Core Structures in Random Graphs and Hypergraphs

by

Cristiane Maria Sato

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2013

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The k-core of a graph is its maximal subgraph with minimum degree at least k. The study of k-cores in random graphs was initiated by Bollobás in 1984 in connection to k-connected subgraphs of random graphs. Subsequently, k-cores and their properties have been extensively investigated in random graphs and hypergraphs, with the determination of the threshold for the emergence of a giant k-core, due to Pittel, Spencer and Wormald, as one of the most prominent results.

In this thesis, we obtain an asymptotic formula for the number of 2-connected graphs, as well as 2-edge-connected graphs, with given number of vertices and edges in the sparse range by exploiting properties of random 2-cores. Our results essentially cover the whole range for which asymptotic formulae were not described before. This is joint work with G. Kemkes and N. Wormald. By defining and analysing a core-type structure for uniform hypergraphs, we obtain an asymptotic formula for the number of connected 3-uniform hypergraphs with given number of vertices and edges in a sparse range. This is joint work with N. Wormald.

We also examine robustness aspects of k-cores of random graphs. More specifically, we investigate the effect that the deletion of a random edge has in the k-core as follows: we delete a random edge from the k-core, obtain the k-core of the resulting graph, and compare its order with the original k-core. For this investigation we obtain results for the giant k-core for Erdős–Rényi random graphs as well as for random graphs with minimum degree at least k and given number of vertices and edges.

Acknowledgements

I would like to start by thanking my supervisor Nick Wormald for guiding me during my PhD studies. Hania Wormald also has my gratitude for encouraging me so many times.

Thanks to Jane (Pu) Gao, Xavier Pérez-Giménez and Graeme Kemkes, who had very important roles in my development as a researcher.

Thanks to the faculty and staff in the Department of Combinatorics and Optimization.

Thanks to my dear friends and family.

And last, but not least, thanks to Marcel.

Dedication

To my dear grandmothers.

Table of Contents

List of Figures

Chapter 1

Introduction

Random graphs were first used by Erdős [22] in 1947 to obtain bounds for Ramsey numbers. Erdős and Rényi then published a series of seminal papers on random graphs [23, 24, 25]. Subsequently, random graphs have been widely applied in Combinatorics, Computer Science and other areas, constituting a rich research area of their own. Most of the results for random graphs are asymptotic in nature. That is, the results are actually for sequences of random graphs and we are interested in the asymptotic behaviour of random variables. One of the reasons for seeking asymptotic results is that often the graphs arise from a context where they are naturally large, for example, the World Wide Web, Statistical Mechanics, Bioinformatics, etc. Another reason is that, by focusing on the asymptotic behaviour, one can often perceive a big and elegant picture that would be lost if trying to account for every single graph.

A sequence of events $(E_n)_{n\in\mathbb{N}}$ defined in a sequence of probability spaces $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is said to happen asymptotically almost surely (a.a.s.) if the probability of E_n in $\mathcal{P}_n$ goes to 1 as n goes to infinity. For more information on random graphs, we recommend the books by Bollobás [11], by Janson, Łuczak and Ruciński [37] and by Alon and Spencer [2]. We say that a graph is an (n, m)-*graph* if it has vertex set $[n] = \{1, \ldots, n\}$ and m edges. The two models of random graphs that have been studied the most so far are: the binomial random graph $G(n, p)$, which is the random graph with vertex set $[n]$ in which each possible edge is included independently with probability p; and the Erdős-Rényi random graph $G(n, m)$, which has uniform distribution on all (n, m)-graphs. Asymptotic results from one model can be often translated to the other.

A very classical result in random graphs concerns the emergence of large components. The random graph $G(n, m)$ may be seen as a structure evolving with time. Starting from the graph with vertex set $[n]$ and no edges, add edges one by one until the graph is complete, according to the following rule: at any time, the edge added should be chosen uniformly at random from the edges not yet present. In this random process, $G(n, m)$ corresponds to the graph at the time when m edges have been added. Erdős and Rényi [24] showed that a remarkable phenomenon concerning the components of the graph happens in this process. When $m = m(n) < cn/2$, where c is a constant smaller than 1, a.a.s., every component has $O(\log n)$ vertices. Roughly speaking, every component is small. Then, a qualitative change happens. When $m > cn/2$, where c is a constant greater than 1, the largest component is giant a.a.s., while the second-largest component still has $O(\log n)$ vertices a.a.s. (Here we apply the term 'giant' to a subgraph of a graph with vertex set $[n]$ if it has at least εn vertices, for some positive constant ε.) This means that in a relatively short period of time a significant reshaping has occurred in the random graph: before, the graph was a collection of small components, and, after, a great number of these small components joined forming a single giant component. This is one of the first statements of a threshold-type result in random graphs and the pursuit for thresholds has been proved very attractive ever since. One result by Friedgut [29] states that this threshold phenomenon happens for every monotone property of graphs (see the paper by by Bollobás and Thomason [16] for an earlier result). We remark that Erdős and Rényi's result was later sharpened and many properties of the giant component have been discovered (see [9, 44, 53, 56, 15, 21]).

Afterwards, Bollobás [10] started the investigation of the size of the largest k-connected subgraph in $G(n, m)$. Clearly, any giant k-connected subgraph has to be a subgraph of the giant component and so the question is whether there is a large k-connected subgraph in the giant. Bollobás [10] proved that this is indeed true for $m > Cn$ for a large enough constant C (with no attempt to optimise C made at that time). At this point, he initiated the study of k-cores in random graphs. The k-core of a graph is its maximal subgraph with minimum degree at least k. Bollobás's proof consists of basically two steps: showing that the giant component contains a giant k-core a.a.s. and then showing that such a k-core is k-connected a.a.s. This naturally raises the following question: when does a giant k-core appear and is it k-connected? More specifically, it is of importance whether it appears around the same time when the giant component is born or later. The case of 2-connected graphs was settled before long. Pittel [52] proved that, for $m > cn/2$ where c is a constant greater than 1, the giant component contains a giant 2-connected subgraph.

In the ensuing years, results estimating the moment when the k-core is born were proved for $k \geq 3$ (see [19, 50]). A surprising result by Łuczak [46, 45] states the following: in the random process for $G(n, m)$, a.a.s., the first nonempty k-core that appears is k-connected and has at least $0.0002n$ vertices. In other words, the first nonempty k-core is born giant and k-connected a.a.s. Consequently, by finding a threshold for the appearance of a nonempty k-core, one immediately obtains a threshold for the appearance of a giant k-connected subgraph. We remark that the first nonempty 2-core is quite small: the first nonempty 2-core is simply the first cycle and Janson [34] showed that its length is bounded in probability. Pittel, Spencer and Wormald [54] proved a result estimating the time of emergence of the k-core quite precisely. Roughly speaking, they defined a constant c_k and show that, if the average degree $c = 2m/n$ is below c_k, then the k-core of $G(n, m)$ is empty a.a.s., and if c is above c_k, then the k-core is giant. Even more than that, they estimate quite precisely the number of vertices in the k-core at the moment it emerges. The strategy used by Pittel, Spencer and Wormald was to analyse the behaviour of a deletion procedure applied to $G(n,m)$. This deletion procedure receives a graph and finds its k-core by iteratively deleting vertices of degree less than k until the graph remaining has minimum degree at least k. After the result in [54], the emergence of k-cores has been studied in a number of scenarios and proofs using a variety of techniques were discovered (see [20, 28, 51, 17, 40, 35, 36, 59]).

Properties of k-cores of random graphs have been extensively studied. In some cases, the focus is on the k-core close to the time of its emergence and, in others, the focus is on later stages. Pittel, Spencer and Wormald [54] proved much more than a threshold for the emergence of the k-core and its number of vertices when it emerges: from their results, a concentration result for the number of vertices (and edges) in the k-core during the whole graph process is immediate. From now on, we will also use the expression 'k-cores' to denote graphs with minimum degree at least k, since such graphs are their own k-cores. We say that a graph that is an (n,m)-graph and has minimum degree at least k is an (n,m,k)-*core.* From Łuczak's result that we mentioned before [46, 44], it can easily be deduced that a k-core sampled uniformly from all (n,m,k)-cores is k-connected a.a.s. More recently, Achlioptas and Molloy [1] showed that, for $m \sim Cn/2$ where $C \neq c_k$ is a constant, the k-core of $G(n,m)$ is found in $O(\log n)$ rounds a.a.s. by a deletion procedure for finding the k-core of a graph. This deletion procedure is similar to the one used by Pittel, Spencer and Wormald in [54] that we mentioned before. In this case, in each round, the deletion procedure removes all vertices with current degree less than k from the graph. If, after a round, the graph remaining is a k-core, the deletion procedure ends. (We remark that their result is actually for hypergraphs.) Sometimes the k-core is studied in connection with some other property of interest. For example, as we mentioned before, when k-cores were first studied by Bollobás, he was interested in k-connected subgraphs in random graphs. Recently, Chan and Molloy [18] proved that, for large k and $c \in (c_{k+1}, c_{k+1} + 2\sqrt{k \log k})$, a.a.s., the $(k+1)$-core of $G(n,p)$ with $p = c/n$ either contains a k-factor (that is, a spanning k-regular subgraph) or it is k-factor-critical (that is, the subgraph obtained by removing any vertex from the k-core contains a k-factor) depending on the parity of k times the number of vertices in the k-core. This provides an upper bound for the time of the appearance of a k-regular subgraph, which is a topic that has been receiving attention recently (see [13, 57, 30]).

In Chapter 5, we examine further questions concerning the giant k-core of $G(n,m)$. More precisely, we study robustness aspects of the k-core of $G(n,m)$ for $k \geq 3$ and $2m > c_k n + \omega(n^{3/4})$. We investigate the effect that the deletion of a random edge in the k-core of $G(n,m)$ has on the k-core as such. Consider the following process: after the deletion of a random edge in the k-core of $G(n,m)$, we obtain the k-core of the resulting graph and compare its number of vertices with the original k-core. If the new k-core is much smaller than the original one, it could be said that the original one was not robust as a k-core. We prove that the k-core of $G(n,m)$ is quite robust for $2m > c_k n + \omega(n^{3/4})$. In fact, our results are applicable in a more general setting. We study this process when the k-core is sampled uniformly at random from all (n,m,k)-cores. We define a constant c'_k and analyse the behaviour of the process for $c = 2m/n$ above and below c'_k. We show that, for $c < c'_k - \varepsilon$ and any $h(n) \to \infty$, the new k-core is either empty or has at least $n - h(n)$

vertices a.a.s. Moreover, if $c \to k$, we show that the new k-core is empty a.a.s. For $c > c'_k + \psi(n)$ with $\psi(n) = \omega(n^{-1/4})$ and any $h(n) = \omega(\psi(n)^{-1})$, we prove that the new k-core has at least $n - h(n)$ vertices a.a.s.

As we will see in the next paragraphs, k-cores are also related to enumeration problems. Enumerating graphs with some property of interest is a fundamental problem in graph theory. There are many variants that can be considered. Both labelled and unlabelled graphs have been studied, with more results being obtained for the labelled case. One can consider graphs with given number of vertices, with given number of vertices and edges, with a given degree sequence, etc. The results we mention here all concern the labelled case.

The enumeration of (n, m)-graphs with no isolated vertices, that is, 1-cores, was addressed by Korshunov [42] and Bender, Canfield and McKay [6]. Wright [67, 68, 69, 70] published a series of papers dating from 1977 to 1983 concerning the enumeration of connected (n, m)-graphs (and other graph enumeration problems). He obtained an asymptotic formula for the number of connected (n, m)-graphs when $m - n = o(n^{1/3})$. Bender, Canfield and McKay [7] obtained an asymptotic formula for the number of connected (n, m)-graphs when $m - n \to \infty$. Later, the formula in [7] was rederived with some improvements in the error bounds by Pittel and Wormald [56]. Pittel and Wormald followed a strategy that made use of random 2-cores. Any connected graph can be decomposed into two parts: its 2-core and a rooted forest with the vertices of the 2-core as its roots. A rooted forest with roots $r_1, \ldots, r_t$, which are vertices in the forest, is a forest such that each component contains exactly one root r_i. Pittel and Wormald [56] obtained an asymptotic formula for connected $(n, m, 2)$-cores and combined it with a formula for the number of rooted forests, obtaining an asymptotic formula for the number of connected (n, m)-graphs. In order to obtain a formula for the number of connected $(n, m, 2)$-cores in the sparse range $m - n = o(n)$, Pittel and Wormald [56] described a model of 2-cores with given degree sequence, called the kernel configuration model. Given a degree sequence $\mathbf{d}$ such that each coordinate has value at least 2, let $\mathbf{d}'$ be the restriction of $\mathbf{d}$ to the coordinates of value at least 3. In the kernel configuration model, the first step is to generate a random multigraph, which is called a *kernel*, with degree sequence $\mathbf{d}'$. Then a random 2-core (possibly with loops and multiple edges) without isolated cycles is generated by replacing some edges by paths (that is, by 'inserting' vertices of degree 2 in the edges). Roughly speaking, they reduce the problem of enumerating connected 2-cores to computing the probability that this random 2-core is connected and simple, and they prove that this probability is asymptotic to 1. We remark that the 2-core is connected if and only if its kernel is connected. The strategy in [56] uses the fact that $m - n \to \infty$: if $m - n$ was bounded, the number of vertices in the kernel would also be bounded and, if following the strategy in [56], one would have to determine the exact number of connected kernels with a fixed number of vertices and edges.

The asymptotic enumeration of connected hypergraphs with given number of vertices and edges is still an open problem for some ranges. We say that a hypergraph is an (n, m, k)-*hypergraph* if it is a k-uniform hypergraph with vertex set $[n]$ and m edges. Let the *excess* of the hypergraph be

$m - n/(k-1)$. We use the expression 'excess' because any connected (n, m, k)-hypergraph must have at least $(n-1)/(k-1)$ edges. Karonski and Łuczak [38] obtained an asymptotic formula for the number of connected (n, m, k)-hypergraphs for $m = n/(k-1) + o(\ln n/\ln\ln n)$, which is a range with very small excess. Later Andriamampianina and Ravelomanana [3] extended this result to $m = n/(k-1) + o(n^{1/3})$, which still has very small excess. Behrisch, Coja-Oghlan and Kang [4] provided an asymptotic formula for the case with linear excess $m = n/(k-1) + \Theta(n)$. This means that there is a gap in the ranges for which asymptotic formulae were found: between the case where the excess is $o(n^{1/3})$ and the case with linear excess $\Theta(n)$. The range with superlinear excess $\omega(n)$ is also unsolved. In this thesis, we study 3-uniform hypergraphs: we obtain an asymptotic formula for the number of connected $(n, m, 3)$-hypergraphs for $m = n/2 + R$ as long as R satisfies $R = o(n)$ and $R = \omega(n^{1/3} \ln^2 n)$, which almost fills the gap between the range with very small excess $o(n^{1/3})$ and the range with linear excess. Our technique is based on the approach that Pittel and Wormald [56] used to the enumerate connected sparse 2-cores as we described above. We define the core of a hypergraph as its maximal subhypergraph such that each hyperedge has at least 2 vertices of degree 2 and we show that any connected hypergraph can be decomposed into two parts: a connected core and a rooted forest. We then define a model that generates random 2-cores similarly to the kernel configuration model and analyse the probability that such a 2-core is connected and simple. We expect to extend our results so that we can completely close the gap between the case with very small excess and the linear case for 3-uniform hypergraphs.

The problem of enumerating 2-connected (n, m)-graphs was studied by a number of authors. Efficient methods to compute the exact number of 2-connected (n, m)-graphs were obtained (see Harary and Palmer [32], Temperley [61], and Wormald and Wright [63]), but these methods do not provide a closed formula. In 1978, Wright [68] described an exact formula in the case $m = n + k$ for fixed k. Later Wright [70, 62] found an asymptotic formula for the sparse range $m - n = o(\sqrt{n})$ with $m - n \to \infty$. An asymptotic formula for the dense case can be easily derived from classical results in random graphs. For $m \geq (1/2 + \varepsilon) n \log n$ where ε is a positive constant and any fixed k, the random graph $G(n, m)$ is k-connected a.a.s. (see Erdős and Rényi [25]). Hence, the number of 2-connected (n, m)-graphs is asymptotic to the number of (n, m)-graphs. In Chapter 3, we obtain an asymptotic formula for the number of 2-connected (n, m)-graphs for $m - n \to \infty$ and $m = O(n \log n)$. Thus, we completely close the gap between $m - n = o(\sqrt{n})$ and $m = \Omega(n \log n)$ for which no asymptotic formula was previously known. Our strategy has random 2-cores at centre stage. As we mentioned before, for $m - n = o(n)$, Pittel and Wormald [56] reduced the problem of enumerating connected 2-cores to computing the probability that a random 2-core is connected and simple. Our strategy for the case $m = O(n)$ is similar but we have to compute the probability that such a random 2-core is 2-connected and simple. When $m = \Omega(n)$, we show that the number of 2-connected (n, m)-graphs is asymptotic to the number of (n, m)-graphs that are 2-cores, for which an asymptotic formula is already known [55]. The reason for the condition that $m - n \to \infty$ is the same as the reason for the condition that $m - n \to \infty$ in the approach in [56] for connected $(n, m, 2)$-cores with $m - n = o(n)$: if the $m - n$ was bounded, the number of vertices in the kernel would be bounded. We use the same techniques to obtain an asymptotic formula for the number of 2-edge-connected (n, m)-graphs in the range $m - n \to \infty$ and $m = O(n \log n)$. We are not aware of any previous results on the asymptotic enumeration of 2-edge-connected graphs with given number of vertices and edges.

This thesis is organized as follows. Chapter 2 contains some basic definitions and known results in probability and random graphs that will be used throughout the thesis. In Chapter 3, we prove an asymptotic formula for the number of 2-connected (and 2-edge-connected) (n, m)-graphs. In Chapter 4, we present an asymptotic formula for the number of connected $(n, m, 3)$-hypergraphs in a sparse range. In Chapter 5, we study the topic of robustness of k-cores after the deletion of a random edge. In Chapter 6, we discuss some future research directions. Appendix A contains some Maple spreadsheets. The results in Chapter 3 are joint work with Graeme Kemkes and Nicholas Wormald and the results in Chapter 4 are joint work with Nicholas Wormald. We remark that, at the end of Chapters 3 to 5, we include a glossary of symbols frequently used in each chapter.

We remark the results in Chapter 3 have already been published in the following journal article:

Kemkes, Graeme and Sato, Cristiane M. and Wormald, Nicholas,
Asymptotic enumeration of sparse 2-connected graphs,
Random Structures & Algorithms, Volume 43 (3), pp. 354–376, 2013
Wiley Subscription Services, Inc., A Wiley Company
ISSN 1098-2418, DOI 10.1002/rsa.20415

Chapter 3 is an extended version of this article, including more proofs and details.

Chapter 2

Preliminaries

In this chapter, we describe some known results and methods that will be applied throughout the thesis. We also define asymptotic notations and describe some of the models of random graphs that we will use.

2.1 Asymptotic notation

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be sequences of reals such that $b_n \geq 0$ for all n. We use the following notation:

- $a_n = O(b_n)$, if there exist a positive constant $C \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $|a_n| \leq Cb_n$ for all $n \geq N$;
- $a_n = o(b_n)$, if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $|a_n| \leq \varepsilon b_n$ for all $n \geq N_\varepsilon$.
- $a_n = \Omega(b_n)$, if there exist a positive constant $C \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $a_n \geq Cb_n$ for all $n \geq N$;
- $a_n = \omega(b_n)$, if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $a_n \geq (1/\varepsilon)b_n$ for all $n \geq N_\varepsilon$;
- $a_n = \Theta(b_n)$, if there exist positive constants $C_1, C_2 \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $C_1 b_n \leq a_n \leq C_2 b_n$ for all $n \geq N$;
- $a_n \sim b_n$, if $a_n - b_n = o(b_n)$.

We also use $O(b_n)$ to denote a function a_n without specifying it, and we use the same for $o(\cdot), \Omega(\cdot), \Theta(\cdot)$. For example, if we say that $a_n + O(b_n) = c + o(d_n)$, this means that there exists a function $f_n = O(b_n)$ and a function $g_n = o(d_n)$ such that $a_n + f_n = c_n + g_n$.

Given a sequence of probability spaces $(\mathcal{P}_n)_{n\in\mathbb{N}}$ we say that an event E_n holds asymptotically almost surely (a.a.s.) if the probability of E_n in $\mathcal{P}_n$ goes to 1 as $n \to \infty$.

2.2 Models of random graphs

The two most classical models for random graphs are: the binomial random graph $G(n,p)$ and the Erdős-Rényi random graph $G(n,m)$. For any finite set S, let K_S denote the complete graph with vertex set S. We say that a graph is a (n,m)-*graph* if it has vertex set $[n]$ and m edges. $G(n,p)$ is the random graph with vertex set $[n]$ such that each edge $uv \in K_{[n]}$ is included independently with probability p. $G(n,m)$ is the random graph with uniform distribution over all (n,m)-graphs.

The *allocation model* is a model of random multigraphs with vertex set $[n]$ and m edges: let $a : [2m] \to [n]$ be a function/allocation chosen uniformly at random among all functions mapping $[2m]$ to $[n]$; build a multigraph with vertex set $[n]$ by adding an edge joining $a(i)$ and $a(m+i)$ for every $i \in [m]$. It is easy to check that every simple (n,m)-graph is generated by $m!2^m$ allocations. This implies that this random multigraph conditioned upon simple graphs has the same distribution as $G(n,m)$. The allocation model was used by Bollobás and Frieze [12], and Chvatál [19].

