})}$ and where ℓ is itself a function of S_0 and L: $\ell = \frac{\ln(L/S_0)}{\sigma}$. The expressions in (5.10) for the characteristic function factor naturally into several components. For example $\frac{\left(\frac{S_0}{L}\right)^{\frac{\theta}{\sigma}}}{\Psi(\theta\sqrt{D})} = \left(\frac{S_0}{L}\right)^{\frac{\theta}{\sigma}} \times \frac{1}{\Psi(\theta\sqrt{D})}$ is the product of the characteristic function of two independent random variables, one the first passage time from S_0 to L and the second the Parisian time beginning at the barrier L. Similarly $M(u,\sigma,\theta,S_0,L)$ applies when we begin above the barrier $S_0 > L$ (so $\ell < 0$) and consists of the characteristic function of the constant D times $1 - 2\Phi\left(\frac{\ell}{\sqrt{D}}\right)$, the probability that the process remains above the level L for the first D units of time, plus

$$\frac{e^{\theta\ell}\Phi\left(\theta\sqrt{D}+\frac{\ell}{\sqrt{D}}\right)+e^{-\theta\ell}\Phi\left(-\theta\sqrt{D}+\frac{\ell}{\sqrt{D}}\right)}{2\Phi\left(\frac{\ell}{\sqrt{D}}\right)}, \tag{5.11}$$

(which is the conditional characteristic function of the Parisian time given that the first passage to the barrier occurs in the first D units of time), multiplied by the probability of that case, $2\Phi\left(\frac{\ell}{\sqrt{D}}\right)$ again multiplied by the characteristic function $\frac{1}{\Psi(\theta\sqrt{D})}$ of the Parisian time beginning at the barrier L. This decomposition will be of value in the simulations below.

Up and in call option formula

The price of an up and in call option can be expressed as follows

$$C_i^u = e^{-rT}\mathbb{E}_Q\left[(S_T - K)^+ \mathbb{1}_{\tau<T}\right]. \tag{5.12}$$

B.2).

Proposition 5.2.1. *The price of an up and in Parisian call can be calculated as* $\mathbb{E}_{\bar{Q}}\left[h(\tau, Z_\tau)\right]$ *where*

$$h(\tau, Z_\tau) = e^{-\left(r+\frac{m^2}{2}\right)T}\left(\widetilde{x}\Phi\left(\widetilde{d}_1\right) - \widetilde{K}\Phi\left(\widetilde{d}_2\right)\right)\mathbb{1}_{\tau<T}, \tag{5.13}$$

where $\widetilde{x} = xe^{(\sigma+m)Z_\tau+\frac{(\sigma+m)^2}{2}(T-\tau)}$ *and* $\widetilde{K} = Ke^{mZ_\tau+\frac{m^2}{2}(T-\tau)}$ *and where* $\widetilde{d}_1 = \frac{\ln(\widetilde{x}/\widetilde{K})+\frac{\sigma^2(T-\tau)}{2}}{\sigma\sqrt{T-\tau}}$ *and* $\widetilde{d}_2 = \widetilde{d}_1 - \sigma\sqrt{T-\tau}$.

Proof. Using earlier notation, the expression (5.12) becomes

$$C_i^u = e^{-rT}\mathbb{E}_Q\left[\left(xe^{\sigma(mT+Z_T^Q)} - K\right)^+\mathbb{1}_{\tau<T}\right].$$

We can rewrite it under $\bar{Q}$:

$$C_i^u = e^{-rT}e^{-\frac{m^2T}{2}}\mathbb{E}_{\bar{Q}}\left[e^{mZ_T^{\bar{Q}}}\left(xe^{\sigma Z_T^{\bar{Q}}} - K\right)^+\mathbb{1}_{\tau<T}\right],$$

then

$$C_i^u = e^{-\left(r+\frac{m^2}{2}\right)T}\mathbb{E}_{\bar{Q}}\left[\left(xe^{(\sigma+m)Z_T^{\bar{Q}}} - Ke^{mZ_T^{\bar{Q}}}\right)^+\mathbb{1}_{\tau<T}\right].$$

Finally note that $Z_\tau^{\bar{Q}}$ is independent of τ and $Z_T^{\bar{Q}} = Z_\tau^{\bar{Q}} + (Z_T^{\bar{Q}} - Z_\tau^{\bar{Q}})$ is the sum of two independent increments. Denote $\widetilde{x} = xe^{(\sigma+m)Z_\tau+\frac{(\sigma+m)^2}{2}(T-\tau)}$ and $\widetilde{K} = Ke^{mZ_\tau+\frac{m^2}{2}(T-\tau)}$ then

$$\mathbb{E}_{\bar{Q}}\left[\left(xe^{(\sigma+m)Z_T^{\bar{Q}}} - Ke^{mZ_T^{\bar{Q}}}\right)^+|\tau, Z_\tau\right] = \widetilde{x}\Phi\left(\widetilde{d}_1\right) - \widetilde{K}\Phi\left(\widetilde{d}_2\right),$$

where $\widetilde{d}_1 = \frac{\ln(\widetilde{x}/\widetilde{K})+\frac{\sigma^2(T-\tau)}{2}}{\sigma\sqrt{T-\tau}}$ and $\widetilde{d}_2 = \widetilde{d}_1 - \sigma\sqrt{T-\tau}$. (5.13) follows. $\square$

Simulation procedure

The price of a Parisian option is given by $\mathbb{E}_{\bar{Q}}[h(Z_\tau, \tau)\mathbb{1}_{\tau<T}]$, an expectation under $\bar{Q}$ (expression (5.13) of Proposition 5.2.1). It uses two random variables, the Parisian time τ and the value Z_τ of the Brownian motion at that time. To determine the price of a Parisian option, we simulate val-

ues of τ. For each value of τ, $(Z_\tau - \ell)^2$ is exponential with parameter $2D$ where $\ell = \frac{\ln(L/S_0)}{\sigma}$. Note that this simulation procedure does not require the discretization of the trajectories of the underlying stock price and gives an estimate of the Parisian time directly. The specific algorithm will be given in Section 5.4.2.

5.2.4 Multi-level Parisian in executive stock options

As mentioned earlier, Monte Carlo techniques are more useful for multi-dimensional problems. The problems presented so far involve one or two dimensions. However Parisian options can be useful in arbitrarily high dimensional problems. We give an illustration of "multi-levels Parisian options" and their potential use in executive compensation.

Description

In their chapter, Bernard and Boyle (2011) describe the compensation package awarded to Merrill Lynch's CEO, Mr John A. Thain in late 2007. The details of the compensation package can be found in Section 5 of Bernard and Boyle (2011) or originally in a Form 8K filed with the SEC, dated November 16, 2007 (available in the Edgar[4] database). This package consists of several tranches of "Parisian-style" options. We describe a generic package made of three tranches to illustrate how the inversion technique described in this chapter is well-suited to this problem. Assume that it has a maturity of T years and that the initial stock price is S_0. The details of the tranches are as follows. Assume $L_3 > L_2 > L_1 > S_0 = K$.

- Tranche One: A payoff $(S_T - K)^+$ is paid at time T if and only if the stock price stays above the first barrier level L_1 for a period of time D before T. Tranche One is a standard Parisian option.
- Tranche Two: A payoff $(S_T - K)^+$ is paid at time T only if Tranche One is granted before T and in addition if and only if the stock price

[4] http://www.sec.gov/edgar.shtml

stays above the second barrier $L_2 > L_1$ for a period of time D after Tranche One is granted and before T.

- Tranche Three: A payoff $(S_T - K)^+$ is paid at time T only if Tranche Two is granted and in addition if and only if the stock price stays above the third barrier $L_3 > L_2$ for a period of time D after Tranche Two is granted and before T.

The price of the second (resp. third) tranche are definitely lower than standard up-and-in Parisian options because there is an additional condition that needs to be satisfied in order to activate the option, that is that Tranche One (resp. Tranche Two) needs first to be activated. Define τ_1 by

$$\tau_1 = \inf\{t > 0 \mid (t - g_t^{L_1}(S))\mathbb{1}_{S_t \geqslant L_1} \geqslant D\},$$

and τ_i for $i = 2$ and $i = 3$ as follows

$$\tau_i = \inf\{t \geqslant \tau_{i-1} + D \mid (t - g_t^{L_i}(S))\mathbb{1}_{S_t \geqslant L_i} \geqslant D\}.$$

τ_i needs to be higher than $\tau_{i-1} + D$, in other words, $g_t^{L_i}(S) > \tau_{i-1}$, which guarantees that the Tranche i only starts after Tranche $i-1$ is activated. The price of the Tranche i is obtained as

$$\mathbb{E}[e^{-rT}(S_T - K)^+\mathbb{1}_{\tau_i \leqslant T}].$$

Note that when the sojourn time D is equal to zero, Parisian options are standard barrier options. For barrier options, there is no difference between the sequential exercise described above and granting three independent barrier options. Indeed to satisfy the second condition, the underlying needs to go up to level L_2 and thus first pass through the level $L_1 < L_2$. The price of the sequential standard barrier contract is thus equal to the sum of the three barrier options prices. In general $D > 0$ and the pricing of Tranche Three is a real challenge because it seems to involve four random variables S_T, τ_1, τ_2 and τ_3. We now show that the no-arbitrage pricing *per se* of these Tranches is still a one-dimensional problem. However the risk management of these tranches for the company and the valuation by "certainty-equivalent" by the CEO are both multidimensional problems as shown below.

Pricing of multi-level Parisian options

It is not possible to develop a closed-form expression for the characteristic function of the multilevel Parisian option and we are only able to obtain conditional characteristic functions. We deal with only continuously-monitored Parisian options, and in practice Parisian options are usually monitored discretely. A Monte Carlo approach is natural and performs well as it will appear later. More generally the firm may be interested in the distribution of costs to a firm where several such options are offered. This will be a function of $(\tau_1, S_{\tau_1}, \tau_2, S_{\tau_2}, \tau_3, S_{\tau_3})$.

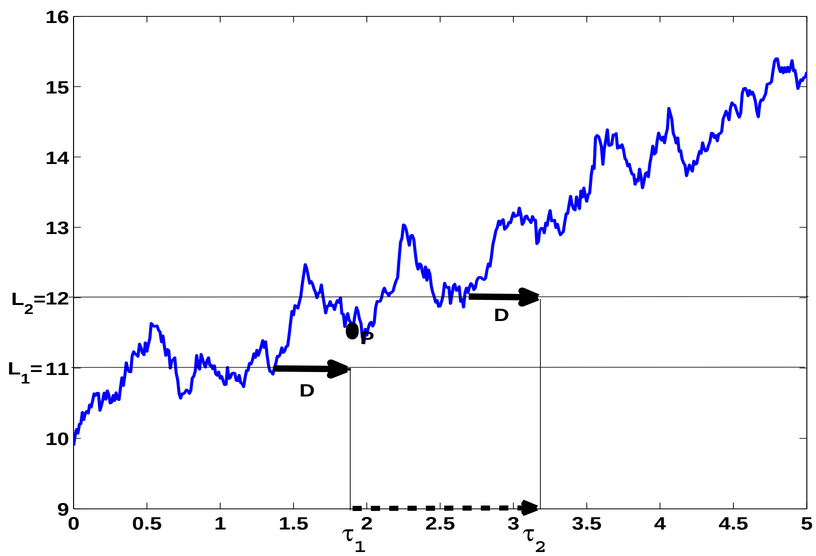


Figure 5.2: Two-level Parisian times

Two-level Parisian times are illustrated in Figure 5.2 with levels $L_1 = 11$ and $L_2 = 12$ and $D = 0.5$. Note that the second arrow illustrating the second tranche must begin to the right of τ_1, the first parisian time. The process renews at the point (τ_1, S_{τ_1}) in that the future of the process after time τ_1 depends only on this point. This implies that the dotted arrow, representing $\tau_2 - \tau_1$, is independent of τ_1 and is another Parisian time. However its distribution depends on S_{τ_1} and more precisely on the ratio between S_{τ_1} and L_2 as illustrated hereafter.

Define $c_i(u) = \mathbb{E}[e^{iu(\tau_i - \tau_{i-1})} | \tau_{i-1}, S_{\tau_{i-1}}]$. Then $c_1(u) = \frac{\left(\frac{S_0}{L_1}\right)^{\frac{\sqrt{-2iu}}{\sigma}}}{\Psi(\sqrt{-2iuD})}$, and

$$\begin{aligned}
c_i(u) &= \mathbb{E}[e^{iu(\tau_i - \tau_{i-1})} \mathbb{1}_{S_{\tau_{i-1}} < L_i} | \tau_{i-1}, S_{\tau_{i-1}}] + \mathbb{E}[e^{iu(\tau_i - \tau_{i-1})} \mathbb{1}_{S_{\tau_{i-1}} \geqslant L_i} | \tau_{i-1}, S_{\tau_{i-1}}] \\
&= \mathbb{E}[e^{iu(\tau_i - \tau_{i-1})} | \tau_{i-1}, S_{\tau_{i-1}}] \mathbb{1}_{S_{\tau_{i-1}} < L_i} + \mathbb{E}[e^{iu(\tau_i - \tau_{i-1})} | \tau_{i-1}, S_{\tau_{i-1}}] \mathbb{1}_{S_{\tau_{i-1}} \geqslant L_i} \\
&= \frac{\left(\frac{S_{\tau_{i-1}}}{L_i}\right)^{\frac{\sqrt{-2iu}}{\sigma}}}{\Psi(\sqrt{-2iuD})} \mathbb{1}_{S_{\tau_{i-1}} < L_i} + M(u, \sigma, \theta, S_{\tau_{i-1}}, L_i) \mathbb{1}_{S_{\tau_{i-1}} \geqslant L_i},
\end{aligned}$$

where $M(\cdot)$ appears in (5.10).

The precise simulation algorithm for multi-level Parisian options is given in Section 5.4.2.

Related multidimensional problems

There are many related problems to the issuance of these CEO compensation that are complex multidimensional problems. First, not all CEOs use the no-arbitrage price to evaluate their compensation package, and some use indifference pricing. Second, the risk management issues for the company which is granting these three tranches to a CEO are more complex than the above pricing.