The *k-core* of a graph is its maximal subgraph that has minimum degree at least k. The k-core of a graph is unique, since the union of two subgraphs with minimum degree at least k also is a subgraph with minimum degree at least k. Graphs with minimum degree at least k are also called k-cores, since they are their own k-cores. A graph is a (n,m,k)-*core* if it is a k-core with vertex set $[n]$ and m edges. We work with random k-cores: let $G_k(n,m)$ be the random k-core with uniform distribution on all (n,m,k)-cores. We will use the allocation model restricted to k-cores: let $a : [2m] \to [n]$ be a function/allocation chosen uniformly at random among the functions such that $|a^{-1}(v)| \geq k$ for any $v \in [n]$; let $G^{\text{multi}} = G_k^{\text{multi}}(n,m)$ be the multigraph on $[n]$ obtained by adding an edge joining $a(i)$ and $a(m+i)$ for every $i \in [m]$. Exactly as in the unrestricted model, every (n,m,k)-core is generated by $m!2^m$ allocations. This implies that $G_k^{\text{multi}}(n,m)$ conditioned upon simple graphs has the same distribution as $G_k(n,m)$.

For any graph H, let $\mathbf{d}(H) \in \mathbb{N}^{V(H)}$ denote the degree sequence of H, that is, $\mathbf{d}(H)_v$ is the degree of v in H. Given any $\mathbf{d} \in \mathbb{N}^n$ such that $\sum_{i=1}^n d_i = 2m$ and $d_i \geq k$ for every i, there are $\frac{(2m)!}{\prod_{i=1}^n d_i!}$ allocations $a : [2m] \to [n]$ such that $|a^{-1}(i)| = d_v$ for every $i \in [n]$. Thus, $\mathbf{d}(G_k^{\text{multi}}(n,m))$ has multinomial distribution conditioned upon each coordinate being at least k and

$$\mathbb{P}\left(\mathbf{d}(G_k^{\text{multi}}(n,m)) = \mathbf{d}\right) = \frac{(2m)!}{\prod_{i=1}^n d_i!}\left(\frac{1}{n}\right)^{2m}\frac{1}{A},$$

where A is the ratio between the number of functions $a : [2m] \to [n]$ such that $|a^{-1}(v)| \geq k$ for every $v \in [n]$ and the number of functions from $[2m] \to [n]$ without restrictions. We denote the multinomial distribution with n coordinates summing to $2m$ conditioned upon each coordinate being at least k by $\text{Multi}_{\geq k}(n, 2m)$.

Next we present some models of random graphs with given degree sequence. The *pairing model* or *configuration model* is a standard model of random (multi)graphs with given degree sequence that was first introduced by Bollobás [8]. For $\mathbf{d} \in \mathbb{N}^n$ with $\sum_{i=1}^n d_i = 2m$, create n sets/bins with $d_1, \ldots, d_n$ points inside them; choose a perfect matching on these points uniformly at random among all perfect matchings. This corresponds to a multigraph (possibly containing loops or parallel edges) with degree sequence $\mathbf{d}$ by contracting each bin into a single vertex. We denote the random graph obtained by $G^{\text{multi}}(\mathbf{d})$. We remark that the random graph $G_k^{\text{multi}}(n, m)$ conditioned upon having degree sequence $\mathbf{d}$ has the same distribution as $G^{\text{multi}}(\mathbf{d})$. Given any pairing corresponding to a simple graph, any permutation of the points inside the bins gives another pairing corresponding to the same graph. The following lemma is then straightforward.

Lemma 2.2.1. Let $\mathbf{d} \in \mathbb{N}^n$ be such that $\sum_{i=1}^n d_i = 2m$. Every (n, m)-graph with degree sequence $\mathbf{d}$ is generated by exactly $\prod_{j=1}^n d_j!$ pairings.

This lemma implies that $G^{\text{multi}}(\mathbf{d})$ conditioned upon simple graphs has uniform distribution on the set of all (n, m)-graphs with degree sequence $\mathbf{d}$. Computing the probability that $G^{\text{multi}}(\mathbf{d})$ is simple has been addressed by a number of authors (see [48, 49]). In this thesis, McKay's result [48] will be strong enough for our purposes. Let $\eta(\mathbf{d}) = \sum_{i=1}^n d_i(d_i - 1) / \sum_{i=1}^n d_i$.

Theorem 2.2.2 ([48, Theorem 4.6])**.** Let $\mathbf{d} \in \mathbb{N}^n$ be such that $\sum_{i=1}^n d_i = 2m$. The probability that $G^{\text{multi}}(\mathbf{d})$ is simple is

$$\exp\left(-\eta(\mathbf{d})/2 - \eta(\mathbf{d})^2/4 + O\left(\frac{\max_i d_i^4}{m}\right)\right). \tag{2.1}$$

We remark that [48, Theorem 4.6] is more general than the version stated above as it allows to forbid a set of edges of appearing.

The model we describe next was introduced by Pittel and Wormald [56]. A *pre-kernel* is a graph with minimum degree at least 2 with no components that are cycles. The *kernel* of a pre-kernel is obtained by iteratively deleting a vertex of degree 2 and joining its neighbours by an edge until there are no vertices of degree 2. Note that the kernel can have loops and multiple edges even when the pre-kernel is simple. The *kernel configuration model* is a model of random (multi)graphs with given degree sequence with minimum degree at least 2. This model was introduced by Pittel and Wormald [56] and it has this name because the kernel is generated with the pairing model and then the vertices of degree 2 are added by splitting edges. For each i with $d_i \geq 3$ create a set/bin with d_i points. Choose, uniformly at random, a perfect matching on the union of these sets of points. Assign the remaining numbers $\{i : d_i = 2\}$ to the edges of the perfect matching and, for each edge, choose a linear order for these numbers. The assignment and the linear ordering are chosen uniformly at random. The pairing and assignment (with the linear orderings) are called the *configuration*. A multigraph is then constructed by contracting each bin into a vertex, which produces the *kernel*, and then placing the vertices of degree 2 on the edges of the kernel according to the assignment and linear orderings (which produces the *pre-kernel*).

Given a configuration corresponding to a simple graph, by permuting the points in the bins of the kernel, we obtain another configuration corresponding to the same graph. From this, the following lemma is straightforward.

Lemma 2.2.3 ([56, Corollary 2])**.** Let $\mathbf{d} \in \mathbb{N}^n$ be such that $\sum_{i=1}^n d_i = 2m$ and $d_i \geq 2$ for all i. Each simple pre-kernel with vertex set $[n]$ and degree sequence $\mathbf{d}$ is produced by $\prod_{i \in R(\mathbf{d})} d_i!$ configurations, where $R = R(\mathbf{d}) := \{i \in [n] : d_i \geq 3\}$.

2.3 Hoeffding's inequality

In this section we state Hoeffding's inequality, which is a classical concentration result for a random variable that is the sum of independent random variables.

Theorem 2.3.1 ([33])**.** Let $X_1, \ldots, X_n$ be independent random variables such that $a_i \leq X_i \leq b_i$ for $1 \leq i \leq n$ and let $X = \sum_{i=1}^n X_i$. Then, for $t > 0$,

$$\mathbb{P}\left(|X - \mathbb{E}(X)| \geq tn\right) \leq 2\exp\left(-\frac{2t^2n^2}{\sum_{i=1}^n |b_i - a_i|}\right).$$

2.4 Taylor's approximation

Taylor's approximation is a well-known result that is used to approximate the value of a function in a neighbourhood of a point x by using its power series expansion at x.

For every $n, k \in \mathbb{N}$, let $A(n,k) := \{(a_1, \ldots, a_n) \in \mathbb{N}^n : \sum_{i=1}^n a_i = k,\ a_i \geq 0\}$, that is, $A(n,k)$ is the set of vectors with n coordinates where each coordinate is a nonnegative integer and the sum of the coordinates is k. For a function $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in A_k$, let

$$D^a f(x) = \frac{\partial^k f}{\partial^{a_1} x_1 \ldots \partial^{a_n} x_n}.$$

Theorem 2.4.1 ([31, Theorem 1.23])**.** Let $U \subseteq \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a function with continuous k-th partial derivatives on U. Let $x, y \in U$ be distinct points such that the line segment joining x and y is contained in U. Then there exists a point z in the open line segment joining x and y such that

$$f(y) = \sum_{k'=0}^{k} \sum_{a \in A(n,k')} \frac{D^a f(x)}{\prod_{i=1}^n a_i!} \prod_{i=1}^n (y_i - x_i)^{a_i} + \sum_{a \in A(n,k)} \frac{D^a f(z)}{\prod_{i=1}^n a_i!} \prod_{i=1}^n (y_i - x_i)^{a_i}.$$

2.5 Stirling's approximation

Stirling's approximation is a classical result concerning the asymptotic behaviour of the factorial function (for more information, see [26, Section 2.9]):

Lemma 2.5.1. We have that

$$n! = \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n\Big(1 + O\Big(\frac{1}{n}\Big)\Big).$$

2.6 The subsubsequence principle

The subsubsequence principle is a well known and elementary result about sequence convergence. For more information, see the book [37].

Theorem 2.6.1 (Subsubsequence principle)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $x \in \mathbb{R}$ be a fixed point. If, for every subsequence of $(x_n)_{n\in\mathbb{N}}$, there exists a subsubsequence that converges to x, then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x.

Proof. Let $a = \limsup x_i$. Then there exists a subsequence $(y_i)_{i\in\mathbb{N}}$ of $(x_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} y_i = a$. By hypothesis, there exists a subsequence $(z_i)_{i\in\mathbb{N}}$ of $(y_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} z_i = x$. Since $\lim_{i\to\infty} y_i = a$, this implies that $x = a$. The same argument for $\liminf x_i$, shows that $\liminf x_i = x = \limsup x_i$ and so $\lim_{i\to\infty} x_i = x$. □

We will use this principle a number of times in this thesis. Here is an example of how it can be applied. Suppose we are working with the random graph $G(n, m)$ where m is a function of n, and we prove that some property $A = A_n$ holds a.a.s. for $m = O(n \log n)$ and a.a.s. for $m = \omega(n \log n)$. Then, is it true that the property holds a.a.s. without restrictions on m? The answer is yes and we explain how to use the subsubsequence principle to deduce it. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n = \mathbb{P}(A_n)$ and let $(x_{n_k})_{n_k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$. Then either $\limsup_{k\to\infty} m(n_k)/(n_k \log n_k) = \infty$ or $\limsup_{k\to\infty} m(n_k)/(n_k \log n_k) = \alpha$, where α is some constant. In the first case, there exists a subsequence of $(x_{n_k})_{n_k\in\mathbb{N}}$ such that $m(n_k)/(n_k \log n_k) \to \infty$ as $k \to \infty$ and $\mathbb{P}(A) \to 1$ in this subsequence since $m(n_k) = \omega(n_k \log n_k)$. In the second case, for n large enough $m(n_k)/(n_k \log n_k) \le \alpha$ and $\mathbb{P}(A) \to 1$ as $k \to \infty$ since $m(n_k) = O(n_k \log n_k)$. Thus, by the subsubsequence principle, $(x_n)_{n\in\mathbb{N}}$ also converges to 1, that is, A_n holds a.a.s.

2.7 Uniformity

In this section, we present two simple but very useful lemmas. They are used to deduce uniform error bounds. Many times during this thesis, we follow an approach similar to the one in Pittel and Wormald [56] for sparse graphs: they proved results for random graphs with given degree sequence and then combined these results into a result for a random graph with given number of edges. In order to do that, it was necessary to obtain uniform error bounds in the estimates of the probabilities of certain events for the random graphs with given degree sequence.

Lemma 2.7.1. For every $n \in \mathbb{N}$, let $\mathcal{D}_n$ be a finite set and let w_n be a function mapping $\mathcal{D}_n$ to $\mathbb{R}$. Suppose that there is a function $f_n : \mathcal{D}_n \to \mathbb{R}$ for each $n \in \mathbb{N}$ such that, for every $(d_n)_{n\in\mathbb{N}}$ with $d_n \in \mathcal{D}_n$ for all $n \in \mathbb{N}$, we have that $|w_n(d_n)/f_n(d_n) - 1| \to 0$. Then there exists a function $\phi(n) = o(1)$ such that $|w_n(d_n)/f_n(d_n) - 1| \le \phi(n)$ for every $(d_n)_{n\in\mathbb{N}}$ with $d_n \in \mathcal{D}_n$ for all $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$, since the set $\mathcal{D}_n$ is finite, there is d_n^* such that $|w(d_n^*)/f_n(d_n) - 1| = \max\{|w(d_n)/f_n(d_n) - 1| : d_n \in \mathcal{D}_n\}$. We may let $\phi(n) = |w(d_n^*)/f_n(d_n^*) - 1|$. Then $\phi(n) = o(1)$ because $d_n^* \in \mathcal{D}_n$ for every $n \in \mathbb{N}$. So, for every $(d_n)_{n\in\mathbb{N}}$ such that $d_n \in \mathcal{D}_n$ for all $n \in \mathbb{N}$, we have that $|w(d_n)/f_n(d_n) - 1| \le \phi(n) = o(1)$. □

Here is an example of how this lemma will be applied. Suppose that we define a set $\mathcal{D}_n \subseteq \mathbb{N}^n$ for each n and that we prove that, for any function $d(n)$ such that $d(n) \in \mathcal{D}_n$ for each n, the random graph $G(d(n))$ (that has uniform distribution on all graphs with vertex set $[n]$ and degree sequence $d(n)$) has a property $A = A_n$ a.a.s. Is it true that, upon conditioning the degree sequence of $G(n,m)$ to be in $\mathcal{D}_n$, we have that $G(n,m)$ has property A a.a.s.? The answer is yes and we explain why. For each n the set $\mathcal{D}_n$ is finite and so by Lemma 2.7.1 there exists a function $\phi(n) = o(1)$ such that the probability that A_n holds is at least $1 - \phi(n)$ for all $d(n)$ such that $d(n) \in \mathcal{D}_n$ for each n. Let $\mathbf{d}(G(n,m))$ denote the degree sequence of $G(n,m)$. Then, we can conclude that

$$\begin{aligned}
\mathbb{P}\left(A_n \,\middle|\, \mathbf{d}(G(n,m)) \in \mathcal{D}_n\right) &= \sum_{d\in\mathcal{D}_n} \mathbb{P}\left(A_n \,\middle|\, \mathbf{d}(G(n,m)) = d\right) \mathbb{P}\left(\mathbf{d}(G(n,m)) = d \,\middle|\, \mathbf{d}(G(n,m)) \in \mathcal{D}_n\right) \\
&\ge \sum_{d\in\mathcal{D}_n} (1 - \phi(n))\, \mathbb{P}\left(\mathbf{d}(G(n,m)) = d \,\middle|\, \mathbf{d}(G(n,m)) \in \mathcal{D}_n\right) \\
&= 1 - \phi(n).
\end{aligned}$$

In summary, Lemma 2.7.1 allows us to deduce results for $G(n,m)$ (and other random graphs) from results about the random graph with given degree sequence.

In the next lemma, we show how to handle the case where we obtain only bounds for a function rather than its asymptotic value.

Lemma 2.7.2. For every $n \in \mathbb{N}$, let $\mathcal{D}_n$ be a finite set and let w_n be a function from $\mathcal{D}_n$ to $\mathbb{R}$. Suppose that there is function $f : \mathbb{N} \to \mathbb{R}$ such that, for every $(d_n)_{n\in\mathbb{N}}$ with $d_n \in \mathcal{D}_n$ for all $n \in \mathbb{N}$, we have $w_n(d_n) \geq c(d_n)f(n)$, where $c(d_n) = \Omega(1)$. Then there exists a function $\phi(n) = \Omega(f(n))$ such that $w_n(d_n) \geq \phi(n)$ for every $(d_n)_{n\in\mathbb{N}}$ with $d_n \in \mathcal{D}_n$ for all $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$, since the set $\mathcal{D}_n$ is finite, there is d_n^* that minimises $c(d_n)$. We may let $\phi(n) = c(d_n^*)f(n) = \Omega(f(n))$ since $c(d_n^*) = \Omega(1)$. □

Here is an example of how this lemma will be applied. Suppose that we define a set $\mathcal{M}_n$ integers in $[\binom{n}{2}]$ for each n. Suppose that we prove that the random graph $G(n, m)$ has a property $A = A_n$ with probability $\Omega(1)$, where m is a function of n such that $m(n) \in \mathcal{M}_n$ for all n. Is is true that $\sum_{m\in\mathcal{M}_n} \mathbb{P}(G(n, m) \text{ has } A) = \Omega(|\mathcal{M}_n|)$? The answer is yes and we explain why. For each n the set $\mathcal{M}_n$ is finite and so by Lemma 2.7.2 there exists a function $\phi(n) = \Omega(1)$ such that A holds with probability at least $\phi(n)$ for all m such that $m(n) \in \mathcal{M}_n$ for all n. Thus, $\sum_{m\in\mathcal{M}_n} \mathbb{P}(G(m, n) \text{ has } A) \geq \phi(n)|\mathcal{M}_n| = \Omega(|\mathcal{M}_n|)$.

2.8 Method of moments

The method of moments is a classical method for proving asymptotic convergence of distributions. Basically, it states that if the moments of a variable X_n converge to the moments of a random variable Z, then X_n converges in distribution to Z as long as Z is determined by its moments (not all random variables are determined by their moments). For more information, we recommend the book by Janson, Łuczak and Ruciński [37, Chapter 6]. Here we will only state a special case for Poisson random variables. It is actually a method of factorial moments.

Theorem 2.8.1 ([11, Theorem 1.22])**.** Let $(\lambda_n)_{n\in\mathbb{N}}$ be a bounded sequence of nonnegative reals. Let $(X_n)_{n\in\mathbb{N}}$ be nonnegative integer-valued random variables. Suppose that, for any fixed integer $k \geq 1$,

$$\mathbb{E}\left([X_n]_k\right) = \lambda_n^k + o(1),$$

where $[X_n]_k := (X_n)(X_n - 1)\ldots(X_n - k + 1)$. Then, for all fixed integers $j \geq 0$,

$$\mathbb{P}(X_n = j) = e^{-\lambda_n}\frac{\lambda_n^j}{j!} + o(1).$$

One example of how this method can be used is to show that the distribution of the number of cycles of length k (for fixed $k \geq 3$) for random d-regular graphs is asymptotically Poisson with mean $(d-1)^k/(2k)$ (see [8, 64]).

As one would expect, the hard part of applying Theorem 2.8.1 is estimating the factorial moments. For the summation of indicator random variables the factorial moment can be written in a more friendly way. For $X = X_n = \sum_{i=1}^{t} X_n(i)$, where each $X_n(i)$ is an indicator random variable, one can write the factorial moment $\mathbb{E}([X]_k)$ as follows:

$$\mathbb{E}([X]_k) = \sum_{(i_1,\ldots,i_k)\in\mathcal{I}_k} \mathbb{E}\left(\prod_{j=1}^{k} X(i_j)\right), \tag{2.2}$$

where $\mathcal{I}_k = \{(i_1,\ldots,i_k) \in [r]^k : i_j \neq i_{j'} \text{ for all } j \neq j'\}$. The proof is a very simple induction on k.

2.9 Differential equation method for random graph processes

The basic idea behind differential equation methods is quite intuitive. Suppose we want to analyse the behaviour of some random variables in a random discrete process. One obtains a system of differential equations by writing the expected change in the random variables per unit of time and setting the derivatives to be as suggested by the expected change. The aim is to show that the solution to the system closely approximates the random variables a.a.s. This general approach has been applied in a number of results (see [65]), one of the most successful being the emergence of a giant k-core (see [54, 17]). Wormald [65] described a general-purpose theorem that we will use in this thesis. In preparation for the statement of this result we need a few definitions.

We say that a function $f : \mathbb{R}^j \to \mathbb{R}$ satisfies a *Lipschitz condition* on a set $D \subseteq \mathbb{R}^j$ if there exists a constant L such that

$$|f(u_1,\ldots,u_j) - f(v_1,\ldots,v_j)| \leq L \max_{1\leq i\leq j} |u_i - v_i|,$$

for every $(u_1,\ldots,u_j),(v_1,\ldots,v_j) \in D$. Such constant L is called a *Lipschitz constant* for f in D.

For any set $D \subseteq \mathbb{R}^{j+1}$ and random variables $Y_1(t),\ldots,Y_j(t)$, we define the *stopping time* $T_D = T_D(Y_1(t),\ldots,Y_j(t))$ as the minimum t such that $(t/n, Y_1(t)/n,\ldots,Y_j(t)/n) \notin D$.

For each $n \in \mathbb{N}$, let $S^{(n)}$ be a set. We denote the *history* $(q_0,...,q_t) \in (S^{(n)})^{t+1}$ to time t by h_t. Let $S^{(n)+}$ denote the set of all histories h_t for every $t \in \mathbb{N}$. We are now ready to state Wormald's general-purpose result. In the next statement, upper case letters Y and H are used for the random variables corresponding to the deterministic parameters denoted by their lower case counterparts.

Theorem 2.9.1 ([65, Theorem 5.1])**.** Let a be a positive integer. For $1 \leq \ell \leq a$, let $y_\ell : S^{(n)+} \to \mathbb{R}$ and $f_\ell : \mathbb{R}^{a+1} \to \mathbb{R}$, such that for some constant C_0 and all ℓ, $|y_\ell(h_t)| \leq C_0 n$ for all $h_t \in S^{(n)+}$ for all n. Assume the following three conditions hold, where in (i) and (iii) D is some bounded connected open set containing the closure of

$$\left\{(0,z_1,\ldots,z_a) : \mathbb{P}(Y_\ell(0) = z_\ell n, 1 \leq \ell \leq a) \neq 0 \text{ for some } n\right\}$$

(i) (Boundedness hypothesis) For some functions $\beta = \beta(n) \leq 1$ and $\gamma = \gamma(n)$, the probability that

$$\max_{1 \leq \ell \leq a} |Y_\ell(t+1) - Y_\ell(t)| \leq \beta,$$

conditional upon H_t, is at least $1 - \gamma$ for $t \leq T_D$.