A company that offers the above package to its CEO might be interested in the distribution of the aggregate payments X

$$X := (S_T - K)^+ \mathbb{1}_{\tau_1 \leqslant T} + (S_T - K)^+ \mathbb{1}_{\tau_2 \leqslant T} + (S_T - K)^+ \mathbb{1}_{\tau_3 \leqslant T}, \qquad (5.14)$$

in order to compute $\mathbb{E}_P(f(X))$ where f is a function (possibly non-linear) and where the expectation is taken under the real probability measure P.

The three cash-flows of the above sum are dependent and X is a function of $(S_T, \tau_1, \tau_2, \tau_3)$. It is straightforward to sequentially simulate the four variables that we are interested in. We start by simulating τ_1, S_{τ_1}, then τ_2, S_{τ_2} given τ_1, S_{τ_1}, finally τ_3, S_{τ_3} given τ_2, S_{τ_2} and finally S_T given S_{τ_3}. We can then estimate any quantity involving the joint distribution of $(S_T, \tau_1, \tau_2, \tau_3)$. This would not be straightforward with deterministic inversion techniques. This is explained for instance in Bernard and Le Courtois (2012): CEOs usually evaluate compensation packages by indifference pricing, finding the amount C which makes them indifferent between receiving the cash amount C and receiving the compensation package. Note that the use of power utility functions is standard for the valuation of executive stock options (see Chance (2009), and Hall and Murphy (2000)). Assume for instance that an executive portfolio contains an initial amount of cash C, n units of stocks S, and m multi-level Parisian packages consisting of three tranches as described above yielding the payoff X at time T given by (5.14). The final expected utility of this manager is given by $\mathbb{E}_P\left(U\left(Ce^{rT} + nS_T + mX\right)\right)$, where $\mathbb{E}_P$ is the expectation in the physical world, r is the risk-free rate and U is the CEO's utility function. The value of the compensation package X is the amount of cash V that should be granted to an executive in order to achieve the same level of expected utility. Therefore, $V(C, n, m, \gamma)$ (denoted hereafter by V) is the solution of the following equation:

$$\mathbb{E}_P\left[U\left(Ce^{rT} + nS_T + mX\right)\right] = \mathbb{E}_P\left[U\left((C+V)e^{rT} + nS_T\right)\right]. \qquad (5.15)$$

The determination of the value V of the compensation package is usually done via Monte Carlo techniques. Since X depends jointly on τ_1, τ_2, τ_3 and S_T, it is clearly a high dimensional problem.

Both the risk management of the cash-flow X in (5.14) and the equation (5.15) are computed under the real probability measure P. Under the real measure P, the underlying stock follows $dS_t/S_t = (\mu - q)dt + \sigma dZ_t^P$, and by the Girsanov theorem, we have the following result

$$\left.\frac{d\bar{Q}}{dP}\right|_{\mathcal{F}_T} = \exp\left(-m_P Z_T^{\bar{Q}} + \frac{m_P^2 T}{2}\right),$$

where $m_P = \frac{\mu - q - \frac{\sigma^2}{2}}{\sigma}$, and where $Z_t^{\bar{Q}} = Z_t^P + m_P t$. Then,

$$\begin{aligned}\mathbb{E}_P\left[f(\tau_1, \tau_2, \tau_3, S_T)\right] &= \mathbb{E}_{\bar{Q}}\left[\exp\left(m_P Z_T^{\bar{Q}} - \frac{m_P^2 T}{2}\right) f(\tau_1, \tau_2, \tau_3, S_T)\right] \\ &= e^{-\frac{m_P^2 T}{2}} \mathbb{E}_{\bar{Q}}\left[\left(\frac{S_T}{x}\right)^{\frac{m_P}{\sigma}} f(\tau_1, \tau_2, \tau_3, S_T)\right].\end{aligned}$$

Therefore all simulations will be done under $\bar{Q}$.

Parisian options as performance based stock options have not only appeared in actual CEO performance packages (as the one granted to the Merrill Lynch CEO in 2007) but have also been proved to have superior properties to the standard CEO compensation packages (Bernard and Le Courtois (2012)). After the scandals of big banks' compensation before their bankruptcy, there is a real need to rethink the design of executive compensation to encourage managers to take the right decisions. It seems that path-dependent packages may be useful in this area. A multi-level Parisian options is only one example.

5.3 Simulation using the characteristic function

Here[5] we provide the main ideas of our simulation method. Assume that we only have information about the characteristic function of the random variable Y, but not its density function $f_Y(.)$. Clearly acceptance-rejection method does not work because we can not bound $f_Y(.)$ without knowing any information about it, and naturally we are led to using importance sampling methods. The advantage of the importance sampling method is that we can simulate from a reference density (usually a simple one, such as the Uniform distribution we choose later), and then attach weights to the simulated values. The weights are constructed from the Radon-

[5]This paragraph is not in the publication Bernard, Cui and McLeish (2012). It is included here to better reflect the goal of our method.

Nikodym derivatives, and the numerator of it contains the $f_Y(.)$, which we do not know. To bypass this difficulty, we use the "randomized importance sampling" (RIS), which requires us to unbiasedly estimate the importance weights. There is a classical Fourier series expansion of the probability density function(see Proposition 5.3.1), and our aim is to introduce a novel randomization to unbiasedly estimate $f_Y(.)$ using random but finite terms of the series. Then we can unbiasedly estimate expectations of functionals of Y.

5.3.1 Distributions with bounded support

Proposition 5.3.1. *Suppose Y is a continuous random variable on the interval $(-\pi, \pi)$, with the probability density function $f_Y(y)$ and the characteristic function*

$$\phi_Y(u) = \phi_1(u) + i\phi_2(u).$$

Then the Fourier expansion of the probability density function of Y is given by

$$\begin{aligned} f_Y(y) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}[\phi_1(n)\cos(ny) + \phi_2(n)\sin(ny)] \qquad (5.16)\\ &= \frac{1}{2\pi} + \frac{1}{\pi}Re\left(\sum_{n=1}^{\infty}\phi_Y(n)e^{-iny}\right), \end{aligned}$$

where $y \in [-\pi, \pi]$.

The proof of Proposition 5.3.1 is standard. See for example formula (4) on p283 of Madan and Seneta (1990) with $a = 0$ and $u = 1$. Pointwise convergence of the Fourier series is a subject of considerable research but for simplicity let us assume that $f_Y \in \mathcal{L}^2$ so that convergence holds at least in $\mathcal{L}^2$ (in fact, the Fourier series converge a.e.[6]). We will assume throughout sufficient smoothness (e.g. piecewise continuous) of the probability density function so that the Fourier series is absolutely convergent, $\sum |\phi_Y(n)| < \infty$.

[6] This is the Carleson Theorem, see Carleson (1966).

Then the limit is continuous and coincides with $f_Y(y)$ almost everywhere (see Rudin (1966), Section 9.4). So in this chapter we will not need to distinguish notationally between the probability density function $f_Y(y)$ and the limit of its Fourier series. Absolute convergence of the Fourier series holds under very weak conditions, for example if f_Y is a function of bounded variation and satisfies a Hölder condition

$$|f_Y(x) - f_Y(y)| \leqslant C\,|x-y|^{\alpha} \text{ for some } \alpha > 0.$$

Our objective is not a precise value of $f(x)$ but an unbiased estimator of it and the following is the main result.

Theorem 5.3.1. *Suppose Y is a bounded random variable $a \leqslant Y \leqslant b$ with piecewise continuous probability density function $f_Y(y)$. Let the characteristic function of Y be*

$$\begin{aligned}\phi_Y(u) &= \mathbb{E}[e^{iuY}] = \phi_1(u) + i\phi_2(u)\\ &= e^{i\theta(u)}\,|\phi_Y(u)|\,, \quad \textit{for a suitable real function } \theta(u),\end{aligned}$$

and assume that $k = \sum_{n=n_0}^{\infty} |\phi_Y(u_n)| < \infty$ where $u_n = \frac{2\pi n}{b-a}$ and $n_0 \in \mathbb{N}^*$ (positive integer). Suppose M is a random variable such that

$$P(M = u_n) = \frac{1}{k}\,|\phi_Y(u_n)| \;\; \textit{for } n \geqslant n_0 \textit{ and } n \in \mathbb{N}^*. \tag{5.17}$$

Then

$$\begin{aligned}\widetilde{f}_Y(y;M) &= \frac{1}{b-a} + \frac{2}{b-a} Re\left(\sum_{n=1}^{n_0-1} \phi_Y\left(u_n\right) e^{-iu_n y}\right) + \frac{2k}{b-a}\cos(My - \theta(M))\\ &= \frac{1}{b-a} + \frac{2}{b-a}\sum_{n=1}^{n_0-1}\left[\phi_1(u_n)\cos(u_n y) + \phi_2(u_n)\sin(u_n y)\right]\\ &\quad + \frac{2k}{b-a}\cos(\theta(M) - My)\end{aligned} \tag{5.18}$$

is an unbiased estimator of $f_Y(y)$ at continuity points y of $f_Y(y)$ with standard error less than $\frac{2k}{b-a}$.

Proof. Define $X = cY - d$ where $c = \frac{2\pi}{b-a}$, $d = \frac{a+b}{b-a}\pi$. Then $-\pi \leqslant X \leqslant \pi$, $\phi_X(u) = e^{-iud}\phi_Y(cu)$ and $\phi_Y(cu) = e^{i\theta(cu)} \left|\phi_Y(cu)\right|$.

$$\begin{aligned} f(x) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\phi_1(n)\cos(nx) + \phi_2(n)\sin(nx)\right] \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\mathrm{Re}\left(\sum_{n=1}^{\infty}\phi_X(n)e^{-inx}\right). \end{aligned}$$

Therefore since $f_Y(y) = cf_X(cy - d)$

$$\begin{aligned} f_Y(y) &= \frac{c}{2\pi} + \frac{c}{\pi}\mathrm{Re}\left(\sum_{n=1}^{\infty}\phi_X(n)e^{-in(cy-d)}\right) \\ &= \frac{c}{2\pi} + \frac{c}{\pi}\mathrm{Re}\left(\sum_{n=1}^{\infty}e^{-ind}\phi_Y(cn)e^{-in(cy-d)}\right) \\ &= \frac{c}{2\pi} + \frac{c}{\pi}\mathrm{Re}\left(\sum_{n=1}^{n_0-1}\phi_Y(cn)e^{-incy}\right) + \frac{ck}{\pi}\mathrm{Re}\left(\sum_{n=n_0}^{\infty}e^{-incy}e^{i\theta(cn)}\frac{|\phi_Y(cn)|}{k}\right) \\ &= \frac{c}{2\pi} + \frac{c}{\pi}\mathrm{Re}\left(\sum_{n=1}^{n_0-1}\phi_Y(cn)e^{-incy}\right) + \frac{ck}{\pi}\mathbb{E}[\mathrm{Re}\left(e^{i(\theta(M)-My)}\right)] \\ &= \frac{c}{2\pi} + \frac{c}{\pi}\mathrm{Re}\left(\sum_{n=1}^{n_0-1}\phi_Y(cn)e^{-incy}\right) + \frac{ck}{\pi}\mathbb{E}[\cos(\theta(M) - My)] \\ &= \frac{1}{b-a} + \frac{2}{b-a}\mathrm{Re}\left(\sum_{n=1}^{n_0-1}\phi_Y(\frac{2\pi n}{b-a})e^{-in\frac{2\pi y}{b-a}}\right) + \frac{2k}{b-a}\mathbb{E}[\cos(\theta(M) - My)] \\ &= \frac{1}{b-a} + \frac{2}{b-a}\sum_{n=1}^{n_0-1}\left[\phi_1(u_n)\cos(u_n y) + \phi_2(u_n)\sin(u_n y)\right] + \frac{2k}{b-a}\mathbb{E}[\cos(\theta(M) - My)]. \end{aligned}$$

$$
\begin{aligned}
Var(\widetilde{f}_Y(y;M)) &= \frac{4k^2}{(b-a)^2} Var(\cos(\theta(M) - My) \\
&= \frac{c^2k^2}{\pi^2} Var[\mathrm{Re}\left(e^{i(\theta(M)-My)}\right)] \\
&\leqslant \frac{c^2k^2}{\pi^2} \mathbb{E}[\mathrm{Re}\left(e^{i(2\theta(M)-2My)}\right)] \\
&\leqslant \frac{c^2k^2}{\pi^2} \mathrm{Re}\left(\sum_{n=n_0}^{\infty} e^{-2incy} e^{2i\theta(cn)} \frac{|\phi_Y(cn)|}{k}\right) \\
&\leqslant \frac{c^2k}{\pi^2} \mathrm{Re}\left(\sum_{n=n_0}^{\infty} |\phi_Y(u_n)|\right) \leqslant \frac{4k^2}{(b-a)^2}.
\end{aligned}
$$

This completes the proof. □

Remark 5.3.1. *Using* $\cos(\theta(M)-My) = \cos(\theta(M))\cos(My)+\sin(\theta(M))\sin(My)$*, we have* $Re\left(e^{i\theta(M)}e^{-iMy}\right) = Re\left(e^{-iMy}\frac{\phi_Y(M)}{|\phi_Y(M)|}\right)$*. In the special case when the distribution is symmetric about* 0*, so that the characteristic function is real,* $\theta(M) = 0$ *if* $\phi_Y(M) > 0$ *and otherwise* $\theta(M) = -\pi$*. In this case*

$$
\begin{aligned}
\cos(My - \theta(M)) &= \cos(My) sgn(\phi_Y(M)) \\
\textit{and } Re\left(\phi_Y(u_n)e^{-iu_n y}\right) &= \phi_Y(u_n)\cos(u_n y).
\end{aligned}
$$

The above expansion in Theorem 5.3.1 does not work very well for Parisian options because the characteristic function $\frac{1}{\Psi(\sqrt{-2iu})}$ of the Parisian times starting at the barrier is not absolutely convergent. An alternative expansion that works better for Parisian options is the Fourier-cosine expansion of the density. This (see Fang and Oosterlee (2008)) alternative expression for a density on the interval $[a, b]$ is:

$$
f(x) \simeq \frac{1}{b-a} + \frac{2}{b-a}\mathrm{Re}\sum_{n=1}^{\infty} \phi(u_n)e^{-iu_n a}\cos(u_n(x-a)) \quad \text{where } u_n = \frac{\pi n}{b-a}. \tag{5.19}
$$