(ii) (Trend hypothesis) For some function $\lambda_1 = \lambda_1(n) = o(1)$, for all $\ell \leq a$

$$\left| \mathbb{E}\left(Y_\ell(t+1) - Y_\ell(t) \middle| H_t\right) - f_\ell\big(t/n, Y_1(t)/n, \ldots, Y_a(t)/n\big) \right| \leq \lambda_1$$

for $t < T_D$.

(iii) (Lipschitz hypothesis) Each function f_ℓ is continuous, and satisfies a Lipschitz condition on

$$D \cap \big\{(t, z_1, \ldots, z_a) : t \geq 0\big\},$$

with the same Lipschitz constant for each ℓ.

Then the following are true.

(a) For $(0, \hat{z}_1, \ldots, \hat{z}_a) \in D$ the system of differential equations

$$\frac{dz_\ell}{dx} = f_\ell(x, z_1, \ldots, z_a), \qquad \ell = 1, \ldots, a$$

has a unique solution in D for $z_\ell : \mathbb{R} \to \mathbb{R}$ passing through

$$z_\ell(0) = \hat{z}_\ell,$$

$1 \leq \ell \leq a$, and which extends to points arbitrarily close to the boundary of D;

(b) Let $\lambda > \lambda_1 + C_0 n\gamma$ with $\lambda = o(1)$. For sufficiently large constant C, with probability $1 - O(n\gamma + \frac{\beta}{\lambda}\exp(-\frac{n\lambda^3}{\beta^3}))$,

$$Y_\ell(t) = nz_\ell(t/n) + O(\lambda n)$$

uniformly for $0 \leq t \leq \gamma n$ and for each ℓ, where $z_\ell(x)$ is the solution in (a) with $\hat{z}_\ell = \frac{1}{n}Y_\ell(0)$, and $\sigma = \sigma(n)$ is the supremum of those x to which the solution can be extended before reaching with ℓ^∞-distance $C\lambda$ of the boundary of D.

Wormald also describes a result that allows the use of additional stopping times:

Theorem 2.9.2 ([65, Theorem 6.1])**.** For any set $\hat{D} = \hat{D}(n) \subseteq \mathbb{R}^{a+1}$, define the stopping time $T_{\hat{D}} = T_{\hat{D}(n)}(Y_1, \ldots, Y_a)$ to be the minimum t such that $(t/n, Y_1(t)/n, ..., Y_a(t)/n) \notin \hat{D}$. Assume that the first two hypotheses of Theorem 2.9.1 apply only with the restricted range $t < T_{\hat{D}}$ of t. Then the conclusions of the theorem hold as before, except with $0 \leq t \leq \sigma n$ replaced by $0 \leq t \leq \min\{\sigma n, T_{\hat{D}}\}$.

2.10 Properties of truncated Poisson random variables

Given a nonnegative real number λ and a nonnegative integer k, we say that a random variable Y is a *truncated Poisson* with parameters (k, λ) if, for every $j \in \mathbb{N}$,

$$\mathbb{P}(Y = j) = \begin{cases} \dfrac{\lambda^j}{j! f_k(\lambda)}, & \text{if } j \geq k; \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

where

$$f_k(\lambda) := e^\lambda - \sum_{i=0}^{k-1} \frac{\lambda^i}{i!}. \tag{2.4}$$

Note that the function f_k plays a normalising role in the probability. We use $\text{Po}(k, \lambda)$ to denote the distribution of a truncated Poisson random variable with parameters (k, λ). A truncated Poisson random variable with parameters $(0, \lambda)$ simply is a Poisson random variable with mean λ.

Truncated Poisson random variables have been used to generate degree sequences for random graphs with minimum degree constraints. Recall that $G_k^{\text{multi}}(n, m)$ is a (n, m, k)-core generated by the allocation model restricted to k-cores as described in Section 2.2. As we discussed in Section 2.2, $\mathbf{d}(G_k^{\text{multi}}(n, m))$ has distribution $\text{Multi}_{\geq k}(n, 2m)$ (multinomial distribution conditioned upon each coordinate being at least k). The following is a well-known relation between $\text{Multi}_{\geq k}(n, 2m)$ and $\text{Po}(k, \lambda)$ (see, for example, [17]):

Lemma 2.10.1. Let k be a nonnegative integer and let λ be a nonnegative real number. The distribution $\text{Multi}_{\geq k}(n, 2m)$ is the same as the distribution of a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ where the Y_i's are independent random variables with distribution $\text{Po}(k, \lambda)$ conditioned upon the event Σ that $\sum_{i=1}^n Y_i = 2m$.

Proof. For any $\mathbf{d} \in \mathbb{N}^n$ with $\sum_{i=1}^n d_i = 2m$,

$$\mathbb{P}\left(\mathbf{Y} = \mathbf{d} \,\middle|\, \Sigma\right) = \frac{1}{\mathbb{P}(\Sigma)} \prod_{i=1}^n \frac{\lambda^{d_i}}{d_i! f_k(\lambda)} = \frac{1}{\mathbb{P}(\Sigma)} \frac{1}{\prod_{i=1}^n d_i!} \frac{\lambda^{2m}}{f_k(\lambda)^n},$$

that is, $\mathbb{P}(\mathbf{Y} = \mathbf{d} \mid \Sigma) = \alpha \cdot 1/\prod_{i=1}^n d_i!$ where α is the same for all $\mathbf{d}$. The probability that a random vector $\mathbf{Z}$ with distribution $\text{Multi}_{\geq k}(n, 2m)$ is $\mathbf{d}$, as already discussed in Section 2.2, is

$$\frac{(2m)!}{\prod_{i=1}^n d_i!} \left(\frac{1}{n}\right)^{2m},$$

which is $\beta \cdot 1/\prod_{i=1}^n d_i!$ where β is the same for all $\mathbf{d}$. Since

$$\sum_{\substack{\mathbf{d} \in \mathbb{N}^n \\ \sum_{i=1}^n d_i = 2m}} \mathbb{P}\left(\mathbf{Y} = \mathbf{d} \,\middle|\, \Sigma\right) = \sum_{\substack{\mathbf{d} \in \mathbb{N}^n \\ \sum_{i=1}^n d_i = 2m}} \mathbb{P}\left(\mathbf{Z} = \mathbf{d}\right) = 1,$$

we must have $\alpha = \beta$. □

We will use the following corollary of Lemma 2.10.1 a number of times in this thesis.

Corollary 2.10.2. Let $m = m(n) \geq kn$ be an integer. The distribution of $\mathbf{d}(G_k^{\text{multi}}(n, m))$ is the same as the distribution of a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ where the Y_i's are independent truncated Poisson random variables with parameters (k, λ) conditioned upon the event Σ that $\sum_{i=1}^{n} Y_i = 2m$.

Pittel and Wormald [55] proved many properties about truncated Poisson random variables. In this section, we include for the reader's convenience several of their results that we use quite extensively in this thesis.

Lemma 2.10.3 ([55, Lemma 1])**.** For any integer k and $c = c(n) > k$, there exists a unique positive root $\lambda(k, c)$ of

$$\frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} = c. \tag{2.5}$$

Moreover, $\lambda(k, c)$ satisfies the following:

(a) If $c \to k$, then $\lambda(k, c) = (k + 1)(c - k) + O((c - k)^2)$;

(b) $\lambda(k, c) \leq c$ always;

(c) if $c \to \infty$, then $\lambda(k, c) \sim c$.

Note that the first part of Lemma 2.10.3 is equivalent to saying that, for any $c > k$, there exists $\lambda > 0$ such that the expectation of a random variable with distribution $\text{Po}(k, \lambda)$ is c.

By continuity, we define $\lambda(k, k) = 0$. For any positive integer k and $c > k$, let

$$\eta_c = \frac{\lambda(k, c) f_{k-2}(\lambda(k, c))}{f_{k-1}(\lambda(k, c))}. \tag{2.6}$$

A trivial relation between η_c and c is:

$$\eta_c \leq c. \tag{2.7}$$

Lemma 2.10.3 is proved by computing $\text{Var}(Y)$, where Y is a truncated Poisson random variable with parameters $(k, \lambda(k, c))$ and relating it to the derivative of $g(\lambda) := \lambda f_{k-1}(\lambda)/f_k(\lambda)$. Pittel and Wormald show that

$$\mathbb{E}\left(Y(Y-1)\right) = c\eta_c \quad \text{and} \quad \text{Var}(Y) = c(1 + \eta_c - c). \tag{2.8}$$

Then, they relate the derivative of $g(\lambda)$ and $\text{Var}(Y)$:

$$\frac{\mathrm{d}\, g(\lambda)}{\mathrm{d}\, \lambda} = \frac{1}{\lambda} \text{Var}(Y) > 0,$$

for $\lambda > 0$. This shows that $g(\lambda)$ is an increasing function. Moreover, as λ goes to 0,

$$\frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} = k + \frac{\lambda}{k+1} + O(\lambda^2) \to k. \tag{2.9}$$

and, as $\lambda \to \infty$, we have that $g(\lambda) \to \infty$. This shows that $\lambda(k, c)$ is an increasing function of c and it is defined for all $c > k$. Thus, Pittel and Wormald [55] proved the following.

Lemma 2.10.4. $\lambda(k, c)$ is a strictly increasing function of c. Moreover, $\lambda(k, c) \to 0$ if $c \to k$, and $\lambda(k, c) \to \infty$ if $c \to \infty$.

The first derivative of $g(\lambda)$ is obviously a continuous function. From this, one obtains the following lemma:

Lemma 2.10.5. Let γ and k be positive integer constants with $\gamma > k$. Let $\alpha(n), \beta(n)$ be function such that $k < \alpha(n) < \beta(n) < \gamma$ and $|\alpha(n) - \beta(n)| = o(\phi)$ where $\phi = o(1)$. Then $|\lambda(k, \alpha) - \lambda(k, \beta)| = o(\phi)$.

By Lemma 2.10.3(a) and (2.9), the following approximations for $\lambda(k, c)$ and η_c are immediate, by using the definition of η_c and computing its series with $\lambda(k, c)$ around 0.

Lemma 2.10.6. For $c \to k$,

$$\lambda(k, c) = (k + 1)(c - k) + O((c - k)^2). \tag{2.10}$$

and

$$\eta_c = 1 + \lambda(k, c)/2 + O(\lambda(k, c)^2). \tag{2.11}$$

Pittel and Wormald use (2.8) to bound the value of the variance of truncated Poisson random variables:

Lemma 2.10.7 ([55, Lemma 2])**.** Uniformly for all $c \in (k, \infty)$, if Y is a truncated Poisson random variable with parameters $(k, \lambda(k, c))$,

$$\mathrm{Var}(Y) = c(1 + \bar{\eta}_c - c) = \Theta(\lambda(k, c)) = \Theta(c - k). \tag{2.12}$$

Moreover, if $c \to k$, then

$$c(1 + \bar{\eta}_c - c) \sim c - k. \tag{2.13}$$

We remark that (2.13) is not stated in [55, Lemma 2], but is part of its proof in [55, Equation (20)].

Usually, when generating degree sequences for random graphs with truncated Poisson random variables with parameters $(k, \lambda(k, c))$, we want the sum of the degrees to be cn (the average degree times n). Pittel and Wormald estimated the probability of this event.

Theorem 2.10.8 ([55, Theorem 4])**.** Let k be a positive integer and let $c = c(n) \geq k$. Let Σ denote the event that $\sum_{i=1}^{n} Y_i = cn$ where $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is a vector of independent truncated Poisson random variables with parameters $(k, \lambda(k, c))$. Let $r = cn - kn$. If $r \to \infty$ and $r = O(n \log n)$, then

$$\mathbb{P}(\Sigma) = \frac{1 + O(r^{-1})}{\sqrt{2\pi nc(1 + \eta_c - c)}}. \tag{2.14}$$

If $r = O(n^{5/2})$, then

$$\mathbb{P}(\Sigma) = \left(1 + O(r^{5/2} n^{-1})\right) e^{-r} \frac{r^r}{r!}. \tag{2.15}$$

For $\varepsilon > 0$,

$$\mathbb{E}\left(\exp(-\eta(\mathbf{Y})/2 - \eta(\mathbf{Y})^2/4) | \Sigma\right) = (1 + O(n^{1/2-\varepsilon})) \exp(-\eta_c/2 - \eta_c^2/4), \tag{2.16}$$

where $\eta(\mathbf{Y}) = \sum_i Y_i(Y_i - 1)/(cn)$.

Let $k \geq 2$, let $\mathcal{D}_k(n, m)$ denote the set of $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ such that $d_i \geq k$ for all $i \in [n]$ and $\sum_i d_i = 2m$. Let

$$Q_k(n, m) = \sum_{\mathbf{d} \in \mathcal{D}_k(n,m)} \prod_{j=1}^{n} \frac{1}{d_j!}.$$

Pittel and Wormald [55, Equation (10)] proved a nice relation between $Q_k(n, m)$ and the event Σ that $\sum_{i=1}^{n} Y_i = cn$ where $\mathbf{Y} = (Y_1, \ldots, Y_n)$, is a vector of independent truncated Poisson random variables with parameters (k, λ) with $\lambda > 0$:

Lemma 2.10.9. Let k be a positive integer and let $c = c(n) \geq k$. Let Σ be the event that $\sum_{i=1}^{n} Y_i = cn$ where $\mathbf{Y} = (Y_1, \ldots, Y_n)$, is a vector of independent truncated Poisson random variables with parameters (k, λ) with $\lambda > 0$. Then

$$Q_k(n, m) = \frac{f_k(\lambda)^n}{\lambda^{cn}} \mathbb{P}(\Sigma). \tag{2.17}$$

To deal with the function $\eta(\mathbf{Y})$ it is useful to know the expectation and the variance of $Y(Y-1)$, where Y is a truncated Poisson random variable with parameters (k, λ). Pittel and Wormald [55] provided estimates for $\mathbb{E}(Y(Y-1))$ and $\mathrm{Var}(Y(Y-1))$. The proof is not so easily extracted from their paper and, for this reason, we will reproduce it here. We also add a bound on $\mathrm{Var}(Y(Y-1))$ for $\lambda = O(1)$.

Lemma 2.10.10. Let Y be a random variable $\mathrm{Po}(k, \lambda)$, where $k \in \mathbb{N}$ and $\lambda = \lambda_n$ is a positive real. For $\lambda_n = o(1)$, we have that $\mathbb{E}(Y(Y-1)) = k(k-1) + 2\lambda k/(k+1) + O(\lambda^2)$ and $\mathrm{Var}(Y(Y-1)) = \Theta(\lambda_n)$. Moreover, $\mathrm{Var}(Y(Y-1)) = O(1)$ for bounded λ.

Proof of Lemma 2.10.10. By (2.8) and by computing the series of $f_{k-2}(\lambda)$ and $f_k(\lambda)$ for $\lambda \to 0$,

$$\begin{aligned}\mathbb{E}\left(Y(Y-1)\right) &= \frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} \cdot \frac{\lambda f_{k-2}(\lambda)}{f_{k-1}(\lambda)} = \frac{\lambda^2 f_{k-2}(\lambda)}{f_k(\lambda)} = \frac{\lambda^2\left(\frac{\lambda^{k-2}}{(k-2)!} + \frac{\lambda^{k-1}}{(k-1)!} + O(\lambda^k)\right)}{\frac{\lambda^k}{k!} + \frac{\lambda^{k+1}}{(k+1)!} + O(\lambda^{k+2})} \\ &= k(k-1) + \frac{2k\lambda}{k+1} + O(\lambda^2)\end{aligned}$$

and so

$$\begin{aligned}\mathrm{Var}(Y(Y-1)) &= \sum_{j\geq k}(j(j-1) - \mathbb{E}\left(Y(Y-1)\right))^2 \frac{\lambda^j}{j!f_k(\lambda)} \\ &= (2k + O(\lambda))^2 \frac{\lambda^{k+1}}{(k+1)!f_k(\lambda)} + \sum_{j\geq k+2}(j(j-1) - k(k-1) + O(\lambda))^2 \frac{\lambda^j}{j!f_k(\lambda)} \\ &= (2k+O(\lambda))^2 \frac{\lambda^{k+1}}{(k+1)!\left(\frac{\lambda^k}{k!} + O(\lambda^{k+1})\right)} + O\left(\sum_{j\geq k+2}(j(j-1))^2 \frac{\lambda^j}{j!\lambda^{k+1}}\right) \\ &= \Theta(\lambda).\end{aligned}$$

The bound $\mathrm{Var}(Y(Y-1)) = O(1)$ follows from the fact that $\mathbb{E}\left(Y^\ell\right) = O(1)$ for any fixed ℓ when $\lambda = O(1)$, which can be obtained by trivial computations. □

Since we use truncated Poisson random variables to generate degree sequences, it will be useful to know how large the maximum degree is likely to be. For any j_0 such that $j_0 > 2e\lambda_c$,

$$\mathbb{P}(Y_i \geq j_0) = \sum_{j\geq j_0} \frac{\lambda_c^j}{j!(e^{\lambda_c} - 1 - \lambda_c)} = O(\exp(-j_0/2)). \tag{2.18}$$

This holds because $j_0 > 2e\lambda_c$ ensures that the ratio between consecutive terms $\lambda_c^j/(j!(e^{\lambda_c} - 1 - \lambda_c))$ and $\lambda_c^{j+1}/((j+1)!(e^{\lambda_c} - 1 - \lambda_c))$ is less than $1/e$ for $j > j_0/2$ and each term is at most 1. (This is the same bound as in [55, (27)].)

Chapter 3

Asymptotic enumeration of sparse 2-connected graphs

In this chapter, we are interested in the enumeration of 2-connected graphs with given number of vertices and edges. As we mentioned in the introduction, efficient methods to compute the exact number of 2-connected graphs with given number of vertices, and given numbers of vertices and edges were described long ago (see Harary and Palmer [32], Temperley [61], and Wormald and Wright [63]). But no closed formula has ever been found. This is one of the reasons why it is worthwhile to seek asymptotic formulae. Another reason is that an asymptotic formula might be much simpler than a closed formula (which may not even exist). The asymptotics in the following are for $n \to \infty$, where n will denote the number of vertices in the graphs in question.

It would be natural to first try to find an asymptotic formula for the number of 2-connected graphs with given number of vertices (instead of given number of vertices and of edges). It turns out this is an easy problem even for k-connected graphs, for any fixed $k \geq 1$. It follows from the well-known fact that almost all graphs with vertex set $[n]$ are k-connected [25]. That is, a graph chosen uniformly at random over all possible graphs with vertex set $[n]$ is k-connected with probability going to 1. An intuitive reason for why this holds is that this random graph is actually $G(n,p)$ with $p = 1/2$, which is a very dense graph. The number of k-connected graphs with vertex set $[n]$ is then asymptotic to the number of simple graphs with vertex set $[n]$, which is simply $2^{\binom{n}{2}}$.

Recall that an (n,m)-graph is any graph with vertex set $[n]$ and m edges. The enumeration of connected (n,m)-graphs has received a lot of attention, with formulae being derived for many ranges of m. One of the best known was proved by Bender, Canfield and McKay [7] whose asymptotic formula works for all $m - n \to \infty$. Their basic approach was to analyse a differential equation arising from a recurrence formula for the number of connected graphs. Afterwards, Pittel and Wormald [56] derived a formula with improved error bounds for some ranges of m. Their proof is somewhat simpler and our approach shares basic features with parts of it, which we will discuss later.

Asymptotic formulae for the number of 2-connected (n, m)-graphs were also found for the ranges $m = n + o(\sqrt{n})$ and $m > (1/2 + \varepsilon)n \log n$, where ε is a positive constant. Wright [70] found an asymptotic formula for the number of 2-connected (n, m)-graphs for the sparse range $m - n = o(\sqrt{n})$ with $m - n \to \infty$. We remark that in Wright's formula there was a constant that Wright did not compute exactly, although he could approximate it very precisely. This constant was later determined by Voblyĭ [62]. Wright [68] also described an exact formula for the number of 2-connected $(n, n + k)$-graphs with fixed k. There is no reason to consider the case $m < n$, since no 2-connected (n, m)-graph exists in this case. For the range $m \geq (1/2 + \varepsilon)n \log n$, where ε is a positive constant, an asymptotic formula for the number of 2-connected (n, m)-graphs can be deduced from known results. For $m \geq (1/2 + \varepsilon)n \log n$, where ε is a positive constant and any fixed k, the random graph $G(n, m)$ is k-connected a.a.s.(see Erdős and Rényi [25]). Hence, the number of 2-connected (n, m)-graphs is asymptotic to the number of simple (n, m)-graphs.

In this chapter, we provide an asymptotic formula for the number of 2-connected (n, m)-graphs with $m - n \to \infty$ and $m = O(n \log n)$. This way, we obtain an asymptotic formula that holds in the entire range for which no asymptotic formula was previously known. We also obtain an asymptotic formula for the number of 2-edge-connected (n, m)-graphs in the same range. The results in this chapter are joint work with G. Kemkes and N. Wormald [39].

Our strategy has random 2-cores at centre stage. Recall that a graph is a (n, m, k)-core if it is an (n, m)-graph that is also a k-core. Every 2-connected graph (with at least 3 vertices) is always a 2-core. In [56], Pittel and Wormald found an asymptotic formula for the number of connected $(n, m, 2)$-cores in the sparse range, as an intermediate step to obtain an asymptotic formula for the number of connected (n, m)-graphs. Roughly speaking, Pittel and Wormald [56] reduced the problem of enumerating connected $(n, m, 2)$-cores to computing the probability that a random $(n, m, 2)$-core with given degree sequence is connected and simple. Our strategy for the case $m = O(n)$ is similar, but we have to compute the probability that such a random 2-core is 2-connected and simple. When $m = \Omega(n)$, we show that the number of 2-connected (n, m)-graphs is asymptotic to the number of $(n, m, 2)$-cores, for which an asymptotic formula was provided by Pittel and Wormald [55]. We remark that the number of 2-cores has been studied prior to that by Wright [68] and others (see e.g. Ravelomanana and Thimonier [58]), but without the generality of the result by Pittel and Wormald [55].