Although this is not exact, the Fourier-cosine expansion is close to the true value of the density when the quantity $h = \frac{\pi}{b-a}$ is small. We choose in this case to unbiasedly estimate the sum of this infinite series using a finite sum. In particular for an integer-valued random variable M such that $P(M \geqslant n) = Q_n$, (5.19) can be unbiasedly estimated using

$$\hat{f}(x) \simeq \frac{1}{b-a} + \frac{2}{b-a}\text{Re}\sum_{n=1}^{M} \frac{\phi(u_n)}{Q_n} e^{-iu_n a}\cos(u_n(x-a)). \tag{5.20}$$

Recall that our objective is not a precise value of $f(x)$ but an unbiased estimator of the importance sampling weight $\breve{w}$, the integerized value of $\frac{f(x)}{\zeta(x)}$. In general we need Q_n to decrease quite slowly to zero so that the variance of $\hat{f}(x)$ is finite. For simplicity, in our Parisian option example we choose a finite minimum and maximum value for the random variable M, $n_{min} \leqslant M \leqslant n_{\max}$ and then, for U generated from the Uniform $\left[\frac{1}{\sqrt{n_{\max}}}, \frac{1}{\sqrt{n_{\min}}}\right]$, set $M = \left\lfloor \frac{1}{U^2} \right\rfloor$. This seems to result in reasonable convergence with $n_{\min} = 5,000$ and $n_{\max} = 20,000$.

5.3.2 Randomized importance sampling

Suppose we sample independent values X_i from a probability density function $\zeta(x)$ such that $\zeta(x) > 0$ whenever $f_Y(x) > 0$. An importance sampling estimator of the expected value of a function $h(Y)$ or $\int h(x) f_Y(x)dx$ is given by

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i)w_i, \text{ where } w_i = \frac{f_Y(X_i)}{\zeta(X_i)}. \tag{5.21}$$

We do not affect the unbiasedness of this estimator if we replace w_i by an unbiased estimator $\hat{w}_i$ of w_i such that $E[\hat{w}_i|X_i] = w_i$ and then estimate the integral using $\frac{1}{n}\sum_{i=1}^{n} h(X_i)\hat{w}_i$. We can easily produce such an unbiased estimator of w_i by replacing $f_Y(X_i)$ in the numerator of (5.21) by $\widetilde{f}_Y(X_i; M_i)$ as defined in Theorem 5.3.1, where M_i is sampled from the distribution (5.17). Attached to the observation X_i is a weight $\hat{w}_i = \frac{\widetilde{f}_Y(X_i;M_i)}{\zeta(X_i)}$. There is often additional advantage to "integerizing" the weights or replacing them by random integers, in part because those observations that end up with weight 0 need not be retained. In fact this is essentially the function of acceptance-rejection: converting importance sampling weights to binary weights and then discarding those which end up with weight 0. To integerize the weights, replace $\hat{w}_i$ by

$$\breve{w}_i = \breve{w}(X) = \lfloor \hat{w}_i \rfloor + \text{Bern}(\hat{w}_i - \lfloor \hat{w}_i \rfloor), \tag{5.22}$$

where $\text{Bern}(\hat{w}_i - \lfloor \hat{w}_i \rfloor)$ represents a Bernoulli random variable B with $P(B = 1) = \hat{w}_i - \lfloor \hat{w}_i \rfloor = 1 - P(B = 0)$.

We often replace weights $\hat{w}_i$ by their self-normalized analogue, $\hat{w}_i / \sum_j \hat{w}_j$ because then scale factors can be ignored, and of course we can do the same with $\breve{w}_i$. The resulting estimators, though no longer strictly unbiased, are very nearly so, for large sample sizes, and they are still consistent as the number of simulations approaches infinity.

What should we use as a candidate distribution $\zeta(x)$? If X is a random variable on a bounded interval (a, b), the simplest choice is the uniform distribution on this interval $\zeta(x) = \frac{1}{b-a}, a < x < b$, so that the weights are proportional to the estimated density:

$$\hat{w}_i \propto \widetilde{f}_Y(X_i; M_i).$$

Suppose, for an arbitrary function h, we wish to generate an integral or conditional expected value[7] $\mathbb{E}[h(Y)|\alpha < Y < \beta]$ over some interval $[\alpha, \beta] \subset [a, b]$. Choose $n_0 \in \mathbb{N}^*$. Since this random variable is on $[\alpha, \beta]$, a natural candidate of reference random variable is the uniform random variable. Using a uniform $[\alpha, \beta]$ distribution, $\zeta(x) = \frac{1}{\beta-\alpha}$, the acceptance-rejection algorithm is given as follows:

1. Generate $X_i \sim \text{Uniform}[\alpha, \beta]$.

2. Generate M_i from the distribution (5.17). With $u_n = \frac{2\pi n}{b-a}$ for $n \geqslant n_0$. define, for a suitable constant of proportionality

$$\hat{w}_i \propto 1+2 \sum_{n=1}^{n_0-1} [\phi_1(u_n)\cos(u_n X_i) + \phi_2(u_n)\sin(u_n X_i)] + 2k\cos(M_i X_i - \theta(M_i)).$$

[7]Such conditional expectations can be very useful in risk management, e.g. in assessing capital requirements.

3. Define $\breve{w}_i = \breve{w}(X_i) = \lfloor \hat{w}_i \rfloor + \text{Bern}(\hat{w}_i - \lfloor \hat{w}_i \rfloor)$.

4. Estimate $\mathbb{E}[h(Y)|\alpha < Y < \beta]$ using

$$\frac{\sum_i \breve{w}_i h(X_i)}{\sum_i \breve{w}_i}.$$

5.3.3 Boundary effects

Recall that the Fourier series expansion of a continuous function on $[a, b]$ results in a continuous function, periodic with period $b - a$ and fails to converge to the function at the boundary if $f(a) \neq f(b)$. To facilitate truncation, we hope that the probability density function is almost completely supported by a finite interval $[a, b]$ with $f(a) \simeq f(b) \simeq 0$. If an interval $[a, b]$ is chosen which is too small, it may fail to contain the bulk of the mass and introduce edge effects. Choice of a too large interval $[a, b]$ will result in spurious values with both positive and negative weights, adding considerable noise to the simulation. The pricing of forward-starting options as well as the simulation of the Parisian time both require the simulation from a density with infinite support. It may be therefore important to control for boundary effects as we discuss below.

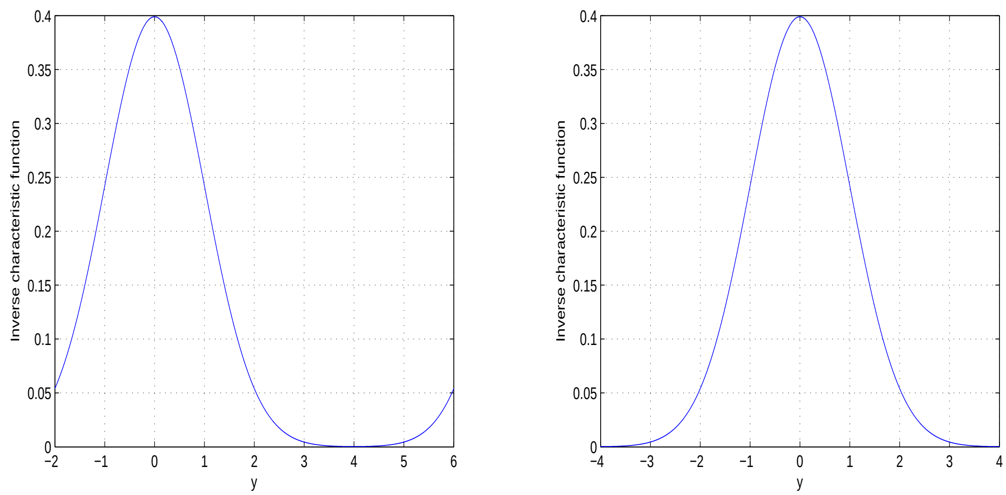
How do we reduce the effect of an arbitrary truncation of the density?