Next we briefly discuss the enumeration of k-connected (n, m)-graphs, with fixed $k \geq 3$. The enumeration of (n, m, k)-cores is related to the enumeration of k-connected (n, m)-graphs since k-connected graphs are always k-cores. An asymptotic formula for the number of k-connected (n, m)-graphs can be easily deduced from the formula for (n, m, k)-cores and some other known results. Łuczak [46] showed that given a degree sequence with minimum degree at least k, under some additional hypotheses on the degree sequence, a graph chosen uniformly at random among all graphs with degree sequence $\mathbf{d}$ is k-connected a.a.s. By showing that these additional hypotheses are innocuous, that is, that the degree sequence of a random (n, m, k)-core satisfies them a.a.s., one can conclude that the random (n, m, k)-core is k-connected a.a.s. Thus, the number of k-connected (n, m)-graphs is asymptotic to the number of (n, m, k)-cores. Interestingly, this strategy fails for 2-connected graphs because it is not true that random 2-cores are 2-connected a.a.s. We also note that one gets an asymptotic formula for the number of k-edge-connected (n, m)-graphs since a k-connected graph is always k-edge-connected and a k-edge-connected graph is always a k-core. That is, the number of k-edge-connected (n, m)-graphs is sandwiched between the number of (n, m, k)-cores and k-connected (n, m)-graphs.

3.1 Main results

Let $T(n, m)$ denote the number of 2-connected (n, m)-graphs (simple graphs with vertex set $[n]$ and m edges). For any positive $k \in \mathbb{N}$, let $f_k(\lambda) = e^\lambda - \sum_{i=0}^{k-1} \lambda^i/i!$ and recall that the function $\lambda(2, c)$ is defined in (2.5) as the unique positive root of $\lambda f_1(\lambda)/f_2(\lambda)$. For any $c > 2$, let

$$\lambda_c = \lambda(2, c), \quad \eta_c = \frac{\lambda_c e^{\lambda_c}}{f_1(\lambda_c)} \quad \text{and} \quad p_c = \frac{\lambda_c^2}{2 f_2(\lambda_c)}.$$

The parameter p_c is the probability that a truncated Poisson random variable with parameters $(2, \lambda_c)$ has value 2. Truncated Poisson random variables simply are Poisson random variables conditioned upon having at least some value (in this case, the value is 2). For the definition of truncated Poisson random variables and some of their properties, see (2.3) and Section 2.10.

Define the odd falling factorial $(2k - 1)!!$ as $(2k - 1)(2k - 3) \cdots 1$, for any integer $k \geq 1$. Throughout this chapter, let $c = 2m/n$ denote the average degree and let $r = 2m - 2n$, which can be seen as an excess function.

Our main result is the following asymptotic formula for $T(n, m)$.

Theorem 3.1.1. Suppose $m = O(n \log n)$ and $r := 2m - 2n \to \infty$. Then

$$T(n, m) \sim (2m - 1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi n c(1 + \eta_c - c)}} \sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right),$$

where $c := 2m/n$.

First we obtain formulae for $T(n, m)$ for three ranges of the average degree: $c \to 2$, bounded $c > 2$, and $c \to \infty$. For each case, we will show that the formulae obtained are asymptotically equivalent to the formula in Theorem 3.1.1. Theorem 3.1.1 is then easily proved using the subsubsequence principle (for more on the subsubsequence principle, see Section 2.6).

Theorem 3.1.2. Suppose $m = O(n \log n)$ and $r := 2m - 2n \to \infty$. Then, for $c := 2m/n$,

(a) if $c \to 2$,

$$T(n,m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi n(c-2)}} \cdot \frac{\sqrt{3r}}{e\sqrt{2m}};$$

(b) if $c = O(1)$ and $c > C_0$ for some constant $C_0 > 2$,

$$T(n,m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi nc(1+\eta_c - c)}} \sqrt{\frac{c-2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right);$$

(c) if $c \to \infty$,

$$T(n,m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi nc}} \exp\left(-\frac{\eta_c}{2} - \frac{\eta_c^2}{4}\right).$$

The proofs for each range in Theorem 3.1.2 follow the same strategy. Computing the value of $T(n,m)$ is the same as computing $\sum_{\mathbf{d}} T(\mathbf{d})$, where $T(\mathbf{d})$ is the number of 2-connected (n,m)-graphs with degree sequence $\mathbf{d}$ and the sum is over all possible degree sequences with $\sum_i d_i = 2m$. We show that computing $T(\mathbf{d})$ can be reduced to computing the probability $A(\mathbf{d})$ that a certain random multigraph with degree sequence $\mathbf{d}$ is 2-connected and simple. Using this we show that approximating $\sum_{\mathbf{d}} T(\mathbf{d})$ can be done by estimating the value of the expectation of a random variable $B(\mathbf{d})$, that is closely related to $A(\mathbf{d})$, when the degree sequence $\mathbf{d}$ is random with a certain distribution.

In order to approximate the expectation, we define a set of 'typical' degree sequences, that is a set containing the random degree sequence a.a.s., and show that for such degree sequences the value of $B(\mathbf{d})$ can be determined a.a.s. with uniform error bounds and that the degrees outside this set have no significant contribution for the expectation. This way, we obtain an asymptotic formula for $T(\mathbf{d})$ for 'typical' degree sequences and an asymptotic formula for $T(n,m)$.

We use $\mathcal{D}(n,m)$ to represent the set of degree sequences $\mathbf{d} := (d_1, \ldots, d_n)$ such that $\sum_{i=1}^n d_i = 2m$ and $d_i \geq 2$ for all $i \in [n]$. For $\mathbf{d} \in \mathcal{D}(n,m)$, define

$$\eta(\mathbf{d}) = \frac{1}{2m} \sum d_j(d_j - 1)$$

and, for every integer j, let $D_j = D_j(\mathbf{d})$ denote $|\{i : d_i = j\}|$, that is $D_j(\mathbf{d})$ is the number of vertices of degree j.

The next result states the asymptotic formula for 'typical' degree sequences in each range.

Theorem 3.1.3. Suppose $m = O(n \log n)$ and $r := 2m - 2n \to \infty$. Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a vector of independent truncated Poisson random variables with parameters $(2, \lambda_c)$.

(a) Suppose further that $c = c(n) := 2m/n \to 2$. Let $\psi(n) = r^{1-\varepsilon}$ for some $\varepsilon \in (0, 1/4)$. If $\mathbf{d} = \mathbf{d}(n) \in \mathcal{D}(n,m)$ satisfies

  (i) $|D_2 - \mathbb{E}(D_2(\mathbf{Y}))| \le \psi(n)$,
  (ii) $|D_3 - \mathbb{E}(D_3(\mathbf{Y}))| \le \psi(n)$,
  (iii) $|\sum_i \binom{d_i}{2} - \mathbb{E}(\sum_i \binom{Y_i}{2})| \le \psi(n)$, and
  (iv) $d_i \le 8\log(n - D_2(\mathbf{d}))$ for every i,

  then
$$T(\mathbf{d}) \sim \frac{\sqrt{3r}}{e\sqrt{2m}} \cdot \frac{(2m-1)!!}{\prod_{j=1}^n d_j!}.$$

(b) Suppose further that $c = O(1)$ and $c > C_0$ for some constant $C_0 > 2$. Let $\psi(n) = 1/n^{\varepsilon}$ for some $\varepsilon \in (0, 1/4)$. If $\mathbf{d} = \mathbf{d}(n) \in \mathcal{D}(n,m)$ satisfies

  (i) $d_i \le 6\log n$ for every i,
  (ii) $|\eta(\mathbf{d}) - \eta_c| \le \psi(n)$ and
  (iii) $|D_2(\mathbf{d})/n - p_c| \le \psi(n)$,

  then
$$T(\mathbf{d}) \sim \frac{(2m-1)!!}{\prod_{j=1}^n d_j!}\sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right).$$

(c) If $c \to \infty$ and $\mathbf{d} = \mathbf{d}(n) \in \mathcal{D}(n,m)$ satisfies $\max d_i \le n^{\epsilon}$ for some $\epsilon \in (0, 0.01)$ then
$$T(\mathbf{d}) \sim \frac{(2m-1)!!}{\prod_{j=1}^n d_j!} \exp\left(-\frac{\eta(\mathbf{d})}{2} - \frac{\eta(\mathbf{d})^2}{4}\right).$$

We remark that the case $r = o(\sqrt{n})$ with $r \to \infty$ has been solved by Wright [70] (see also Voblyĭ [62]):
$$T(n,m) = \frac{\sqrt{3}}{e\sqrt{2\pi}} n^{n+3r/2-1/2} e^{r-n} (9r^2/2)^{-r/2} (1 + O(r^{-1}) + O(r^2/n)).$$
To compare Wright's formula to our own, we compute $\lambda_c = 3r/n - (3/2)(r/n)^2 + (6/5)(r/n)^3 + O((r/n)^4)$ for $r = o(n^{2/3})$, and then from Theorem 3.1.2(c) the following is immediate, confirming Wright's formula for the the case $r = o(n^{1/2})$.

Corollary 3.1.4. Suppose that $r := 2m - 2n = o(n^{2/3})$ and $r \to \infty$. Then
$$T(n,m) \sim \frac{\sqrt{3}}{e\sqrt{2\pi}} n^{n+3r/2-1/2} e^{r-n+3r^2/(8n)} (9r^2/2)^{-r/2}.$$

Let $T'(n,m)$ denote the number of 2-edge-connected (n,m)-graphs. Our methods for the enumeration of 2-connected graphs can be easily adapted to obtain an asymptotic formula for $T'(n,m)$.

Theorem 3.1.5. Suppose $m = O(n \log n)$ and $r := 2m - 2n \to \infty$. Then

$$T'(n,m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi nc(1+\eta_c - c)}} \sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4} + \frac{\lambda_c^3}{2(e^{\lambda_c} - 1)^2}\right).$$

This chapter is organized as follows. In Section 3.2, we reduce the problem of finding the number of 2-connected (n,m)-graphs to computing the expectation of a random variable in a probability space of random degree sequences. This random variable is related to the probability that a random multigraph generated with given degree sequence is 2-connected and simple. The models of random multigraphs that we will deal with are the kernel configuration model and the pairing model, which are defined in Section 2.2. In Section 3.3, we prove that, a.a.s., the random multigraph with minimum degree at least 3 generated by the pairing model is 2-connected if and only if it 2-edge-connected. We prove Theorem 3.1.2 in Sections 3.4, 3.5 and 3.6. More specifically, we prove Theorems 3.1.2(a) and 3.1.3(a) in Section 3.4, Theorems 3.1.2(b) and 3.1.3(b) in Section 3.5, and Theorems 3.1.2(c) and 3.1.3(c) in Section 3.6. In Section 3.7, we combine the formulae obtained in Theorem 3.1.2 to obtain Theorem 3.1.1. In Section 3.8, we explain how to obtain the formula for 2-edge-connected (n,m)-graphs for $m - n \to \infty$ and $m = O(n \log n)$.

3.2 Enumeration and random graphs

In this section we show how to reduce the enumeration problem, which is a deterministic problem, to the computation of the expected value of a random variable in a probability space of random degree sequences. The approach we use is the same as in [55].

Let

$$Q(n,m) = \sum_{\mathbf{d} \in \mathcal{D}(n,m)} \prod_{j=1}^{n} \frac{1}{d_j!}.$$

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a vector of independent truncated Poisson random variables with parameters $(2, \lambda_c)$ and let Σ denote the event that $\sum_i Y_i = 2m$. The function $Q(n,m)$ and $\mathbb{P}(\Sigma)$ are related: by Lemma 2.10.9 and Theorem 2.10.8(a),

$$Q(n,m) = \frac{f_2(\lambda)^n}{\lambda^{cn}} \mathbb{P}(\Sigma) = \frac{1 + O(r^{-1})}{\sqrt{2\pi c(1+\eta_c - c)}}. \tag{3.1}$$

We will use $\mathbf{Y}$ to generate degree sequences. Many times in the following sections we will work with 'typical' degree sequences. We say that a subset of $\tilde{\mathcal{D}} \subseteq \mathcal{D}(n,m)$ is a set of typical degree sequences when the probability that the vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ conditioned upon Σ is in $\tilde{\mathcal{D}}$ goes to 1. The two propositions we will present now show how to estimate $T(n, m)$ by computing the expectation of a random variable in the probability space of $\mathbf{Y}$ conditioned upon Σ. In both cases, this random variable is closely related to the probability that a random graph with degree sequence $\mathbf{Y}$ (conditioned upon Σ) is 2-connected and simple.

The first proposition concerns graphs generated with the pairing model (see Section 2.2 for the description of the pairing model). In the first proposition, the random variable is the probability that the random graph generated with the pairing model with given degree sequence is 2-connected and simple.

The second proposition concerns graphs generated with the kernel configuration model (see Section 2.2 for the description of this model). Recall that $D_2(\mathbf{d})$ is the number of occurrences of 2's in $\mathbf{d}$. For $\mathbf{d} \in \mathcal{D}(n, m)$, let $m' = m'(\mathbf{d}) := m - D_2(\mathbf{d})$, that is, m' is the number of edges in the kernel, and let $R = R(\mathbf{d}) := \sum_{i:d_i \geq 3} d_i$. In the second result, the random variable will be $\sqrt{m'(\mathbf{d})}\,\mathbb{P}(\mathbf{2cs}(\mathbf{d}))$, where $\mathbf{2cs}(\mathbf{d})$ is the event that a random graph generated with the kernel configuration model and degree sequence $\mathbf{d}$ is 2-connected and simple.

In summary, we basically reduced the enumeration problem to the computation of the probability that random graphs (generated by the pairing model or the kernel configuration model) are 2-connected and simple and the degree sequence is generated by truncated Poisson random variables conditioned upon the degrees having the correct summation. The proofs for both results follow the same argument as the proof in [55, Equation (13)] which states for 2-cores that

$$C(n, m) \sim (2m - 1)!!Q(n, m)\mathbb{E}\left(U(\mathbf{Y})|\Sigma\right), \tag{3.2}$$

where $C(n, m)$ denotes the number of (simple) 2-cores on $[n]$ with m edges and $U(\mathbf{Y})$ is the probability that the random graph generated with the pairing model and degree sequence $\mathbf{Y}$ is simple.

Proposition 3.2.1. Let $U'(\mathbf{d})$ be the probability that a random pairing with degree sequence $\mathbf{d}$ is 2-connected and simple. We have that

$$T(\mathbf{d}) = \frac{(2m - 1)!!}{\prod_{j=1}^{n} d_j!} U'(\mathbf{d}), \text{ for } \mathbf{d} \in \mathcal{D}(n, m) \tag{3.3}$$

and

$$T(n, m) = (2m - 1)!!Q(n, m)\mathbb{E}\left(U'(\mathbf{Y})|\Sigma\right). \tag{3.4}$$

Proof. For $\mathbf{d} \in \mathcal{D}(n, m)$, the number of pairings corresponding to a given simple graph is $\prod_{j=1}^{n} d_j!$ by Lemma 2.2.1. Since, in the pairing model each matching is chosen with uniform probability and $(2m-1)!!$ is the number of perfect matchings on $2m$, this implies (3.3). We now prove (3.4). The argument is basically the same as in [55, Equation (13)]. We have that

$$\begin{aligned}
T(n,m) &= \sum_{\mathbf{d}\in\mathcal{D}(n,m)} T(\mathbf{d}) = (2m-1)!! \sum_{\mathbf{d}\in\mathcal{D}(n,m)} \frac{U'(\mathbf{d})}{\prod_{j=1}^n d_j!} \quad \text{by (3.3)} \\
&= \frac{(2m-1)!! f_2(\lambda_c)^n}{\lambda_c^{2m}} \sum_{\mathbf{d}\in\mathcal{D}(n,m)} U'(\mathbf{d}) \prod_{j=1}^n \frac{\lambda_c^{d_j}}{d_j! f_2(\lambda_c)} \quad \text{since } \sum_{j=1}^n d_j = 2m \\
&= \frac{(2m-1)!! Q(n,m)}{\mathbb{P}(\Sigma)} \sum_{\mathbf{d}\in\mathcal{D}(n,m)} U'(\mathbf{d})\, \mathbb{P}(\mathbf{Y}=\mathbf{d}) \quad \text{by (3.1)} \\
&= (2m-1)!! Q(n,m) \mathbb{E}\left(U'(\mathbf{Y})|\Sigma\right).
\end{aligned}$$

□

Proposition 3.2.2. We have that

$$T(\mathbf{d}) = \frac{(2m-1)!!\sqrt{m'(\mathbf{d})/m}\,\mathbb{P}(\mathbf{2cs}(\mathbf{d}))}{\prod_{i=1}^n d_i!}\left(1 - O\left(\frac{1}{m'(\mathbf{d})}\right)\right), \text{uniformly for } \mathbf{d}\in\mathcal{D}(n,m) \quad (3.5)$$

and

$$T(n,m) = (2m-1)!! Q(n,m)\sqrt{m^{-1}}\mathbb{E}\left(w(\mathbf{Y})|\Sigma\right), \quad (3.6)$$

where

$$w(\mathbf{d}) = \mathbb{P}(\mathbf{2cs}(\mathbf{d}))\sqrt{m'(\mathbf{d})} \quad (3.7)$$

and $\mathbf{2cs}(\mathbf{d})$ is the event that the pre-kernel generated by the kernel configuration model is 2-connected and simple.

Proof. By Lemma 2.2.3, each simple pre-kernel is produced by $\prod_{i\in R(\mathbf{d})} d_i!$ configurations, where $R = R(\mathbf{d}) := \{i \in [n] : d_i \geq 3\}$. There are $(2m'-1)!!$ ways of generating the kernel and $(m-1)!/(m'-1)!$ ways of adding the degree-2 vertices. Thus,

$$T(\mathbf{d}) = \frac{(2m'-1)!!(m-1)!}{(m'-1)!\prod_{i\in R(\mathbf{d})} d_i!}\mathbb{P}(\mathbf{2cs}(\mathbf{d})) = \frac{(2m'-1)!!(m-1)!2^{D_2}}{(m'-1)!\prod_{i=1}^n d_i!}\mathbb{P}(\mathbf{2cs}(\mathbf{d})).$$

Using the fact that $(2k-1)!! = (2k)!/(2^k k!)$ for any integer $k \geq 1$,

$$\begin{aligned}
\frac{(2m'-1)!!(m-1)!2^{D_2}}{(m'-1)!} &= \frac{(2m-1)!!m'}{m} \cdot \frac{m!2^m}{(2m)!} \cdot \frac{(2m')!}{m'!2^{m'}} \cdot \frac{m!2^{D_2}}{m'!} \\
&= \frac{(2m-1)!!m'}{m} \cdot \frac{(2m')!m!^2 2^{m+D_2}}{(2m)!m'!^2 2^{m'}}.
\end{aligned}$$

By Stirling's approximation, we have that

$$\begin{aligned}
\frac{(2m')!m!^2 2^{m+D_2}}{(2m)!m'!^2 2^{m'}} &= \sqrt{\frac{m}{m'}}\frac{(2m')^{2m'}m^{2m}2^{m+D_2}}{(2m)^{2m}(m')^{2m'}2^{m'}}\left(1+O\left(\frac{1}{m'}\right)\right)\\
&= \sqrt{\frac{m}{m'}}\frac{2^{2m'+m+D_2}}{2^{2m+m'}}\left(1+O\left(\frac{1}{m'}\right)\right)\\
&= \sqrt{\frac{m}{m'}}\left(1+O\left(\frac{1}{m'}\right)\right),
\end{aligned}$$

since $m' = m + D_2$. Thus,

$$\begin{aligned}
T(\mathbf{d}) &= \frac{(2m'-1)!!(m-1)!2^{D_2}}{(m'-1)!\prod_{i=1}^n d_i!}\,\mathbb{P}(\mathbf{2cs}(\mathbf{d}))\\
&= \frac{(2m-1)!!m'\,\mathbb{P}(\mathbf{2cs}(\mathbf{d}))}{\prod_{i=1}^n d_i!m}\cdot\sqrt{\frac{m}{m'}}\left(1+O\left(\frac{1}{m'}\right)\right)\\
&= \frac{(2m-1)!!}{\prod_{i=1}^n d_i!}\sqrt{\frac{m'}{m}}\,\mathbb{P}(\mathbf{2cs}(\mathbf{d}))\left(1+O\left(\frac{1}{m'}\right)\right),
\end{aligned}$$

and the constants in the error term are independent of $\mathbf{d}$. Thus, we proved (3.5). Since $m'(\mathbf{d}) = m - D_2(\mathbf{d}) \geq m - n = r/2$, the error term $O(1/m')$ above can be replaced by $O(1/r)$, uniformly for $\mathbf{d}$. We now prove (3.6). The proof is very similar to the proof of (3.4):

$$\begin{aligned}
T(n,m) &= \sum_{\mathbf{d}\in\mathcal{D}(n,m)} T(\mathbf{d})\\
&= \sum_{\mathbf{d}\in\mathcal{D}(n,m)} \frac{(2m-1)!!}{\prod_{i=1}^n d_i!}\sqrt{\frac{m'(\mathbf{d})}{m}}\,\mathbb{P}(\mathbf{2cs}(\mathbf{d}))\left(1+O\left(\frac{1}{r}\right)\right) \quad \text{by (3.5)}\\
&= \frac{(2m-1)!!f_2(\lambda_c)^n}{\lambda_c^{2m}\sqrt{m}}\sum_{\mathbf{d}\in\mathcal{D}(n,m)}\sqrt{m'(\mathbf{d})}\,\mathbb{P}(\mathbf{2cs}(\mathbf{d}))\prod_{j=1}^n\frac{\lambda_c^{d_j}}{d_j!f_2(\lambda_c)}\left(1+O\left(\frac{1}{r}\right)\right)\\
&= \frac{(2m-1)!!Q(n,m)}{\mathbb{P}(\Sigma)\sqrt{m}}\sum_{\mathbf{d}\in\mathcal{D}(n,m)}\mathbb{P}(\mathbf{2cs}(\mathbf{d}))\sqrt{m'(\mathbf{d})}\,\mathbb{P}(\mathbf{Y}=\mathbf{d})\left(1+O\left(\frac{1}{r}\right)\right) \quad \text{by (3.1)}\\
&= (2m-1)!!Q(n,m)\mathbb{E}\left(w(\mathbf{Y})|\Sigma\right)\left(1+O\left(\frac{1}{r}\right)\right).
\end{aligned}$$

□

3.3 Relation of vertex-connectivity and edge-connectivity

In this section, we show that a.a.s. the kernel generated with pairing model is 2-connected if and only if it is 2-edge-connected (with constraints in the maximum degree). For arbitrary graphs with at least 3 vertices, 2-connectivity implies 2-edge-connectivity and so one direction is trivial. For the other direction, we will show that every cut-vertex has to be in a bridge a.a.s.