For a simple example consider the standard normal characteristic function $\phi_Y(t) = e^{-t^2/2}$ and suppose we truncate the distribution at $[a, b]$. The expansion (5.16) becomes

$$f_Y(y) \propto 1 + 2\sum_{n=1}^{\infty} e^{-u_n^2/2} \cos(u_n y) \text{ with } u_n = \frac{2\pi n}{b-a}, n = 1, 2, ...$$

These values are plotted below for various choices of a and b in Figure 5.3. When $a = -b$, we get a reasonable representation of the truncated density function but when we choose an asymmetric interval such as $[-2, 6]$ (see

Panel A in Figure 5.3) one which contains almost 98% of the mass of the distribution, the edge effects are apparent, and indeed near 6 the Fourier approximation to the density increases again. This problem vanishes in Panel B of Figure 5.3 if we choose the interval $[-4, 4]$, one which contains a mass much closer to 1 and in this case $f_Y(b) = f_Y(-a)$.



Panel A Panel B

Figure 5.3: Panel A represents the inverse of the standard normal characteristic function on the interval $[-2, 6]$, Panel B represents the inverse of the standard normal characteristic function on the interval $[-4, 4]$.

A common approach to reducing boundary effects in time series analysis is to taper the signal (see Brillinger (1975)) or to artificially introduce a convolution $f * g$ and while this usually improves the mean squared estimates, it also introduces bias, something that is undesirable when a large but unknown number of simulations is contemplated. We can deal with this problem by taking some care in the selection of the interval $[a, b]$, and sometimes transforming the distribution to one with a smaller interval of support. An automated choice of interval $[a, b]$ based on the estimated distribution can be used, but special care is needed for heavy-tailed distributions. This issue is further discussed in the next section.

We use this technique to invert the conditional characteristic function of $\int_0^T V_s ds$ in the pricing of forward-starting options and it works well. The case of the Parisian option is more difficult because the moments of τ are infinite.

5.4 Application to option pricing

5.4.1 Pricing in the Heston model

This section presents numerical results obtained by applying the inversion method presented in the previous section. We first price standard call options in the Heston model since there exists a very accurate technique to price them that we could use for the sake of comparison. We then present prices for forward-starting options (that are mildly path-dependent derivatives).

Standard call options pricing

In Tables 5.1, 5.2 and 5.3, we give the results of $M = 20,000$ simulations of option prices for some given sets of parameters and for a range of values of the volatility parameter σ. We then compare the prices obtained by our inversion method with benchmark prices: the "FO price" is computed by the Cosine-Fourier expansion of Fang and Oosterlee (2008) and the "BK price" refers to prices computed by Broadie and Kaya (2006). We also report the CPU time in minutes. Here Table 5.1 and 5.2 are based on parameters in Broadie and Kaya (2006). Table 5.3 is based on parameters of Table 4 and 5 of Fang and Oosterlee (2008).

For option pricing in the Heston model, when the Feller condition is not satisfied, the evaluation of the relevant characteristic function usually takes substantial CPU time. This is observed in Table 7.1 of Ruijter and Oosterlee (2012), where it can be seen that when the Feller condition is not satisfied, the convergence of the numerical scheme is significantly slower. In the numerical test cases, we check whether the Feller condition is satisfied

σ	MC inversion (std. dev.)	CPU time (minutes)	FO Price	BK Price
100%	35.03 (0.032)	1.684	34.9998	34.9998
42.43%	35.79 (0.040)	1.646	35.7675	NA
30%	35.89 (0.045)	1.626	35.8667	NA
21.21%	35.85 (0.046)	1.637	35.9164	NA

Table 5.1: Prices for a standard call option in the Heston model when the parameters are set as follows: $\kappa = 2$, $V_0 = 0.09$, $\theta = 0.09$, $r = 0.05$, $\rho = -0.3$, $T = 5$. Our inversion technique by Monte Carlo is implemented with 20,000 simulations.

σ	MC inversion (std. dev.)	CPU time (minutes)	FO Price	BK Price
61%	6.82 (0.02)	1.6976	6.8047	6.8061
34.35%	6.91 (0.02)	1.6281	6.9211	NA
17.17%	6.97 (0.03)	1.6341	6.9480	NA

Table 5.2: Prices for a standard call option in the Heston model when $\kappa = 6.21$, $V_0 = 0.010201$, $\theta = 0.019$, $r = 0.0319$, $\rho = -0.7$, $T = 1$. Our inversion technique by Monte Carlo is implemented with 20,000 simulations.

for each set of parameter values. For example, in Table 5.1 here, the first set of parameters, when $\sigma = 1$, we have $2\kappa\theta/\sigma^2 - 1 = -0.64 < 0$ and the Feller condition is not satisfied, and it can be seen from the table that the corresponding CPU time is higher than that of the other cases (but not significantly higher).

T	MC inversion (std. dev.)	CPU time (minutes)	FO Price
1	5.784 (0.013)	1.698	5.7852
10	22.336 (0.038)	1.694	22.3189

Table 5.3: Prices for a standard call option in the Heston model when $\kappa = 1.5768$, $V_0 = 0.0175$, $\theta = 0.0398$, $r = 0$, $\rho = -0.5711$ and $\sigma = 0.5751$. Our inversion technique by Monte Carlo is implemented with 20,000 simulations.

Our method is faster than the Broadie and Kaya (2006) method, which is well-known to be computationally intensive and slow. However, it is much slower than the COS method presented by Fang and Oosterlee (2008). Their method converges with considerable accuracy in a few milliseconds but is based on deterministic numerical techniques (Fourier expansion methods). Numerical techniques normally require more programming and the information obtained is one-dimensional whereas a simulation allows the estimation of a number of parameters with a single run, including error estimates. For the pricing of Heston call options, Fang and Oosterlee (2008)'s approach is certainly superior. This example is to illustrate our inversion technique. The Monte Carlo method cannot be expected to be as computationally efficient as a deterministic approach for a one-dimensional problem. The next section on Parisian options shows that our inversion technique can handle multidimensional problems.

Forward-starting options

For forward-starting option, we take the following table from Table 3 on p245 of Kruse and Nogel (2005). We compare our result using the unbiased simulation with their closed-form formulae by running $1,000$ simulations. They use 170 terms of the series to approximate the Bessel function (see their equation (67)). The pricing results are given in Table 5.4.