Proposition 3.3.1. Let $\mathbf{d} \in \mathcal{D}(n, m)$ satisfying $n \geq 3$ and $3 \leq \delta = d_1 \leq \cdots \leq d_n = \Delta \leq n^{0.04}$. Let K be the kernel of the random multigraph produced by the pairing model using degree sequence $\mathbf{d}$. A.a.s., K is 2-connected if and only if it is 2-edge-connected.

This proposition is an easy consequence of the following lemmas:

Lemma 3.3.2. A.a.s., no subgraph of K with s vertices, $2 \leq s \leq n^{0.4}$, has more than $1.2s$ edges.

Lemma 3.3.3. A.a.s., each subset of K with s vertices, $n^{0.3} \leq s \leq n/2$ has more than δ neighbours.

Proof of Proposition 3.3.1. Suppose that v is a cut-vertex in K not in a bridge. Then v decomposes K into components W_1 and W_2 with $|W_1| \leq |W_2|$. Note that v sends at least 2 edges to W_1 and at least 2 edges to W_2. (Otherwise v would be in a bridge).

Suppose that $|W_1| = 1$. Then the number of edges induced by $W_1 \cup \{v\}$ is at least 3 (since $\delta \geq 3$) which is $\frac{3}{2}|W_1 \cup \{v\}|$. On the other hand, if $|W_1| \geq 2$, the number of edges induced by $W_1 \cup \{v\}$ is at least $(3|W_1| + 2)/2 \geq 1.25|W_1 \cup \{v\}|$. Thus, for $|W_1 \cup \{v\}| \leq n^{0.4}$, we conclude that such v a.a.s. does not exist, by Lemma 3.3.2. Otherwise, $|W_1| \geq n^{0.3}$ and such v a.a.s. does not exist by Lemma 3.3.3.

So a.a.s., K has a bridge if it has a cut-vertex. The converse is deterministically true for multigraphs with at least three vertices, and the proposition follows. □

We remark that the proof of Proposition 3.3.1 can be easily adapted to show that, a.a.s., for any bridge, at least one of its endpoints is incident with no non-loop edges, apart from the bridge itself.

We now prove Lemmas 3.3.2 and 3.3.3. We observe that these lemmas can be proved by closely following Łuczak's proofs of properties of (simple) graphs with given degree sequence in [46, Section 12.3]). For the proofs of these lemmas, let the kernel K be generated by choosing a perfect matching M uniformly at random on the points of sets/bins $S_1, \ldots, S_n$ with $d_1, \ldots, d_n$ points in them, and then contracting each bin S_i into a single vertex i. Let P denote the set of the $2m$ points inside the bins. Let $\Phi(k)$ denote the number of perfect matchings on $[2k]$ for any nonnegative integer k. It is straightforward that $\Phi(k) = (2k-1)!! = (2k)!/(2^k k!)$.

Proof of Lemma 3.3.2. Let $\mathcal{S} = \{S \subseteq [n] : 2 \le |S| \le n^{0.4}\}$. For each $S \in \mathcal{S}$, let X_S denote the indicator variable that S induces more than $1.2|S|$ edges and let $X = \sum_{S\in\mathcal{S}} X_S$. We will show that $\mathbb{E}(X) = o(1)$. Assuming this, by Markov inequality,

$$\mathbb{P}(X \ge 1) \le \mathbb{E}(X) = o(1),$$

which proves the lemma.

For $S \in \mathcal{S}$, let $E_S = \{xy : x \in S_i,\ y \in S_j,\ i, j \in S, x \ne y\}$. For $S \in \mathcal{S}$ with $s = |S|$,

$$\mathbb{E}(X_S) \le \sum_{\ell\ge 1.2s} \sum_{\substack{E\subseteq E_S\\|E|=\ell}} \mathbb{P}(E \subseteq M) = \sum_{\ell\ge 1.2s} \sum_{\substack{E\subseteq E_S\\|E|=\ell}} \frac{\Phi(m-\ell)}{\Phi(m)}. \tag{3.8}$$

The number of sets $E \subseteq E_S$ of size ℓ is bounded by $\binom{(s\Delta)^2}{\ell}$ because there are at most $s\Delta$ points in S. Thus, combining this with (3.8) yields

$$\begin{aligned}
\mathbb{E}(X_S) &\le \sum_{\ell\ge 1.2s} \binom{(s\Delta)^2}{\ell} \frac{\Phi(m-\ell)}{\Phi(m)} = \sum_{\ell\ge 1.2s} \binom{(s\Delta)^2}{\ell} \frac{[m]_\ell 2^\ell}{[2m]_{2\ell}} \le \sum_{\ell\ge 1.2s} \binom{(s\Delta)^2}{\ell} \left(\frac{1}{2m-2\ell}\right)^\ell \\
&\le \sum_{\ell\ge 1.2s} \left(\frac{es^2\Delta^2}{\ell(2m-2\ell)}\right)^\ell, \quad \text{since } \binom{s^2\Delta^2}{\ell} \le \left(\frac{es^2\Delta^2}{\ell}\right)^\ell \\
&\le \sum_{\ell\ge 1.2s} \left(\frac{es\Delta^2}{1.2n}\right)^\ell, \quad \text{since } 1.2s \le \ell \le s^2 \le n^{0.8} \\
&\le 2\left(\frac{s\Delta^2}{\alpha n}\right)^{1.2s},
\end{aligned}$$

where the last inequality follows from the formula for the sum of the terms of a geometric progression with ratio $s\Delta^2/(\alpha n) \le (\alpha n^{.52})^{-1} \to 0$, where $\alpha = 1.2/e$. We also use the fact that

$(1-s\Delta^2(\alpha n))^{-1} \leq \alpha n/(\alpha n - n^{0.48}) \leq 2$ for sufficiently large n. Thus,

$$
\begin{aligned}
\mathbb{E}(X) = \sum_{S\in\mathcal{S}} \mathbb{E}(X_S) = \sum_{s=2}^{n^{0.4}} \sum_{\substack{S\in\mathcal{S}\\|S|=s}} \mathbb{E}(X_S) \leq \sum_{s=2}^{n^{0.4}} \sum_{\substack{S\in\mathcal{S}\\|S|=s}} 2\left(\frac{s\Delta^2}{\alpha n}\right)^{1.2s} &= 2\sum_{s=2}^{n^{0.4}} \binom{n}{s}\left(\frac{s\Delta^2}{\alpha n}\right)^{1.2s} \\
&\leq 2\sum_{s=2}^{n^{0.4}} \left(\frac{es^{0.2}\Delta^{2.4}}{\alpha^{1.2}n^{0.2}}\right)^{s}, \quad \text{since} \binom{n}{s} \leq \left(\frac{en}{s}\right)^{s} \\
&\leq 2\sum_{s=2}^{n^{0.4}} \left(\frac{e}{\alpha^{1.2}n^{0.024}}\right)^{s}, \quad \text{since } s \leq n^{0.4} \text{ and } \Delta \leq n^{0.04} \\
&\leq 2\frac{1}{1-\frac{e}{\alpha^{1.2}n^{0.024}}}\left(\frac{e}{\alpha^{1.2}n^{0.024}}\right)^{2}, \quad \text{by sum of g.p.} \\
&= O\left(1/n^{0.048}\right),
\end{aligned}
$$

and we are done. □

Proof of Lemma 3.3.3. Let $\mathcal{S}$ denote the subsets of $[n]$ with size at most δ. For each $S \in \mathcal{S}$, let $\mathcal{W}_S$ denote the set of subsets $W \subseteq [n] \setminus S$ of size at least $n^{0.3}$ and at most $(n-|S|)/2$. For any $S \in \mathcal{S}$ and $W \in \mathcal{W}_S$, let $X_{S,W}$ denote the indicator variable for the event that all edges with exactly one end in W have the other end in S. That is, $X_{S,W}$ indicates the event that S separates W from the rest of the graph. Let $X = \sum_{S\in\mathcal{S}} \sum_{W\in\mathcal{W}_S} X_{S,W}$.

We will show that it suffices to prove that $\mathbb{E}(X) = o(1)$. By Markov's inequality this would imply $\mathbb{P}(X \geq 1) \leq \mathbb{E}(X) = o(1)$. So we need to show that the event that there is a set W' of size in $[n^{0.3}, n/2]$ with at most δ neighbours (outside W') is contained in the event that there exist $S \in \mathcal{S}$ and $W \in \mathcal{W}_S$ such that $X_{S,W} = 1$. So suppose there is such a set W' and let S' denote its set of neighbours. If $|W'| \leq (n-|S'|)/2$, then $W' \in \mathcal{W}_{S'}$ and so $X_{S',W'} = 1$. If $|W'| \geq (n-|S'|)/2$, then, for $W := [n] \setminus (S' \cup W')$, we have that W also has all its neighbours in S'. Moreover, $|W'| \leq n - |S'| - (n-|S'|)/2 = (n-|S'|)/2$ and $W' \geq n - |S'| - n/2 \geq n/2 - \Delta \geq n^{0.3}$ for sufficiently large n since $\delta \leq \Delta \leq n^{0.04}$. This implies $W \in \mathcal{W}_{S'}$ and so $X_{S',W} = 1$.

For $(S,W) \in \mathcal{S} \times \mathcal{W}_S$, let $\mathcal{E}(S,W)$ be the set of all $E \subseteq \{xy \in (S_i, S_j) : i \in W, j \in W \cup S\}$ such that each point in a bin corresponding to a vertex in W has degree 1 in E and each point in a bin corresponding to a vertex in S has degree 0 or 1 in E. In other words, $\mathcal{E}(S,W)$ is the set of all possible edges (as pairs of points) with at least one end in W covering all points in W so that S separates W from the rest of the graph. The size of any set $E \in \mathcal{E}$ is at least $3|W|/2$ since $\delta \geq 3$ and, on the other hand,

$$|E| \leq m - \frac{3}{2}\big(n - |S| - |W|\big) \geq m - \frac{3}{2}|W|,$$

since $|W| \le (n-|S|)/2$. Moreover,

$$\frac{3}{2}|W| \le \frac{3n}{4} \le \frac{m}{2}. \tag{3.9}$$

For $\ell \in [\lceil 3|W|/2 \rceil, m - \lceil 3|W|/2 \rceil]$, the number of sets $E \in \mathcal{E}(S,W)$ of size ℓ is bounded by $2^{\delta\Delta}\Phi(\ell)$. This is because we have to choose 2ℓ points in the bins of W and S and a perfect matching on these points. All the points in bins in W have to be selected for $E \in \mathcal{E}(S,W)$. A rough upper bound for the number of ways of selecting $2\ell - \sum_{i\in W} d_i$ points in bins in S is $2^{\sum_{i\in S} d_i} \le 2^{|S|\Delta} \le 2^{\delta\Delta}$ (this is a very rough upper bound). Moreover, there are $\Phi(\ell)$ ways of matching the selected points.

$$\begin{aligned}
\mathbb{E}\left(X_{S,W}\right) &= \sum_{\ell=\lceil 3|W|/2\rceil}^{m-\lceil 3|W|/2\rceil} \sum_{\substack{E\in\mathcal{E}(S,W)\\ |E|=\ell}} \mathbb{P}(E \subseteq M) \le 2^{\delta\Delta} \sum_{\ell=\lceil 3|W|/2\rceil}^{m-\lceil 3|W|/2\rceil} \frac{\Phi(\ell)\Phi(m-\ell)}{\Phi(m)} \\
&= 2^{\delta\Delta} \sum_{\ell=\lceil 3|W|/2\rceil}^{m-\lceil 3|W|/2\rceil} \frac{(2\ell)!(2m-2\ell)!m!}{\ell!(m-\ell)!(2m)!} = 2^{\delta\Delta} \sum_{\ell=\lceil 3|W|/2\rceil}^{m-\lceil 3|W|/2\rceil} \binom{m}{\ell}\binom{2m}{2\ell}^{-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}\left(X_{S,W}\right) &\le 2^{\delta\Delta} \sum_{\ell=\lceil 3|W|/2\rceil}^{m-\lceil 3|W|/2\rceil} \binom{m}{\ell}^{-1}, \quad \text{since } \binom{2m}{2\ell} \ge \binom{m}{\ell}^2 \\
&\le 2^{\delta\Delta} m \binom{m}{\lceil 3|W|/2\rceil}^{-1}, \quad \text{by (3.9)} \\
&\le 2^{\delta\Delta} m \binom{\lceil 3n/2\rceil}{\lceil 3|W|/2\rceil}^{-1}, \quad \text{since } m \ge \frac{3n}{2}.
\end{aligned}$$

For any $S \in \mathcal{S}$, we have that there are at most $\binom{n}{w}$ sets of size w in $\mathcal{W}_S$. Thus, together with the above, we get

$$
\begin{aligned}
\mathbb{E}\left(X\right) &\leq \sum_{S \in \mathcal{S}} \sum_{w=\lceil n^{0.3} \rceil}^{\lfloor (n-|S|)/2 \rfloor} \binom{n}{w} 2^{\delta\Delta} m \binom{\lceil 3n/2 \rceil}{\lceil 3w/2 \rceil}^{-1} \\
&\leq 2^{\delta\Delta} m \sum_{S \in \mathcal{S}} \sum_{w=\lceil n^{0.3} \rceil}^{\lfloor (n-|S|)/2 \rfloor} \binom{\lceil n/2 \rceil}{\lceil w/2 \rceil}^{-1}, \quad \text{since } \binom{\lceil 3n/2 \rceil}{\lceil 3w/2 \rceil} \geq \binom{\lceil n/2 \rceil}{\lceil w/2 \rceil} \binom{n}{w}, \\
&\leq 2^{\delta\Delta} m \sum_{S \in \mathcal{S}} \sum_{w=\lceil n^{0.3} \rceil}^{\lfloor (n-|S|)/2 \rfloor} \binom{\lceil n/2 \rceil}{\lceil \lceil n^{0.3} \rceil /2 \rceil}^{-1}, \\
&\leq 2^{\delta\Delta} m 2^n n \binom{\lceil n/2 \rceil}{\lceil \lceil n^{0.3} \rceil /2 \rceil}^{-1}, \quad \text{since the number of choices for } S \text{ is at most } 2^n; \\
&\leq 2^{\delta\Delta} m 2^n n \left(\frac{1}{n^{0.6}}\right)^{n^{0.3}/2}, \quad \text{since } \binom{a}{b} \geq \left(\frac{a}{b}\right)^a \text{ for any nonnegative integers } a, b; \\
&\leq 2^{n^{1.08}} n^{2.04} \left(\frac{1}{n}\right)^{0.3 n^{0.3}} \qquad \text{since } \Delta \leq n^{0.04} \\
&= O\left(\left(\frac{1}{n}\right)^{0.2 n^{0.3}}\right) = o(1),
\end{aligned}
$$

and we are done. □

3.4 The case $c \to 2$.

In this section, we obtain an asymptotic formula for the number $T(n,m)$ of 2-connected (n,m)-graphs for the range $c = 2m/n \to 2$ from above, proving Theorem 3.1.2(a). We also obtain a formula for the number $T(\mathbf{d})$ of 2-connected graphs with degree sequence $\mathbf{d}$ with $\mathbf{d} \in \mathcal{D}(n,m)$ satisfying some constraints, proving Theorem 3.1.3(a). In Proposition 3.2.2, we have already shown how to obtain $T(\mathbf{d})$ by computing $\mathbb{P}(\mathbf{2cs}(\mathbf{d}))$, the probability that the graph generated with kernel configuration model is 2-connected and simple, and how to obtain $T(n,m)$ by computing $\mathbb{E}\left(\sqrt{m'(\mathbf{Y})}\,\mathbb{P}(\mathbf{2cs}(\mathbf{Y}))\middle|\Sigma\right)$, where $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is a vector of independent truncated Poisson random variables with parameters $(2, \lambda_c)$ and Σ is the event that $\sum_i Y_i = 2m$.

We will define a set of typical degree sequences $\tilde{\mathcal{D}}$ so that, for $\mathbf{d} \in \tilde{\mathcal{D}}$, we have $\mathbb{P}(\mathbf{2cs}(\mathbf{d})) \sim 1/e$ and so we obtain $T(\mathbf{d})$ for typical degree sequences. We then proceed to show that the degree sequences outside $\tilde{\mathcal{D}}$ have no significant contribution to $\mathbb{E}\left(\sqrt{m'(\mathbf{Y})}\,\mathbb{P}(\mathbf{2cs}(\mathbf{Y}))\middle|\Sigma\right)$ and so we can obtain an asymptotic formula for $T(n,m)$.

As expected, the definition of $\tilde{\mathcal{D}}$ requires some functions of the degree sequence to be concentrated around their expected value. So first we will analyse the expected value of some of these functions. Let

$$\mu_2 = \mathbb{E}\left(D_2(\mathbf{Y})\right), \quad \mu_3 = \mathbb{E}\left(D_3(\mathbf{Y})\right), \quad \mu = \mathbb{E}\left(\sum_{i=1}^{n} \binom{Y_i}{2}\right).$$

Lemma 3.4.1. We have $\mu_2 = n - r + o(r)$, $\mu_3 = r + o(r)$ and $\mu = n + 2r + o(r)$.

Proof. Let $r(\mathbf{Y}) = \sum_{i=1}^{n} Y_i - 2n$ and $n'(\mathbf{Y}) = n - D_2(\mathbf{Y})$. Note that $r(\mathbf{Y})$ may not coincide with $r = 2m - 2n$ because we are not conditioning on Σ, which is the event that $\sum_i Y_i = 2m$. But

$$\mathbb{E}\left(r(\mathbf{Y})\right) = \mathbb{E}\left(\sum_{i=1}^{n} Y_i\right) - 2n = \sum_{i=1}^{n} c - 2n = 2m - 2n = r. \tag{3.10}$$

Note that

$$\sum_{i=1}^{n} Y_i = \sum_{i \in R(\mathbf{Y})} Y_i + 2D_2(\mathbf{Y}) \geq \sum_{i \in R(\mathbf{Y})} 3 + 2n - 2n'(\mathbf{Y}) = 3n'(\mathbf{Y}) + 2n - 2n'(\mathbf{Y}) = n'(\mathbf{Y}) + 2n.$$

Hence, $n'(\mathbf{Y}) \leq r(\mathbf{Y})$.

Thus,

$$D_2(\mathbf{Y}) = n - n'(\mathbf{Y}) \geq n - r(\mathbf{Y}). \tag{3.11}$$

and so, by (3.10),

$$\mathbb{E}\left(D_2(\mathbf{Y})\right) \geq n - r. \tag{3.12}$$

Moreover, $D_2(\mathbf{Y}) \le n - D_3(\mathbf{Y})$, which implies that

$$\mathbb{E}\left(D_2(\mathbf{Y})\right) \le n - \mathbb{E}\left(D_3(\mathbf{Y})\right). \tag{3.13}$$

Since $n-r = n+o(n)$ and $n-D_3(\mathbf{Y}) \le n$, we conclude that $\mathbb{E}\left(D_2(\mathbf{Y})\right) = n+o(n)$. Using (2.10),

$$\mu_3 = \mathbb{E}\left(D_3(\mathbf{Y})\right) = \frac{\lambda_c^3}{3!(e^{\lambda_c} - 1 - \lambda_c)} n = \frac{\lambda_c}{3}\mathbb{E}\left(D_2(\mathbf{Y})\right) = \left(\frac{r}{n} + O\left(\frac{r^2}{n^2}\right)\right)(n + o(n))$$
$$= r + o(r) + O(r^2/n) = r + o(r).$$

By (3.13), $\mathbb{E}\left(D_2(\mathbf{Y})\right) \le n - \mathbb{E}\left(D_3(\mathbf{Y})\right) = n - r + o(r)$. So by (3.12), we conclude that $\mu_2 = \mathbb{E}\left(D_2(\mathbf{Y})\right) = n - r + o(r)$.

By (2.8), we have that $\mathbb{E}\left(Y_1(Y_1 - 1)\right) = c\eta_c$. By (2.10) and (2.11), this implies that

$$\mu = \mathbb{E}\Big(\sum_{i=1}^{n} \binom{Y_i}{2}\Big) = \frac{n}{2}\mathbb{E}\left(Y_1(Y_1 - 1)\right) = \frac{nc\eta_c}{2} = \frac{2n + r}{2}\Big(\frac{3r}{2n} + 1 + O\Big(\frac{r^2}{n^2}\Big)\Big)$$
$$= n + 2r + O(r^2/n) = n + 2r + o(r).$$

□

We now define a set of 'typical' degree sequences. For any function $\psi(n) : \mathbb{N} \to \mathbb{R}_+$ such that $\psi(n) = o(r)$, let

$$\tilde{\mathcal{D}}_n(\psi) = \tilde{\mathcal{D}}(\psi) := \Big\{\mathbf{d} \in \mathcal{D}(n, m) : |D_2(\mathbf{d}) - \mu_2| \le \psi(n);\ |D_3(\mathbf{d}) - \mu_3| \le \psi(n);$$
$$\Big|\sum_{i=1}^{n} \binom{d_i}{2} - \mu\Big| \le \psi(n);\ \max_i d_i \le 8 \log n'(\mathbf{d})\Big\}.$$

and define $\tilde{\mathcal{D}}^c(\psi) = \mathcal{D}(n, m) \setminus \tilde{\mathcal{D}}(\psi)$.