K	MC inversion (std.dev.)	Kruse-Nogel price	Crude MC (std.dev.)	CPU time (minutes)
.5	50.51 (0.417)	50.21	50.25 (0.07)	0.88739
.75	27.81 (0.35)	26.95	26.98 (0.06)	0.88427
1	9.25 (0.195)	9.01	9.00 (0.06)	0.87933
1.25	1.10 (0.048)	1.01	1.03 (0.03)	0.87985

Table 5.4: Pricing of a forward-starting call option in the Heston stochastic volatility model. Parameters are set to $S_0 = 100$, $\kappa = 4$, $V_0 = 0.09$, $\theta = 0.06$, $r = 0$, $\rho = -0.9$, $T_1 = 1$, $T_2 = 2$. The strike K is given in the first column.

When the interest rates are also stochastic, van Haastrecht and Pelsser

(2011a) and Ahlip and Rutkowski (2009) both propose expressions for the price of forward starting options. It is straightforward to apply our inversion technique to this more complicated model because they provide the expression for the characteristic function to invert.

5.4.2 Parisian options in the Black-Scholes model

One-level Parisian options

The algorithm, one-level Parisian option, assuming $S_0 \neq L$ Since the distribution of the first passage time to a barrier for geometric Brownian motion is simple, we can conduct our simulation conditional on the event that the first passage time is less than T. We repeat the following for each batch of n simulations:

1. There are two cases depending on the sign of ℓ where ℓ is given by (5.7).

   (a) Case $\ell > 0$, so $S_0 < L$. Randomly generate a Binomial variable n_s with parameters $(n, 1-p)$ where $p = 2\Phi(\frac{\ell}{\sqrt{T}}) - 1$ is the probability that the first passage time (FPT) is greater than T. Then n_s is the number of occasions when $FPT < T$. Repeat n_s times:

      i. generate a random variable τ uniformly distributed on the interval $[D, T]$.

      ii. Evaluate an approximately unbiased estimator $\hat{f}_0(\tau|FPT < T)$ of the conditional density $f_0(\tau|FPT < T)$ at the generated values of τ where f_0 denotes the probability density function of the Parisian times. This is done by using (5.20) to invert the characteristic function

$$\frac{e^{-\theta\ell}\Phi\left(\theta\sqrt{T} - \frac{\ell}{\sqrt{T}}\right) + e^{\theta\ell}\Phi\left(-\theta\sqrt{T} - \frac{\ell}{\sqrt{T}}\right)}{2\Phi(-\frac{\ell}{\sqrt{T}})\Psi(\sqrt{D}\theta)}$$

of the conditional Parisian time given that the first passage time is less than T.

iii. Assign weight $\hat{w} = (T - D)\hat{f}_0(\tau|FPT < T)$ to the observed value of τ. This is the likelihood ratio of the Parisian time probability density $\hat{f}_0(\tau|FPT < T)$ divided by the Uniform$[D, T]$ importance pdf.

iv. For the remaining $n - n_s$ values of τ, since the first passage time is greater than T, so is the Parisian time τ. In these cases, since they will not appear in the pricing of the option, we can assign an arbitrary large value of τ, for example $\tau = 10,000$ and corresponding weight $w = 1$.

(b) If $\ell < 0$, so $S_0 > L$, then we start above the barrier and either stay above, or strike the barrier within D units of time. The probability of staying above the barrier for D units of time is $p = 2\Phi(\frac{|\ell|}{\sqrt{D}}) - 1$.

i. Generate n_s, a binomial random variable with parameters $(n, 1-p)$. This is the number of times the first passage time is less than D.

ii. Repeat n_s times. Generate τ exactly as in part (a) but conditional on the first passage time being less than D, i.e. with characteristic function (5.11)

iii. Assign weight $\hat{w} = (T-D)\hat{f}_0(\tau|FPT < D)$ to the generated value of τ. Again this is the ratio of the estimated Parisian time pdf divided by the Uniform pdf.

iv. If the first passage time is greater than D, then $\tau = D$ so the remaining $n-n_s$ observations are assigned values $\tau = D$ and weights $w = 1$.

2. Divide all weights by n. For arbitrary integrable function g supported on $[0, T]$, $\mathbb{E}(g(\tau))$ can now be unbiasedly estimated by $\sum_i w_i g(\tau_i)$.

3. Generate the stock price S_τ at those Parisian times which are less than or equal to T.

(a) In case 1(a) the probability density function of $Y = \sigma^{-1} \ln(S_\tau/L)$ under the risk neutral measure is given by

$$\left(\frac{y}{D}\right) \exp\left(-\frac{y^2}{2D}\right) \text{ fo } y > 0, \tag{5.23}$$

which implies that S_τ can be generated as $Le^{\sigma\sqrt{-2D\ln(U)}}$, where U is a $U[0,1]$ random variable.

(b) Exactly the same method generates S_τ in case 1(b) when the first passage time $FPT < D$. However, in case $1(b)$ when $FPT > D$, $\tau = D$ and we need to generate S_τ conditional on the path $S_u \geqslant L$ for all $0 < u < D$. If $f_D(x)$ denotes the unconditional probability density function of S_D, then the probability density function of S_D, conditional on staying above the barrier, is proportional to its unconditional density multiplied by a factor, i.e.

$$f_D(x)\left(1 - \left(\frac{L}{x}\right)^\lambda\right), \quad x > L, \text{ where } \lambda = 2\frac{\ln(S_0/L)}{\sigma^2 D}, \tag{5.24}$$

(see e.g. McLeish (2005), p238). It is easy to generate from such a distribution using acceptance-rejection.

4. Generate S_T as $S_T = S_\tau e^{\sigma\sqrt{T-\tau}W}$ for W a standard normal random variable independent of τ.

5. Estimate the Monte Carlo option price of the Parisian up and in option with

$$e^{-\left(r+\frac{m^2}{2}\right)T} \sum_{i=1}^{n} w\left[\left(\frac{S_T}{S_0}\right)^{m/\sigma} (S_T - K)^+ \mathbb{1}_{\tau<T}\right].$$

Numerical results

We use as a benchmark the table in Bernard and Boyle (2011), denoted by "BB price" in Table 5.5 below.

Table 5.5: Prices of continuous Parisian options

Prices of up and in continuous Parisian call options C_i^u for different input parameters. The barrier L is either 120 or 180 and the volatility $\sigma = .15, .30, .45$. Other parameters are set as follows. The initial value of the underlying is $S_0 = 100$, the maturity is $T = 3$, the constant risk-free rate is $r = 4\%$, the continuously compounded dividend yield is equal to $q = 0.4\%$, the sojourn time is $D = 1/12$ and the strike price is set to $K = 100$. 100,000 simulations are used in the inversion by Monte Carlo.

		σ =15%	σ=30%	σ=45%
L=120	BB price	14.02	24.10	33.37
	MC inversion	14.00 (0.03)	24.17 (0.04)	33.31 (0.07)
L=180	BB price	2.132	16.07	28.78
	MC inversion	2.11 (0.02)	16.12 (0.04)	28.86 (0.06)

Multi-level Parisian options

We implement the inversion technique presented in this chapter in an iterative way. We first simulate τ_1, S_{τ_1}, we then simulate τ_2, S_{τ_2} conditional on these observations and then τ_3, S_{τ_3} to finally obtain the price of the third tranche.