We will determine $\mathbb{P}(\mathbf{2cs}(\mathbf{d}))$ asymptotically for 'typical' degree sequences:

Proposition 3.4.2. Let $\psi = o(r)$ and let $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$. Then

$$\mathbb{P}(\mathbf{2cs}(\mathbf{d})) = \frac{1}{e} + o(1).$$

The proof of Theorem 3.1.3(a) is now straightforward. By Proposition 3.2.2 (Equation (3.5)) and Proposition 3.4.2,

$$T(\mathbf{d}) \sim \frac{(2m-1)!!\sqrt{m'(\mathbf{d})/m}\,\mathbb{P}(\mathbf{2cs}(\mathbf{d}))}{\prod_{i=1}^{n} d_i!} \sim \frac{1}{e}\sqrt{\frac{m'(\mathbf{d})}{m}}\frac{(2m-1)!!}{\prod_{i=1}^{n} d_i!}$$

and so to prove Theorem 3.1.3(a) it suffices to show that $m'(\mathbf{d}) \sim \frac{3r}{2}$. Indeed, using Lemma 3.4.1 for μ_2 and the fact that $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$, we have that

$$m'(\mathbf{d}) = m - D_2(\mathbf{d}) = m - \mu_2 + O(\psi) = m - \mu_2 + o(r) = \frac{3r + o(r)}{2}. \tag{3.14}$$

This finishes the proof of Theorem 3.1.3(a).

We now prove Theorem 3.1.2(a). Note that, for any arbitrary sequence $(\mathbf{d}_n)_{n\in\mathbb{N}}$ with $\mathbf{d}_n \in \tilde{\mathcal{D}}_n(\psi)$ for every $n \in \mathbb{N}$, we have that $w(\mathbf{d}_n) \sim e^{-1}\sqrt{3r/2} =: t(n)$ by Proposition 3.4.2 and (3.14). Since $\tilde{\mathcal{D}}_n(\psi)$ is a finite for every n, Lemma 2.7.1 implies that there exists a function $h(n) = o(1)$ such that, for every $(\mathbf{d}_n)_{n\in\mathbb{N}}$ such that $\mathbf{d}_n \in \tilde{\mathcal{D}}_n(\psi)$ for every $n \in \mathbb{N}$, we have that $|w(\mathbf{d}_n)/t(n) - 1| \leq h(n) = o(1)$. This implies that

$$\begin{aligned}\left|\mathbb{E}\left(w(\mathbf{Y})|\tilde{\mathcal{D}}(\psi)\right) - \frac{1}{e}\sqrt{\frac{3r}{2}}\right| &= \left|\sum_{\mathbf{d}\in\tilde{\mathcal{D}}(\psi)} w(\mathbf{d})\,\mathbb{P}(\mathbf{Y} = \mathbf{d}\,|\,\tilde{\mathcal{D}}(\psi)) - \frac{1}{e}\sqrt{\frac{3r}{2}}\right| \\ &\leq \sum_{\mathbf{d}\in\tilde{\mathcal{D}}(\psi)} \left|\left(w(\mathbf{d}) - \frac{1}{e}\sqrt{\frac{3r}{2}}\right)\right| \mathbb{P}(\mathbf{Y} = \mathbf{d}\,|\,\tilde{\mathcal{D}}(\psi)) \\ &\leq \sum_{\mathbf{d}\in\tilde{\mathcal{D}}(\psi)} h(n)\,\mathbb{P}(\mathbf{Y} = \mathbf{d}\,|\,\tilde{\mathcal{D}}(\psi)) = h(n) = o(1).\end{aligned} \tag{3.15}$$

For any $\mathbf{d} \in \mathcal{D}(n, m)$,

$$\begin{aligned}r = 2m - 2n &= 2(m'(\mathbf{d}) + D_2(\mathbf{d})) - 2(n'(\mathbf{d}) + D_2(\mathbf{d})) \\ &= 2m'(\mathbf{d}) - 2n'(\mathbf{d}) \geq 2m'(\mathbf{d}) - \frac{2}{3}m'(\mathbf{d}) = \frac{m'(\mathbf{d})}{3},\end{aligned}$$

because $2m'(\mathbf{d}) \geq 3n'(\mathbf{d})$. Thus, for any $\mathbf{d} \in \mathcal{D}(n, m)$,

$$w(\mathbf{d}) = \mathbb{P}(\mathbf{2cs}(\mathbf{d}))\sqrt{m'(\mathbf{d})} \leq \sqrt{3r}. \tag{3.16}$$

Let $\psi(n) = r^{1-\varepsilon}$ for some $\varepsilon \in (0, 1/4)$. We will show that the set $\tilde{\mathcal{D}}(\psi)$ is indeed a set of typical degree sequences. More precisely, we will show

$$\mathbb{P}(\tilde{\mathcal{D}}(\psi)|\Sigma) = 1 + O(\sqrt{r}/n) + O(r^{2\varepsilon-1/2}). \tag{3.17}$$

Together with (3.15) and (3.16), this implies that

$$\begin{aligned}\mathbb{E}\left(w(\mathbf{Y})|\Sigma\right) &= \mathbb{E}\left(w(\mathbf{Y})|\tilde{\mathcal{D}}(\psi)\right)\mathbb{P}(\tilde{\mathcal{D}}(\psi)|\Sigma) + \mathbb{E}\left(w(\mathbf{Y})|\tilde{\mathcal{D}}^c(\psi)\right)\mathbb{P}(\tilde{\mathcal{D}}^c(\psi)|\Sigma) \\ &= \mathbb{E}\left(w(\mathbf{Y})|\tilde{\mathcal{D}}(\psi)\right)(1 - O(\sqrt{r}/n) - O(r^{2\varepsilon-1/2})) + o(\sqrt{r}) \\ &\sim \frac{1}{e}\sqrt{\frac{3r}{2}}.\end{aligned} \tag{3.18}$$

In Lemma 2.10.7, Equation (2.13) tells us that $c(1+\eta_c-c)\sim c-2$. Together with (3.6) in Proposition 3.2.2, (3.1) and (3.18),

$$\begin{aligned}T(n,m) &= (2m-1)!!Q(n,m)\sqrt{m^{-1}}\mathbb{E}\left(w(\mathbf{Y})|\Sigma\right)\\ &\sim (2m-1)!!\frac{f_2(\lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi nc(1+\eta_c-c)}}\sqrt{m^{-1}}\mathbb{E}\left(w(\mathbf{Y})|\Sigma\right)\\ &\sim (2m-1)!!\frac{f_2(\lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi n(c-2)}}\frac{1}{e}\sqrt{\frac{3r}{2}},\end{aligned}$$

proving Theorem 3.1.2(a).

So it suffices to prove (3.17). Let $p^{(i)}$ denote the probability that a variable with distribution $\mathrm{Po}(2,\lambda_c)$ has value i. Recall that $\eta(\mathbf{d})=(\sum_{i=1}^n d_i(d_i-1))/(\sum_{i=1}^n d_i)$. First we will study the first three conditions in the definition of $\widetilde{\mathcal{D}}(\psi)$. Let F be the event that $\mathbf{Y}$ fails to satisfy any of the three conditions.

Using Chebyshev's inequality, we have that

$$\mathbb{P}(|D_2(\mathbf{Y})-\mu_2|\geq\psi(n))\leq\frac{p^{(2)}(1-p^{(2)})n}{\psi(n)^2}\quad\text{and}\quad\mathbb{P}(|D_3(\mathbf{Y})-\mu_3|\geq\psi(n))\leq\frac{p^{(3)}(1-p^{(3)})n}{\psi(n)^2}. \tag{3.19}$$

By Lemma 2.10.3(a), we have that $\lambda_c\sim 3(c-2)=3r/n=o(1)$ and so

$$p^{(2)}(1-p^{(2)})\leq 1-p^{(2)}=1-\frac{\lambda_c^2}{2f_2(\lambda_c)}=\frac{e^{\lambda_c}-1-\lambda_c-\lambda_c^2/2}{e^{\lambda_c}-1-\lambda_c}=\frac{\lambda_c^3/6}{\lambda_c^2/2}(1+O(\lambda_c))\sim\frac{r}{n}$$

and

$$p^{(3)}(1-p^{(3)})\leq p^{(3)}=\frac{\lambda_c^3}{6f_2(\lambda_c)}=\frac{\lambda_c^3}{6(e^{\lambda_c}-1-\lambda_c)}\leq\frac{\lambda_c^3}{6\frac{\lambda_c^2}{2}}=\frac{\lambda_c}{3}\sim\frac{r}{n}.$$

Together with (3.19), this implies

$$\mathbb{P}\left(|D_2(\mathbf{Y})-\mu_2|\geq\psi(n)\right)=O\left(\frac{r}{\psi(n)^2}\right)\quad\text{and}\quad\mathbb{P}\left(|D_3(\mathbf{Y})-\mu_3|\geq\psi(n)\right)=O\left(\frac{r}{\psi(n)^2}\right). \tag{3.20}$$

By Lemma 2.10.10 and Lemma 2.10.3(a),

$$\mathrm{Var}(Y_i(Y_i-1))=\Theta(\lambda_c)=\Theta(r/n).$$

So using the fact that the Y_i's are independent and Chebyshev's inequality,

$$\mathbb{P}\left(\Big|\sum_{i=1}^n\binom{Y_i}{2}-\mu\Big|\geq\psi(n)\right)=O\left(\frac{r}{\psi(n)^2}\right).$$

Together with (3.20) this implies that $\mathbb{P}(F) = O(r/\psi(n)^2)$.

By Theorem 2.10.8 and Lemma 2.10.7,

$$\mathbb{P}(\Sigma) \sim \frac{1}{\sqrt{2\pi nc(1+\eta_c - c)}} \sim \frac{1}{\sqrt{2\pi r}} = \Omega\left(\frac{1}{\sqrt{r}}\right).$$

Thus,

$$\mathbb{P}(F|\Sigma) \leq \frac{\mathbb{P}(F)}{\mathbb{P}(\Sigma)} = O(\sqrt{r})O\left(\frac{r}{\psi(n)^2}\right) = O\left(\frac{r^{3/2}}{\psi(n)^2}\right).$$

Now consider the last condition in the definition of $\tilde{\mathcal{D}}(\psi)$: $\max_i d_i \leq 8\log n'(\mathbf{d})$. If the first condition in the definition of $\tilde{\mathcal{D}}(\psi)$ holds, then, using Lemma 3.4.1, we have $D_2(\mathbf{d}) = n - r + \phi(n)$ for some function $\phi(n) = o(r)$ and so $n'(\mathbf{d}) = r - \phi(n)$. Let F' denote the event that the first condition holds but the last condition fails. Thus, $\mathbb{P}(F') \leq \mathbb{P}\big(\max_i Y_i \geq 8\log(r - \phi(n))\big)$. For $r \leq \sqrt{n}$, by Lemma 2.10.3(a),

$$\mathbb{E}\left(D_j(\mathbf{Y})\right) = \frac{\lambda_c^2}{j! f_2(\lambda_c)} \lambda_c^{j-2} = O\bigg(\frac{r^{j-2}}{n^{j-2}}\bigg),$$

for every $j \geq 3$. Thus, using Markov's inequality and the union bound,

$$\mathbb{P}\big(D_j(\mathbf{Y}) \geq 1 \text{ for some } j \geq 4\big) \leq n \cdot O(1/n^2) = O(1/n).$$

For $r > \sqrt{n}$, it is easy to bound the tail probability of Y_i. By (2.18),

$$\mathbb{P}\Big(Y_i \geq 8\log(r - \phi(n))\Big) = O\Big(\exp\Big(-4\log\big(r - \phi(n)\big)\Big)\Big) = O\Big(\exp(-4\log r)\Big) = O\left(\frac{1}{n^2}\right).$$

Thus, $\mathbb{P}(\max_i Y_i > 8\log(r - \phi(n))) = O(1/n)$. Since $\mathbb{P}(\Sigma) = \Omega(1/\sqrt{r})$, we conclude that

$$\mathbb{P}(F'|\Sigma) \leq O(\sqrt{r})O(1/n) = O(\sqrt{r}/n).$$

Hence

$$\mathbb{P}(\tilde{\mathcal{D}}(\psi)|\Sigma) \geq 1 - \mathbb{P}(F|\Sigma) - \mathbb{P}(F'|\Sigma) = 1 + O\bigg(\frac{r\sqrt{r}}{\psi(n)^2}\bigg) + O\bigg(\frac{\sqrt{r}}{n}\bigg) = 1 + O\Big(r^{2\varepsilon - 1/2}\Big) + O\bigg(\frac{\sqrt{r}}{n}\bigg),$$

and we proved (3.17). This finishes the proof of Theorem 3.1.3(a).

3.4.1 2-connected simple pre-kernels

In this section, we estimate the probability that the graph generated with kernel configuration model with degree sequence $\mathbf{d}$ is 2-connected and simple, thus proving Proposition 3.4.2. Let $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$. Let G be the pre-kernel obtained by the kernel configuration model with degree sequence $\mathbf{d}$ and let K denote the kernel. Since the pre-kernel G is obtained from K by subdividing edges, any loop in K either remains a loop in G or it receives at least one vertex and then the vertex incident to the loop in K becomes a cut-vertex in G. Hence, G is 2-connected and simple if and only G is simple and K is 2-connected and loopless (but K is permitted to have multiple edges). Let B denote the event that G is simple and K is 2-edge-connected and has no loops. The maximum degree in K is at most $8\log(n') < (n')^{0.04}$ and so by Proposition 3.3.1

$$\mathbb{P}(B) = \mathbb{P}(\mathbf{2cs}(\mathbf{d})) + o(1),$$

and thus it suffices to show that $\mathbb{P}(B) \sim 1/e$.

First we compute the probability that G is simple. We use a result in [56]:

Lemma 3.4.3 ([56, Lemma 5])**.** Let $\hat{\mathcal{D}}(n,m)$ be the subset of $\mathcal{D}(n,m)$ such that $\mathbf{d} \in \hat{\mathcal{D}}(n,m)$ if $\max_i d_i \le 6\log n$ and $\sum_{i\in R(\mathbf{d})} \binom{d_i}{2} < 4r$. The graph G generated with kernel configuration model and degree sequence $\mathbf{d}$ is connected and simple with probability $1 + O(r^{-1} + rn^{-1})$ uniformly for $\mathbf{d} \in \hat{\mathcal{D}}(n,m)$ with $r \to \infty$ and $r = o(n)$.

We remark that we only use the fact that the probability that G is simple is $1 + o(1)$ and so Lemma 3.4.3 states more than we actually need. By looking at its proof in [56], one can easily see that the probability that G is simple is $1 + O(r/n)$.

We have that $8\log n'(\mathbf{d}) \le 5(\log n)$ for n sufficiently large, depending only on ψ. This is because $n'(\mathbf{d}) = r + o(r) = o(n)$ for $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$. So we have that $\max d_i \le 6\log n$ and, by Lemma 3.4.1 and the definition of $\tilde{\mathcal{D}}(\psi)$,

$$\sum_{i\in R(\mathbf{d})} \binom{d_i}{2} = \sum_{i=1}^{n} \binom{d_i}{2} - D_2(\mathbf{d}) = n + 2r + o(r) - (n - r + o(r)) = 3r + o(r) < 4r,$$

for large n (the required size of n depending only on m and ψ). Thus, the probability of G being simple is $1 + o(1)$ by Lemma 3.4.3 and so it suffices to show that the probability that the kernel is 2-edge-connected and loopless is asymptotic to $1/e$.

For a random pairing with a given degree sequence such that each entry has value at least 3, the probability of being 2-edge-connected was investigated by Łuczak in [46].

Theorem 3.4.4 ([46, Theorem 12.1])**.** Let H be a graph obtained with pairing model with degree degree sequence $\mathbf{d} \in \mathcal{D}(n,m)$ such that each entry has value at least 3. Then

(i) With probability going 1 as m goes to infinity, all 2-edge connected maximal subgraphs in H, except at most one, are loops in vertices of degree 3.

(ii) If $D_3/m \to \alpha$ as $m \to \infty$, where D_3 is the number of vertices of degree 3, then the probability that H is 2-edge-connected goes to $\exp(-1.5\alpha)$ as $M \to \infty$.

We remark that [46, Theorem 12.1] is stronger and has more statements than Theorem 3.4.4, but we only included the statements we will use. Recall that $m'(\mathbf{d}) = m - D_2(\mathbf{d})$ is the number of edges in the kernel. Using Lemma 3.4.1 for μ_2 and the fact that $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$,

$$m'(\mathbf{d}) = m - D_2(\mathbf{d}) = m - \mu_2 + O(\psi) = m - \mu_2 + o(r) = \frac{3r + o(r)}{2}.$$

Applying this to K, we have

$$\frac{D_3(\mathbf{d})}{m'} = \frac{r + o(r)}{(3/2)r + o(r)} \sim \frac{2}{3}$$

so the probability that K is 2-edge-connected goes to $1/e$ by Theorem 3.4.4(ii). Note that K being 2-edge-connected implies that there are no loops on vertices of degree 3 in K. The expected number of loops in K on vertices of degree at least 4 is

$$\begin{aligned}
\sum_{i:d_i \geq 4} \binom{d_i}{2} \frac{(2m'-3)!!}{(2m'-1)!!} &= \sum_{i:d_i \geq 4} \binom{d_i}{2} \frac{1}{2m'-1} \\
&= \left(\sum_{i=1}^{n} \binom{d_i}{2} - D_2 - 3D_3 \right) \frac{1}{2m - 2D_2 - 1} \\
&= \frac{n + 2r - (n-r) - 3r + o(r)}{2m - 2(n-r) + o(r)} \quad \text{by Lemma 3.4.1} \\
&= \frac{o(r)}{3r + o(r)} = o(1)
\end{aligned}$$

and so a.a.s. no such loops exist. We conclude that $\mathbb{P}(B) \sim 1/e$ and so $\mathbb{P}(\mathbf{2cs}(\mathbf{d})) \sim \frac{1}{e}$.

Observation 1. An alternative to compute the probability that the graph generated by the pairing model with a degree sequence with minimum entry at least 3 is bridgeless and loopless is to directly modify Łuczak's proof: when computing the expected number of bridges (Y in his notation), one could leave out the bridges whose deletion would create a component with a single vertex and a single edge. By doing so, it is easy to see that the expected value of Y would be $o(1)$. It follows directly from this that the only bridges are a.a.s. adjacent to vertices of degree 3 with a loop. Then, one would only need to compute the probability of having no loops.

3.5 The case c bounded away from 2, and bounded

In this section, we obtain an asymptotic formula for the number $T(n,m)$ of 2-connected (n,m)-graphs for the range $c = 2m/n = O(1)$ bounded away from 2 from above, proving Theorem 3.1.2(b). We also obtain a formula for the number $T(\mathbf{d})$ of 2-connected graphs with degree sequence $\mathbf{d}$ with $\mathbf{d} \in \mathcal{D}(n,m)$ satisfying some constraints, proving Theorem 3.1.3(b). In Proposition 3.2.2, we have already shown how to obtain $T(\mathbf{d})$ by computing $\mathbb{P}(\mathbf{2cs}(\mathbf{d}))$, the probability that a graph generated with the kernel configuration model is 2-connected and simple, and how to obtain $T(n,m)$ by computing $\mathbb{E}\left(\sqrt{m'(\mathbf{Y})}\,\mathbb{P}(\mathbf{2cs}(\mathbf{Y}))\middle|\Sigma\right)$, where $\mathbf{Y} = (Y_1,\ldots,Y_n)$ is a vector of independent truncated Poisson random variables with parameter $(2,\lambda_c)$ and Σ is the event that $\sum_i Y_i = 2m$.

We will define a set of typical degree sequences $\tilde{\mathcal{D}}$ so that, for $\mathbf{d} \in \tilde{\mathcal{D}}$, we have $\mathbb{P}(\mathbf{2cs}(\mathbf{d})) \sim \exp(-c/2 - \lambda_c^2/4)$ and so we obtain $T(\mathbf{d})$ for typical degree sequences. We use the following strategy to compute $\mathbb{P}(\mathbf{2cs}(\mathbf{d}))$. Proposition 3.3.1 tells us that the kernel is 2-connected if and only if it is 2-edge-connected. We then use a result by Łuczak [46] that says that every bridge is adjacent to a vertex of degree 3 with a loop. This means that a.a.s. the pre-kernel is 2-connected and simple if and only if the kernel is loopless and for every pair of parallel edges in the kernel at least one of them is subdivided. We use Theorem 2.8.1 (Method of factorial moments for Poisson random variables) to show that the number of loops in the kernel and the number of pairs of parallel edges in the pre-kernel have distribution asymptotic to a Poisson random variable with parameter $c/2 + \lambda_c^2/4$.

Similarly to the case $c \to 2$, in order to compute $T(n,m)$, we only need to show that the degree sequences outside $\tilde{\mathcal{D}}$ have no significant contribution to $\mathbb{E}\left(\sqrt{m'(\mathbf{Y})}\,\mathbb{P}(\mathbf{2cs}(\mathbf{Y}))\middle|\Sigma\right)$.

We start by defining a set of typical degree sequences. Let $\psi : \mathbb{N} \to \mathbb{R}$ be a function such that $\psi(n) = o(1)$. Recall that p_c is the probability that a random variable with distribution $\mathrm{Po}(2,\lambda_c)$ has value 2. Let

$$\tilde{\mathcal{D}}_n(\psi) = \tilde{\mathcal{D}}(\psi) := \Big\{\mathbf{d} \in \mathcal{D}(n,m) : d_i \le 6\log n \ \forall i;\ |\eta(\mathbf{d}) - \eta_c| \le \psi(n);\ |D_2(\mathbf{d}) - p_c n| \le n\psi(n)\Big\}.$$

Later we will choose ψ and show that $\tilde{\mathcal{D}}(\psi)$ is indeed a set of typical degree sequences, that is, we will show that $\mathbf{Y}$ conditioned upon Σ is in $\tilde{\mathcal{D}}(\psi)$ a.a.s. Let $\tilde{\mathcal{D}}^c(\psi) = \{\mathbf{d} \in \mathbb{N}^n : d_i \ge 2\ \forall i; \mathbf{d} \notin \tilde{\mathcal{D}}(\psi)\}$. (Note that if $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$ then $\sum d_i = 2m$ but we do not have this constraint for $\tilde{\mathcal{D}}^c(\psi)$.)

Let $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$ and let $\mathbf{d}'$ be the restriction of $\mathbf{d}$ to the coordinates with value at least 3. Let G be obtained using the kernel configuration model with degree sequence $\mathbf{d}$. Let $n' = n'(\mathbf{d}) := |\{i : d_i \ge 3\}|$ denote the number of vertices of degree at least 3. Let M be the random perfect matching placed on the set of $\sum_{i=1}^{n'} d'_i$ points grouped in bins of size $d'_1, d'_2, \ldots, d'_{n'}$ to obtain the kernel. Let K be the kernel obtained by contracting these bins.

We want to compute the probability that G is 2-connected and simple. Let B be the event that G is simple and that K is 2-edge-connected and has no loops. Since $n' = (1 - p_c)n + o(n) = \Theta(n)$, we have $\max_i d_i \le 6 \log n \le (n')^{0.04}$, and so Proposition 3.3.1 says that, a.a.s., the event B implies the event that K is 2-connected. If K is 2-connected and loopless, it is obvious that G is also 2-connected. In other words, $\mathbb{P}(B \setminus \mathbf{2cs}(\mathbf{d})) = o(1)$ and, since $\mathbf{2cs}(\mathbf{d}) \subseteq B$, we deduce $\mathbb{P}(\mathbf{2cs}(\mathbf{d})) = \mathbb{P}(B) + o(1)$.

Let A denote the event that G has no multiple edges and K has no loops. Theorem 3.4.4(iii) due to Łuczak states that a.a.s. all 2-edge-connected maximal subgraphs in K, except at most one, have a single edge. That is, a.a.s. each 2-edge-connected maximal subgraph, except at most one, consists of a single vertex with a loop and the vertex has degree 3 in K. Hence, $\mathbb{P}(A \setminus B) = o(1)$. Since $B \subseteq A$, we deduce $\mathbb{P}(A) = \mathbb{P}(B) + o(1)$. We estimate the probability of A as follows:

Lemma 3.5.1. We have that

$$\mathbb{P}(A) \sim \exp(-c/2 - \lambda_c^2/4). \tag{3.21}$$

We present the proof for this lemma in Section 3.5.1. Thus, we have

$$\mathbb{P}(\mathbf{2cs}) = \mathbb{P}(A) + o(1) \sim \exp(-c/2 - \lambda_c^2/4). \tag{3.22}$$

The proof of Theorem 3.1.3(b) is now straightforward. For $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$, we have that $|D_2(\mathbf{d}) - p_c n| \le n\psi(n)$. Thus,

$$\sqrt{\frac{m'(\mathbf{d})}{m}} = \sqrt{\frac{m - D_2(\mathbf{d})}{m}} = \sqrt{\frac{(c/2)n - p_c n + o(n)}{(c/2)n}} \sim \sqrt{\frac{c - 2p_c}{c}}$$

since $c > 2$ and $p_c \le 1$. Using this fact together with (3.22),

$$\mathbb{P}(\mathbf{2cs}(\mathbf{d}))\sqrt{m'} \sim \sqrt{m}\sqrt{\frac{c - 2p_c}{c}} \exp(-c/2 - \lambda_c^2/4), \tag{3.23}$$

which together with Proposition 3.2.2 (Equation (3.5)) shows that

$$T(\mathbf{d}) \sim \frac{(2m-1)!!\sqrt{m'(\mathbf{d})/m}\,\mathbb{P}(\mathbf{2cs}(\mathbf{d}))}{\prod_{i=1}^n d_i!} \sim \frac{(2m-1)!!}{\prod_{i=1}^n d_i!}\sqrt{\frac{c - 2p_c}{c}} \exp(-c/2 - \lambda_c^2/4),$$

proving Theorem 3.1.3(b).

We now prove Theorem 3.1.2(b). First we show that

$$\mathbb{P}\left(\mathbf{Y} \in \tilde{\mathcal{D}}^c(\psi)\right) = O\left(\frac{1}{n\psi(n)^2}\right) \quad \text{and} \quad \mathbb{P}\left(\mathbf{Y} \in \tilde{\mathcal{D}}^c(\psi) | \Sigma\right) = O\left(\frac{1}{n^{1/2}\psi(n)^2}\right). \tag{3.24}$$

Since $c = O(1)$ we can use (2.18) with $j_0 = 6\log n$, apply the union bound, and conclude

$$\mathbb{P}(\max_i Y_i > 6\log n) = O\left(\frac{1}{n^2}\right).$$

Note that $D_2(\mathbf{Y})$ has binomial distribution with probability p_c. Using Chebyshev's inequality,

$$\mathbb{P}(|D_2(\mathbf{Y}) - p_c n| \geq n\psi(n)) \leq \frac{p_c(1-p_c)n}{n^2\psi(n)^2} = O\left(\frac{1}{n\psi(n)^2}\right)$$

since $0 \leq p_c \leq 1$

We have that $\lambda_c = O(1)$, since $c = O(1)$ and by Lemma 2.10.3. Using Chebyshev's inequality, together with Lemma 2.10.10 and the facts that the variables Y_i's are independent and $\lambda_c = O(1)$, we get

$$\mathbb{P}(|\eta(\mathbf{Y}) - \eta_c| \geq \psi(n)) = O\left(\frac{1}{n\psi(n)^2}\right).$$

Hence,

$$\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}^c) = O\left(\frac{1}{n\psi(n)^2}\right).$$

By Theorem 2.10.8, we have that

$$\mathbb{P}(\Sigma) \sim \frac{1}{\sqrt{2\pi nc(1+\eta_c - c)}}.$$

Using Lemma 2.10.7, we have that $c(1+\eta_c - c) = O(c-2) = O(1)$. This implies that $\mathbb{P}(\Sigma) = \Omega(1/\sqrt{n})$. Conditioning on Σ, we have

$$\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}^c|\Sigma) \leq \frac{\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}^c)}{\mathbb{P}(\Sigma)} = O\left(\frac{n^{1/2}}{n\psi(n)^2}\right) = O\left(\frac{1}{n^{1/2}\psi(n)^2}\right).$$

This proves (3.24).

Let ε be a constant in $(0, 1/4)$ and let $\psi(n) = n^{-\varepsilon}$. We have that

$$\mathbb{E}\left(w(\mathbf{Y})|\Sigma\right) = \mathbb{E}\left(w(\mathbf{Y})|\tilde{\mathcal{D}}(\psi)\right)\mathbb{P}(\tilde{\mathcal{D}}(\psi)|\Sigma) + \mathbb{E}\left(w(\mathbf{Y})|\Sigma \cap \tilde{\mathcal{D}}^c(\psi)\right)\mathbb{P}(\tilde{\mathcal{D}}^c(\psi)|\Sigma).$$

Note that $w(\mathbf{Y}) \leq \sqrt{m}$ since $\mathbb{P}(\mathbf{2cs}) \leq 1$. By (3.24), we have that $\mathbb{P}(\tilde{\mathcal{D}}^c(\psi)|\Sigma) = O(1/n^{1/2-2\varepsilon})$. So $\mathbb{E}\left(w(\mathbf{Y})|\Sigma \cap \tilde{\mathcal{D}}^c\right)\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma) = O(\sqrt{m}/n^{1/2-2\varepsilon})$.

For any $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$, we have that $w(\mathbf{d}) \sim \sqrt{m(c-2p_c)/c}\exp(-c/2 - \lambda_c^2/4) =: t(n)$ by (3.23) and, since $\tilde{\mathcal{D}}_n(\psi)$ is a finite set for each n, we have that there exists a function $h(n) = o(1)$ such that $|w(\mathbf{d})/t(n) - 1| \leq h(n)$ for any $\mathbf{d} \in \tilde{\mathcal{D}}_n(\psi)$ by Lemma 2.7.1 and so

$$\left|\mathbb{E}\left(w(\mathbf{Y})|\Sigma \cap \tilde{\mathcal{D}}(\psi)\right) - t(n)\right| \leq \sum_{\mathbf{d} \in \tilde{\mathcal{D}}(\psi)} |w(\mathbf{d}) - t(n)|\,\mathbb{P}(\mathbf{Y} = \mathbf{d}\,|\,\Sigma \cap \tilde{\mathcal{D}}(\psi)) \leq h(n)t(n) = o(t(n)).$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left(w(\mathbf{Y})|\Sigma\right) &= \mathbb{E}\left(w(\mathbf{Y})|\Sigma \cap \tilde{\mathcal{D}}(\psi)\right)\left(1 - O\left(\frac{1}{n^{1/2-2\varepsilon}}\right)\right) + O\left(\frac{\sqrt{m}}{n^{1/2-2\varepsilon}}\right) \\
&= \sqrt{m}\sqrt{\frac{c-2p_c}{c}}\exp\left(-\frac{c}{2}-\frac{\lambda_c^2}{4}\right)\left(1+o(1)\right)\left(1 - O\left(\frac{1}{n^{1/2-2\varepsilon}}\right)\right) + O\left(\frac{\sqrt{m}}{n^{1/2-2\varepsilon}}\right) \\
&= \sqrt{m}\sqrt{\frac{c-2p_c}{c}}\exp\left(-\frac{c}{2}-\frac{\lambda_c^2}{4}\right)\left(1+o(1)\right),
\end{aligned}
$$

which together with Proposition 3.2.2 (Equation (3.6)) and (3.1) implies

$$
\begin{aligned}
T(n,m) &= (2m-1)!!Q(n,m)\sqrt{m^{-1}}\mathbb{E}\left(w(\mathbf{Y})|\Sigma\right) \\
&\sim (2m-1)!!\frac{f_2(\lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi nc(1+\eta_c-c)}}\sqrt{m^{-1}}\mathbb{E}\left(w(\mathbf{Y})|\Sigma\right) \\
&\sim (2m-1)!!\frac{f_2(\lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi nc(1+\eta_c-c)}}\sqrt{\frac{c-2p_c}{c}}\exp\left(-\frac{c}{2}-\frac{\lambda_c^2}{4}\right),
\end{aligned}
$$

proving Theorem 3.1.2(b).

3.5.1 Probability of no loops in the kernel and no multiple edges in the pre-kernel.

In this section, we estimate the probability that the kernel has no loops and the pre-kernel has no multiple edges, proving Lemma 3.5.1. Recall that $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$. Let $e_1, \ldots, e_\ell$ denote the pairs of points that would induce loops in K. For every $1 \leq i \leq \ell$, let X_i be the indicator variable for $e_i \in E(K)$. Let $X = \sum_{i=1}^{\ell} X_i$, that is X counts the loops in the kernel. Let $f_1, \ldots, f_t$ denote the pairs of points that would induce double edges in K (here we do not include double loops). For every $1 \leq j \leq t$, let Y_j be the indicator variable for $f_j \subseteq E(G)$. Let $Y = \sum_{j=1}^{t} Y_j$, that is Y counts the pairs of parallel edges in the pre-kernel.

Using Theorem 2.8.1, we will show that $X + Y$ converges in distribution to a Poisson random variable with mean $c/2 + \lambda_c^2/4$. This implies that

$$
\mathbb{P}\left(A\right) = \mathbb{P}\left(X+Y=0\right) = \mathbb{P}\left(\mathrm{Po}(c/2+\lambda_c^2/4)=0\right) + o(1) \sim \exp\left(-\frac{c}{2}-\frac{\lambda_c^2}{4}\right),
$$

which proves Lemma 3.5.1. We need to show, for every positive integer k, that

$$
\mathbb{E}\left([X+Y]_k\right) = \left(\frac{c}{2}+\frac{\lambda_c^2}{4}\right)^k + o(1),
$$

where $[x]_i$ is defined as $x(x-1)\cdots(x-i+1)$. Let $\Phi(x) = (2x-1)!! = (2x)!/(2^x x!)$ for $x \in \mathbb{N}$.

Let M denote the number of edges in K. Considering the first moment, note that for every $1 \leq i \leq \ell$, we have that

$$\mathbb{P}(X_i = 1) = \frac{\Phi(m'-1)}{\Phi(m')} = \frac{1}{2m'-1} \sim \frac{1}{2m'}.$$

For the parallel edges, we need to know the probability that a given set of edges of the kernel is not assigned any vertices of degree 2 in the kernel configuration model. Let

$$\delta = \left(\frac{c - 2p_c}{c}\right)^2 = \left(\frac{\lambda_c}{c}\right)^2. \tag{3.25}$$

For any fixed q and any set of pair of points $\{h_1, \ldots, h_q\}$ in K, the probability that none of these kernel edges is assigned a vertex of degree 2 (and hence become edges of G) can be estimated as follows:

$$\begin{aligned}
&\mathbb{P}\left(\{h_1, \ldots, h_q\} \subseteq E(G) \,\middle|\, \{h_1, \ldots, h_q\} \subseteq E(K)\right) = \prod_{i=0}^{D_2-1} \left(1 - \frac{q}{m'+i}\right) \\
&= \exp\left(\sum_{i=0}^{D_2-1} \log\left(1 - \frac{q}{m'+i}\right)\right) = \exp\left(-q \sum_{i=0}^{D_2-1} \frac{1}{m'+i} + O\left(\frac{D_2}{m^2}\right)\right),
\end{aligned}$$

since $\log(1+x) = 1 + x + O(x^2)$ for $x \to 0$. Thus, using that $\sum_{i=1}^{j} = \log j + \gamma + O(1/j)$ where γ is the Euler-Mascheroni constant we have

$$\begin{aligned}
&\mathbb{P}\left(\{h_1, \ldots, h_q\} \subseteq E(G) \,\middle|\, \{h_1, \ldots, h_q\} \subseteq E(K)\right) \sim \exp\left(-q \sum_{i=1}^{m'+D_2-1} \frac{1}{i} + q \sum_{i=1}^{m'-1} \frac{1}{i}\right) \\
&\sim \exp\Big(-q\log(m'+D_2-1) + q\log(m'-1)\Big) = \left(\frac{m'-1}{m'+D_2-1}\right)^q \\
&\sim \left(\frac{cn/2 - p_c n}{cn/2}\right)^q = \delta^{q/2},
\end{aligned} \tag{3.26}$$

since $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$. Thus, for every $1 \leq j \leq t$, we have that

$$\mathbb{P}\left(Y_j = 1\right) = \mathbb{P}\left(f_j \subseteq E(K)\right) \cdot \mathbb{P}\left(f_j \subseteq E(G) \,\middle|\, f_j \subseteq E(K)\right) \sim \frac{\Phi(m'-2)}{\Phi(m')}\delta \sim \frac{\delta}{(2m')^2}.$$

Recall $\mathbf{d}'$ is the degree sequence of K. Hence,

$$\begin{aligned}
\mathbb{E}\left(X+Y\right) = \mathbb{E}\left(X\right) + \mathbb{E}\left(Y\right) &\sim \ell \cdot \frac{1}{2M} + t \cdot \frac{\delta}{(2M)^2} \\
&= \frac{\sum_{i=1}^{n'} \binom{d_i'}{2}}{2M} + \frac{\delta}{(2M)^2} \sum_{\substack{(i,j) \\ i \neq j}} \binom{d_i'}{2}\binom{d_j'}{2},
\end{aligned}$$

since the number of possible of loops is $\ell = \sum_{i=1}^{n'} \binom{d'_i}{2}$ and the number of possible pairs of parallel edges is $t = \sum_{i \neq j} \binom{d'_i}{2}\binom{d'_j}{2}$. We will use the following lemma, which is proved in the end of the section.

Lemma 3.5.2. Let q be a fixed positive integer. For $d \in \tilde{\mathcal{D}}(\psi)$,

$$\sum_{(i_1,\ldots,i_q)} \prod_{j=1}^{q} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} \sim \left(\frac{c}{2}\right)^q,$$

where the sum is over all $(i_1, \ldots, i_q) \in [n']^q$ where $i_j \neq i_{j'}$ for all $j \neq j'$.

Thus, using the definition of δ,

$$\mathbb{E}\left(X + Y\right) \sim \frac{c}{2} + \delta\left(\frac{c}{2}\right)^2 = \frac{c}{2} + \left(\frac{\lambda_c}{c}\right)^2 \frac{c^2}{4} = \frac{c}{2} + \frac{\lambda_c^2}{4}.$$

It only remains to examine the higher factorial moments. We have that

$$\mathbb{E}\left([X+Y]_k\right) = \sum_{k_1+k_2=k} \binom{k}{k_1} \sum_{y \in I(k_1,k_2)} \mathbb{P}(W(y) = 1)$$

for $y \in I(k_1, k_2)$, where $I(k_1, k_2)$ is the set of tuples $y \in (\{e_1, \ldots, e_\ell\})^{k_1} \times (\{f_1, \ldots, f_t\})^{k_2}$ such that $y_i \neq y_j$ for $i \neq j$ and $\bigcup_{i=1}^{k}\{y_i\}$ induces a matching on the set of points of the kernel configuration model, and $W(y)$ is the indicator variable for the event that $X_i = 1$ for every $e_i \in \{y_1, \ldots, y_k\}$ and $Y_j = 1$ for every $f_j \in \{y_1, \ldots, y_k\}$. (For more details, see the observations in Section 2.8)

Let $I'(k_1, k_2)$ be the set of tuples $y \in I(k_1, k_2)$ such that, in the graph induced by $\bigcup_{i=1}^{k}\{y_i\}$ in K, the degree of every vertex is either 0 or 2. (This is the nonoverlapping case.) Let $I''(k_1, k_2) = I(k_1, k_2) \setminus I'(k_1, k_2)$.

For $y \in I''(k_1, k_2)$, it is easy to see that the graph induced by $\bigcup_{i=1}^{k}\{y_i\}$ in K has more edges than vertices. For any fixed multigraph H with more edges than vertices, the expected number of copies of H in K can be bounded as follows. There are at most $(n')^{|V(H)|}$ ways of assigning the vertices of H to vertices of K. If we assign a vertex with degree d in H to a vertex v in K, then there are at most Δ^d ways of choosing the points inside v to be the points of the vertex in H, where Δ is the maximum degree in K. So there are at most $(n')^{|V(H)|}\Delta^{2|E(H)|} = O((n')^{|V(H)|}(\log n)^{2|E(H)|})$ possible copies of H in K, because $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$. The probability that a set of $|E(H)|$ edges in K is $O((m')^{-|E(H)|})$. Thus, the expected number of copies of H in K is at most

$$O\left(\frac{(n')^{|V(H)|}(\log n)^{2|E(H)|}}{(m')^{|E(H)|}}\right) = O\left(\frac{(n')^{|V(H)|}(\log n)^{2|E(H)|}}{(n')^{|V(H)|+1}}\right) = o(1).$$

From this and the fact that there are $O(1)$ non-isomorphic possible graphs induced by $\bigcup_{i=1}^k \{y_i\}$ in K (since k is fixed), we deduce that

$$\sum_{k_1+k_2=k} \binom{k}{k_1} \sum_{y \in I''(k_1,k_2)} \mathbb{P}(W(y)=1) = o(1).$$

For $I'(k_1, k_2)$, using (3.26) and Lemma 3.5.2,

$$\begin{aligned}
\sum_{y \in I'(k_1,k_2)} \mathbb{P}(W(y)=1) &\sim \sum_{y \in I'(k_1,k_2)} \frac{\delta^{k_2}}{(2m')^{k_1+2k_2}} = |I'(k_1,k_2)| \frac{1}{(2m')^{k_1+2k_2}} \delta^{k_2} \\
&= \sum_{(i_1,\ldots,i_{k_1+2k_2})} \prod_{j=1}^{k_1+2k_2} \binom{d'_{i_j}}{2} \frac{1}{(2m')^{k_1+2k_2}} \cdot \delta^{k_2} \\
&\sim \left(\frac{c}{2}\right)^{k_1+2k_2} \delta^{k_2},
\end{aligned}$$

where the second summation is over the tuples $(i_1, \ldots, i_{k_1+2k_2}) \in [n']^{k_1+2k_2}$ such that $i_j \neq i_{j'}$ whenever $j \neq j'$. Thus,

$$\begin{aligned}
\mathbb{E}\left([X+Y]_k\right) &= o(1) + \sum_{k_1+k_2=k} \binom{k}{k_1} \sum_{y \in I'(k_1,k_2)} \mathbb{P}(W(y)=1) \\
&= \sum_{k_1+k_2=k} \binom{k}{k_1} \left(\frac{c}{2}\right)^{k_1+2k_2} \delta^{k_2} + o(1) \\
&= \left(\frac{c}{2} + \frac{\lambda_c^2}{4}\right)^k + o(1),
\end{aligned}$$

as required to prove Lemma 3.5.1.