We provide values of all three tranches with the following constant barrier levels L_1, L_2 and L_3

$$L_i = K\,(1+\rho)^{1+i}\,, \quad K = S_0, \quad i = 1, 2, 3.$$

Algorithm

We implement the inversion technique presented in this chapter in an iterative way. We are interested in pricing the three tranches of this three-level Parisian option, i.e. finding,

$$\mathbb{E}\left[\left(\frac{S_T}{S_0}\right)^{m/\sigma}(S_T-K)^+\,\mathbb{1}_{\tau_i<T}\right],\ i=1,2,3\ \text{ where } m=\frac{1}{\sigma}\left(r-q-\frac{\sigma^2}{2}\right).$$

The algorithm proceeds as follows.

1. Using the same method as for the one-level Parisian option, simulate τ_1, S_{τ_1} using $L_1 = 112$ and $T = 3$, recording the weight w_1 attached to this simulation. If $\tau_1 > T - D$, then the payoff corresponding to $i = 2$ and 3 will be 0 and the simulation is stopped.

2. If $\tau_1 \leqslant T - D$, conditional on τ_1, S_{τ_1}, we simulate τ_2, S_{τ_2}, w_2 for the second tranche. This is done by repeating the one-level steps with S_0 replaced by S_{τ_1}, L by $L_2 = 125.4$, and T by $T - \tau_1$. The time at which the second tranche is activated is the sum of these two Parisian times, $\tau_1 + \tau_2$. If $\tau_1 + \tau_2 > T$, then the payoff from tranche 2 will be 0 and the simulation is stopped.

3. If $\tau_1 + \tau_2 \leqslant T - D$, then we repeat this process, simulating τ_3, S_{τ_3}, w_3 replacing the initial values or input by S_{τ_2}, L by $L_3 = 140.5$, T by $T - \tau_1 - \tau_2$. The time the third tranche is activated is $\tau_1 + \tau_2 + \tau_3$. If $\tau_1 + \tau_2 + \tau_3 > T$, then the payoff will be 0 and the simulation is stopped. Otherwise, simulate the value of S_T exactly as was done for the one-level Parisian option.

The weight attached to a particular branch of the process $S_{\tau_1}, S_{\tau_2}, S_{\tau_3}, S_T$ is the product $w_1 w_2 w_3$ of the weights associated with each level, w_1, w_2, w_3. Consequently to obtain of the price of the third tranche, for example, we evaluate

$$e^{-\left(r+\frac{m^2}{2}\right)T} \sum_{i=1}^{n} w_1 w_2 w_3 \left[\left(\frac{S_T}{S_0} \right)^{m/\sigma} (S_T - K)^+ \mathbb{1}_{\tau_3 < T} \right].$$

There are many opportunities for variance reduction. For example, the expected value of $\left[\left(\frac{S_T}{S_0} \right)^{m/\sigma} (S_T - K)^+ \mathbb{1}_{\tau_3 < T} \right]$ conditional on the process up to time τ_3 can be expressed using the Black-Scholes formula. We can also replace n_s by its expected value. A control variate involving the first passage time, since it is correlated with the Parisian time, can be used. Since we are more concerned with feasibility than with efficiency, the only concession we make to computational efficiency is to conduct the above simulation sequentially in batches, e.g. n_1 simulations of τ_1, S_{τ_1}, and for each of these values, n_2 simulations of τ_2, S_{τ_2} and finally n_3 simulations of the values $(\tau_3, S_{\tau_3}, S_T)$. Note that the majority of these $n_1 n_2 n_3$ paths may be discarded since many τ_i will be greater than T.

Numerical results

We use the following parameters: $K = S_0 = 100$, $r = 4\%$, $\delta = 0.4\%$, $\rho = 12\%$, $T = 3$ years, and the minimum time D the underlying has to spend above the level L_i is equal to 3 months: $D = 3/12$. We use constant barriers $L_i = K(1+\rho)^{1+i}$, with $\rho = 0.12$, $i = 1, 2, 3$. Then $L_1 = 112, L_2 = 125.4$ and $L_3 = 140.5$. $n = 200$ batches were conducted of $n_1 n_2 n_3 = 80 \times 200 \times 200$ paths. Table 5.6 gives prices for each tranche for different levels of volatility. Since the goal of this chapter is to present unbiased Monte Carlo simulation, we cannot make use of the proposed control variate by Bernard and Boyle (2011) because it is based on an approximation of the price of discrete barrier options and therefore may introduce some bias. Moreover we are simulating the price of "continuously" monitored Parisian options.

Parisian Stock Price			
σ	L_1=112	L_2=125.4	L_3=140.5
15%	14.65 (0.13)	11.53 (0.07)	7.35 (0.05)
30%	24.06 (0.28)	22.07 (0.17)	19.31 (0.15)
45%	32.60 (0.50)	31.25 (0.29)	28.85 (0.28)

Table 5.6: Parisian price with respect to L and to σ estimated by Monte Carlo, and obtained using the inversion technique

5.5 Conclusion of Chapter 5

This chapter presents a novel unbiased inversion of the characteristic function by Monte Carlo simulations. We illustrate the study with the pricing of some standard derivatives, a call option and a forward-starting option in the Heston model as well as Parisian options in the Black-Scholes setting when they are continuously monitored. It can be applied to problems for which the characteristic functions are easily evaluated but the corresponding probability density functions are complicated.

Part IV

Conclusion of the Thesis

This thesis is about martingale properties, probabilistic pricing methods and efficient unbiased Monte Carlo simulation techniques for option pricing problems in stochastic volatility models based on time-homogeneous diffusions. Here are some future research directions corresponding to each chapter of the thesis.

Continuing from Part I Chapter 2, we plan to study deterministic criteria for the martingale properties of time-changed Lévy processes with leverage. This is a general class of models that incorporate many of the popular models in finance. A possible goal is to utilize these deterministic criteria to classify the different notions of arbitrage and to classify different types of stock bubbles in the financial market.

Continuing from Part II Chapter 3, we plan to utilize the first hitting time of an integral functional of a time-homogeneous diffusion and study the pricing of European call and put options written on the discrete realized variance or the continuous quadratic variation. We will also explore theoretical properties of the integrated time-homogeneous diffusions and apply them to drawdowns and drawups, maximal inequalities and optimal stopping problems for diffusions. These properties have important implications in risk management, in American option pricing, and optimal stock selling strategies.

Continuing from Part II Chapter 4, we plan to derive general asymptotic relations for the fair strike of a discrete variance swap, and utilize it to analyze the convergence behavior of the discrete fair strike to the continuous fair strike. We shall also extend our method to derive closed-form formulas and asymptotics of discrete moments swaps, or other exotic discrete volatility derivatives.

Continuing from Part III Chapter 5, we plan to develop and facilitate the adoption of a broader class of models, which are amenable to efficient unbiased Monte Carlo simulation and imputation, for modeling continuous time phenomena in actuarial science and finance. We shall also look at how to debiase an asymptotically unbiased Monte Carlo estimator using similar randomization techniques in McLeish (2011) as applied in Chapter 5. Applications include unbiased simulation of option prices under stochastic volatility models based on time-homogeneous diffusions, and also designing unbiased Multi-level Monte Carlo methods and extending the work of Giles (2008).

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