Proof of Lemma 3.5.2. For every $q \geq 1$, let

$$\begin{aligned}
L_q^{\neq} &= \{(i_1, \ldots, i_q) \in [n']^q : i_j \neq i_{j'} \ \forall j \neq j'\}, \\
L_q^{=} &= \{(i_1, \ldots, i_q) \in [n']^q : i_j = i_{j'} \text{ for some } j \neq j'\}.
\end{aligned}$$

Since $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$, by the definition of η_c and p_c,

$$\begin{aligned}
\frac{\sum_{i=1}^{n'} d'_i(d'_i-1)}{\sum_{i=1}^{n'} d'_i} &= \frac{\sum_{i=1}^{n} d_i(d_i-1) - 2D_2}{\sum_{i=1}^{n} d_i - 2D_2} \sim \frac{\eta_c cn - 2p_c n}{cn - 2p_c n} = \frac{\eta_c c - 2p_c}{c - 2p_c} \\
&= \frac{\eta_c - \dfrac{\lambda_c^2}{cf_2(\lambda_c)}}{1 - \dfrac{\lambda_c^2}{cf_2(\lambda_c)}} = \frac{\dfrac{\lambda_c e^{\lambda_c}}{f_1(\lambda_c)} - \dfrac{\lambda_c}{f_1(\lambda_c)}}{1 - \dfrac{\lambda_c}{f_1(\lambda_c)}} = \lambda_c \left(\frac{e^{\lambda_c} - 1}{f_1(\lambda_c) - \lambda_c}\right) = c.
\end{aligned}$$

So, for every $q \geq 1$,

$$\sum_{(i_1,\ldots,i_q)\in[n']^q} \prod_{j=1}^{q} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2m')^q} = \left(\frac{\sum_i \binom{d'_i}{2}}{2m'} \right)^q \sim \left(\frac{c}{2}\right)^q = \Theta(1). \tag{3.27}$$

For $q \geq 2$, for $\Delta = \max_i d'_i$

$$\begin{aligned} \sum_{(i_1,\ldots,i_q)\in L_q^{=}} \prod_{j=1}^{q} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2m')^q} &\leq q! \cdot \sum_{(i_1,\ldots,i_{q-1})\in[n']^{q-1}} \binom{d'_{i_1}}{2} \prod_{j=1}^{q-1} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2m')^q} \\ &\leq q! \frac{\Delta^2}{4m'} \sum_{(i_1,\ldots,i_{q-1})\in[n']^{q-1}} \prod_{j=1}^{q-1} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2m')^{q-1}} \\ &\sim q! \frac{\Delta^2}{4m'} \left(\frac{c}{2}\right)^{q-1} = o(1), \end{aligned} \tag{3.28}$$

since $\Delta \leq 6 \log n$ and $m' = m - D_2 \sim cn - p_c n = \Omega(n)$ as $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$.

Note that for $q = 1$, we have $[n']^q = L_q^{\neq}$ and we are done by (3.27). So suppose $q \geq 2$. Then $[n']^q$ is the disjoint union of $L_q^{\neq}$ and $L_q^{=}$. Thus, using (3.27) and (3.28),

$$\begin{aligned} &\sum_{(i_1,\ldots,i_q)\in L_q^{\neq}} \prod_{j=1}^{q} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2m')^q} = \\ &\quad = \sum_{(i_1,\ldots,i_q)\in[n']^q} \prod_{j=1}^{q} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2m')^q} - \sum_{(i_1,\ldots,i_q)\in L_q^{=}} \prod_{j=1}^{q} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2m')^q} \\ &\quad = \sum_{(i_1,\ldots,i_q)\in[n']^q} \prod_{j=1}^{q} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2m')^q} + o(1) \sim \left(\frac{c}{2}\right)^q. \end{aligned}$$

□

3.6 The case $c \to \infty$

In this section, we obtain an asymptotic formula for the number $T(n, m)$ of 2-connected (n, m)-graphs for the range $c = 2m/n \to \infty$ and $c = O(\log n)$, proving Theorem 3.1.2(c). We also obtain a formula for the number $T(\mathbf{d})$ of 2-connected graphs with degree sequence $\mathbf{d}$ with $\mathbf{d} \in \mathcal{D}(n, m)$ satisfying some constraints, proving Theorem 3.1.3(c).

Recall set $\mathcal{D}(n, m)$ contains all degree sequences $\mathbf{d}$ such that $\sum_{i=1}^{n} d_i = 2m$ and $d_i \geq 2$ for all $i \in [n]$. Recall that $U(\mathbf{d})$ is the probability of obtaining a simple graph using the pairing model with degree sequence $\mathbf{d}$, and $U'(\mathbf{d})$ is defined similarly, for the event that it is additionally 2-connected.

In view of Proposition 3.2.1, it suffices to estimate $U'(\mathbf{d})$ to approximate $T(\mathbf{d})$ and to estimate $\mathbb{E}\left(U'(\mathbf{Y}|\Sigma)\right)$ to estimate $T(n,m)$, where $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is a vector of independent truncated Poisson random variables with parameter $(2, \lambda_c)$ and Σ is the event that $\sum_i Y_i = 2m$.

We will define a set of typical degree sequences such that $U(\mathbf{d}) \sim U'(\mathbf{d})$. The probability $U(\mathbf{d})$ has been already intensively studied (see [5, 48]). This allows us to easily derive an asymptotic formula for $T(\mathbf{d})$. We then show that $\mathbb{E}\left(U'(\mathbf{d})|\Sigma\right) \sim \mathbb{E}\left(U(\mathbf{d})|\Sigma\right)$, which has also been already determined (see [55]) and so we get an asymptotic formula for $T(n,m)$.

Let $0 < \varepsilon < 0.01$ be a constant, and let

$$\tilde{\mathcal{D}}_n = \tilde{\mathcal{D}} := \{\mathbf{d} \in \mathcal{D}(n,m) : \max d_i \leq n^{\varepsilon}\} \quad \text{and} \quad \tilde{\mathcal{D}}^c := \mathcal{D}(n,m) \setminus \tilde{\mathcal{D}}.$$

By [46, Theorem 12.2(iii)],

$$U(\mathbf{d}) \sim U'(\mathbf{d}), \tag{3.29}$$

when $\mathbf{d}$ is in $\mathcal{D}(n,m)$ and satisfies $D_2(\mathbf{d})/m \to 0$ and $\max_i d_i \leq n^{0.01}$. The condition on D_2 is satisfied by all $\mathbf{d}$ of concern when n is large since $D_2(\mathbf{d}) \leq n$ and $c \to \infty$. Thus (3.29) holds for any sequence $\mathbf{d}(n)$ with $\mathbf{d} \in \tilde{\mathcal{D}}$ and $m/n \to \infty$ where $m = \frac{1}{2}\sum_{i=1}^n d_i$.

By Theorem 2.2.2 due to McKay,

$$U(\mathbf{d}) = \exp\left(-\eta(\mathbf{d})/2 - \eta(\mathbf{d})^2/4 + O\left(\frac{\max_i d_i^4}{m}\right)\right). \tag{3.30}$$

This result, together with Proposition 3.2.1 (Equation (3.3)) shows that

$$T(\mathbf{d}) = \frac{(2m-1)!!}{\prod_{j=1}^n d_j!} U'(\mathbf{d}) \sim \frac{(2m-1)!!}{\prod_{j=1}^n d_j!} \exp\left(-\eta(\mathbf{d})/2 - \eta(\mathbf{d})^2/4\right),$$

proving Theorem 3.1.3(c).

In order to use Proposition 3.2.1 we need to compute $\mathbb{E}\left(U'(\mathbf{Y})|\Sigma\right)$. For any $\mathbf{d} \in \tilde{\mathcal{D}}$, we have that $U(\mathbf{d}) \sim U'(\mathbf{d})$ by (3.29) and, since $\tilde{\mathcal{D}}_n$ is a finite set for each n, we have that there exists a function $h(n) = o(1)$ such that $|U(\mathbf{d})/U'(\mathbf{d}) - 1| \leq h(n)$ for any $\mathbf{d} \in \tilde{\mathcal{D}}_n$ by Lemma 2.7.1 and so

$$\left|\mathbb{E}\left(U'(\mathbf{Y})|\tilde{\mathcal{D}}\right) - \mathbb{E}\left(U(\mathbf{Y})|\tilde{\mathcal{D}}\right)\right| \leq \sum_{\mathbf{d} \in \tilde{\mathcal{D}}} \left|U'(\mathbf{d}) - U(\mathbf{d})\right| \mathbb{P}(\mathbf{Y} = \mathbf{d} \,|\, \tilde{\mathcal{D}}) \leq h(n) = o(1).$$

Thus,

$$\begin{aligned} \mathbb{E}\left(U'(\mathbf{Y})|\Sigma\right) &= \mathbb{E}\left(U'(\mathbf{Y})|\tilde{\mathcal{D}}\right)\mathbb{P}(\tilde{\mathcal{D}}|\Sigma) + \mathbb{E}\left(U'(\mathbf{Y})|\tilde{\mathcal{D}}^c\right)\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma) \\ &= \mathbb{E}\left(U(\mathbf{Y})|\tilde{\mathcal{D}}\right)(1+o(1))\,\mathbb{P}(\tilde{\mathcal{D}}|\Sigma) + O(\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma)) \end{aligned} \tag{3.31}$$

Equation (2.18) implies for any $\beta > 0$

$$\mathbb{P}\left(\max_j Y_j \geq m^\beta\right) \leq \exp(-n^\alpha)$$

for some fixed $\alpha(\beta)$. This shows that $\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma) = O(\exp(-n^\alpha))$ for some fixed positive α. Also, by Theorem 2.10.8 (Equation (2.16)) give us

$$\mathbb{E}\left(\exp(-\eta(\mathbf{Y})/2 - \eta(\mathbf{Y})^2/4)|\Sigma\right) \sim \exp(-\eta_c/2 - \eta_c^2/4) = \exp(-O(\log^2 n)),$$

since $\eta_c \leq c$ by (2.7) and $c = 2m/n = O(\log n)$.

Using (3.30) and the bound on $\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma)$, we may now deduce that the first term in (3.31) dominates the second, and thus

$$\mathbb{E}\left(U'(\mathbf{Y})|\Sigma\right) \sim \mathbb{E}\left(U(\mathbf{Y})|\tilde{\mathcal{D}}\right).$$

Similarly,

$$\mathbb{E}\left(U(\mathbf{Y})|\Sigma\right) = \mathbb{E}\left(U(\mathbf{Y})|\tilde{\mathcal{D}}\right)\mathbb{P}(\tilde{\mathcal{D}}|\Sigma) + O(\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma)) \sim \mathbb{E}\left(U(\mathbf{Y})|\tilde{\mathcal{D}}\right)$$

and so

$$\mathbb{E}\left(U'(\mathbf{Y})|\Sigma\right) \sim \mathbb{E}\left(U(\mathbf{Y})|\Sigma\right). \tag{3.32}$$

By [55, Theorem 3],

$$C(n,m) \sim (2m-1)!!Q(n,m)\exp\left(-\eta_c/2 - \eta_c^2/4\right).$$

Thus, by (3.2),

$$\mathbb{E}\left(U(\mathbf{Y})|\Sigma\right) \sim \exp\left(-\eta_c/2 - \eta_c^2/4\right),$$

and using Proposition 3.2.1 (Equation (3.4)) and (3.1), we get

$$T(n,m) \sim (2m-1)!!\frac{(e^{\lambda_c} - 1 - \lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi nc(1+\eta_c - c)}}\exp\left(-\eta_c/2 - \eta_c^2/4\right). \tag{3.33}$$

Since $c \to \infty$, we have that $\lambda_c \sim c$ by Lemma 2.10.3. This implies that $\eta_c = \lambda_c e^{\lambda_c}/(e^{\lambda_c} - 1) \sim c$. This fact together with (3.33) implies Theorem 3.1.2(c).

3.7 Proof of Theorem 3.1.1

We have already derived formulae for $T(n,m)$ according to the range of m. The ranges we considered were: $c \to 2$, bounded $c > 2$ (bounded away from 2), and $c \to \infty$, and the formulae are described in Theorem 3.1.2. Now we will show how to combine the formulae in these cases into a single formula. More precisely, we will show that the formula obtained in Theorem 3.1.2 for each of the cases is asymptotic to the formula in Theorem 3.1.1 and then Theorem 3.1.1 follows from a straightforward application of the subsubsequence principle (for more on this principle, see Section 2.6). Let

$$t(n,m) = (2m-1)!!\frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi nc(1+\eta_c - c)}}\sqrt{\frac{c-2p_c}{c}}\exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right).$$

The asymptotic formula for bounded $c > 2$ in Theorem 3.1.2 matches $t(n,m)$. Thus, it suffices to check the cases $c \to 2$ and $c \to \infty$

For $c \to 2$, by comparing the formula in Theorem 3.1.2 and $t(n,m)$, it suffices to show

$$\sqrt{\frac{3r}{2m}}\frac{1}{e} \sim \sqrt{\frac{c-2p_c}{c}}\exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right), \tag{3.34}$$

and, for $c \to \infty$, it suffices to show

$$\exp\left(-\frac{\eta_c}{2} - \frac{\eta_c^2}{4}\right) \sim \sqrt{\frac{c-2p_c}{c}}\exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right). \tag{3.35}$$

So suppose $c \to 2$. By (2.10), $\lambda_c = 3(c-2) + O((c-2)^2) = o(1)$. Thus,

$$\exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right) \sim \exp\left(-\frac{c}{2}\right) \sim \frac{1}{e}.$$

By series expansion,

$$p_c = 1 - \frac{\lambda_c}{3} + O(\lambda_c^2) = 1 - \frac{1}{3}\cdot(3(c-2)) + O((c-2)^2) = 3 - c + O\left(\frac{r^2}{n^2}\right).$$

Using $c = 2m/n = 2 + r/n$,

$$\sqrt{\frac{c-2p_c}{c}} = \sqrt{\frac{c-6+2c+O(r^2/n^2)}{c}} = \sqrt{\frac{3(2+r/n)-6+O(r^2/n^2)}{2m/n}} \sim \sqrt{\frac{3r}{2m}},$$

and so (3.34) holds. Now suppose $c \to \infty$. Then $\lambda_c \sim c$ by Lemma 2.10.3(c). From the definition of λ_c we have $c = \lambda_c + O(\lambda_c^2 e^{-\lambda_c})$. Also,

$$\eta_c = \lambda_c \cdot \frac{e^{\lambda_c}}{e^{\lambda_c} - 1} = \lambda_c + O(\lambda_c e^{-\lambda_c}) \quad \text{and} \quad p_c = \frac{\lambda_c^2}{2(e^{\lambda_c} - 1 - \lambda_c)} \to 0.$$

This implies

$$\sqrt{\frac{c - 2p_c}{c}} \sim 1 \quad \text{and} \quad \exp\left(-\frac{\eta_c}{2} - \frac{\eta_c^2}{4}\right) \sim \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right),$$

and so (3.35) holds.

The subsubsequence principle states that if, for every subsequence of $(T(n,m)/t(n,m))_{n \in \mathbb{N}}$, there exists a subsequence of it such that

$$\left|\frac{T(n,m)}{t(n,m)} - 1\right| = o(1), \tag{3.36}$$

then, considering the whole sequence,

$$\left|\frac{T(n,m)}{t(n,m)} - 1\right| = o(1).$$

For any subsequence of $(T(n,m)/t(n,m))_{n \in \mathbb{N}}$, there exists a subsequence of it such that $c = 2m/n \to \infty$ or $c \to 2$ or c is bounded and bounded away from 2 from above. In all cases, we checked that (3.36) holds and so we proved Theorem 3.1.1.

3.8 Enumeration of k-edge-connected graphs

Recall that $T'(n,m)$ denotes the number of 2-edge-connected (n,m)-graphs. In this section, we obtain an asymptotic formula for $T'(n,m)$ for the range $m - n \to \infty$ with $m = O(n \log n)$, thus proving Theorem 3.1.5.

The proof is very similar to the proof for 2-connected graphs and so we will only give an overview of the proof, highlighting the differences. For $c \to \infty$, we have already proved that an asymptotic formula for 2-cores with vertex set $[n]$ and m edges is also valid for 2-connected (n,m)-graphs. Since the number of 2-edge-connected (n,m)-graphs is between the number of 2-cores with vertex set $[n]$ and m edges and the number of 2-connected (n,m)-graphs, we have that the formula in Theorem 3.1.2(c) is an asymptotic formula for $T'(n,m)$. For $c \to 2$, our proof actually shows that the probability that the random graph G generated with kernel configuration model is 2-edge-connected is asymptotic to the probability that the graph is 2-connected. This is because, a.a.s., G is 2-connected if and only of its kernel is 2-edge-connected and loopless, and the probability of multiple edges or loops goes to zero. Hence, the formula in Theorem 3.1.2(a) is also an asymptotic formula for $T'(n,m)$ in this range.

We now discuss the case c bounded away from 2 from above and bounded. Recall that, for the 2-connected case, we have that, a.a.s., the random graph G generated with kernel configuration model was 2-connected and simple if and only if G was simple and the kernel K was 2-edge-connected and loopless. We then used a result by Łuczak that show that a.a.s. every bridge in the kernel is an edge incident to a vertex of degree 3 with a loop. We estimated $\mathbb{P}(X+Y)=0$ by using the method of factorial moments for Poisson random variables (Theorem 2.8.1), where X is the number of loops in K and Y is the number of double edges in G. For the 2-edge-connected case, G is simple and 2-edge-connected if and only if G is simple, the kernel K is 2-edge-connected such that every loop in K on vertices of degree at least 4 receives at least 2 vertices of degree 2. Note that loops are not allowed in the kernel of 2-connected graphs because they either remain as a loop in G and so G is not simple, or they receive vertices making the vertex of the kernel where the loop was a cut vertex. For 2-edge-connected graphs, loops in vertices of degree at least 4 are allowed in the kernel as long as they are subdvided at least twice when obtaining G. Loops in vertices of degree 3 are not allowed since the edge adjacent to vertex that is not the loop is then a bridge. Again, Łuczak's result shows that a.a.s. every bridge in the kernel is an edge incident to a vertex of degree 3 with a loop. Thus we need to estimate $\mathbb{P}(X+Y+Z)=0$, where X is the number of loops in the kernel on vertices of degree 3, Y is the number of double edges in pre-kernel G and Z is the number of loops on vertices of degree at least 4 receiving less than 2 vertices in G. This can be accomplished using Theorem 2.8.1 by estimating the factorial moments. We will only show the first moment here.

Similarly to Section 3.5.1, we only need to estimate the probability for a set of 'typical' degree sequences. In this case we use $\tilde{\mathcal{D}}(\psi)$ with $\psi=o(1)$ as define in Section 3.5.1 with the additional constraint that $|D_3(\mathbf{d})-p^{(3)}n|\le\psi(n)$, where $p^{(3)}$ is the probability that a truncated Poisson random variable with parameters $(2,\lambda_c)$ is 3. We work with $\mathbf{d}\in\tilde{\mathcal{D}}(\psi)$. So redefine $\tilde{\mathcal{D}}(\psi)$ as

$$\tilde{\mathcal{D}}_n(\psi)=\tilde{\mathcal{D}}(\psi):=\Big\{\mathbf{d}\in\mathcal{D}(n,m):d_i\le 6\log n\ \forall i;\ |\eta(\mathbf{d})-\eta_c|\le\psi(n);\\ |D_2(\mathbf{d})-p_cn|\le n\psi(n),\ |D_3(\mathbf{d})-p^{(3)}n|\le n\psi(n)\Big\}.$$

First we estimate $\mathbb{E}\left(X\right)$. For each vertex of degree 3, there are 3 choices for points in the loop. Each loop occurs with probability $2M$. Thus, $\mathbb{E}\left(X\right)=D_3/(2M)\sim p^{(3)}n/(2M)\sim c/2-\lambda_c/2$. The expected value of Y has been already estimated in Section 3.5.1: we have that $\mathbb{E}\left(Y\right)\sim\lambda_c^2/4$. We now compute $\mathbb{E}\left(Z\right)$. Using $\mathbf{d}\in\tilde{\mathcal{D}}$, the probability that an edge in the kernel receives no vertex in the pre-kernel is

$$\frac{[m'-1]^{D_2}}{[m']^{D_2}}=\frac{m'-1}{m-1}\sim\sqrt{\delta},$$

where $[x]^k := x(x+1)\dots(x+k-1)$ and δ is defined in (3.25) as $((c-2p_c)/c)^2 = (\lambda_c/c)^2$. Using $\mathbf{d} \in \tilde{\mathcal{D}}$, the probability that an edge in the kernel receives exactly one vertex in the pre-kernel is

$$\frac{D_2[m'-1]^{D_2-1}}{[m']^{D_2}} = \frac{D_2(m'-1)}{(m-2)(m-1)} \sim (1-\sqrt{\delta})\sqrt{\delta}.$$

This is because the probability that any specific vertex from the ones inserted in the edges of the kernel when obtaining the pre-kernel is the only one inserted in the edge is $[m'-1]^{D_2-1}/[m']^{D_2}$ The number of possible loops in vertices of degree at least 4 is $\sum_i \binom{d'_i}{2} - 3D_3$, where $\mathbf{d}'$ is the degree sequence of the kernel. The probability that each of these loops is present is $1/(2m')$. Thus, using $\mathbf{d} \in \tilde{\mathcal{D}}$ and Lemma 3.5.2,

$$\mathbb{E}(Z) = \frac{\sum_i \binom{d'_i}{2} - 3D_3}{2m'}(2\sqrt{\delta}-\delta) \sim \left(\frac{c}{2} - \frac{3D_3}{2m'}\right)(2\sqrt{\delta}-\delta) \sim \frac{\lambda_c}{2} - \frac{\lambda_c^3}{2(e^{\lambda_c}-1)^2}.$$

The probability that G is 2-edge-connected and simple is then asymptotic to

$$\exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4} + \frac{\lambda_c^3}{2(e^{\lambda_c}-1)^2}\right),$$

and the formula for $T'(n,m)$ can be deduced in the same way as the number of 2-connected graph in Section 3.5.

Similarly to the proof in Section 3.7, the asymptotic formulae obtained for the different ranges of c are easily combined using the subsubsequence principle, finishing the proof of Theorem 3.1.5.

Glossary for Chapter 3

$\mathbf{2cs}(\mathbf{d})$	event that a random graph generated with kernel configuration model with degree sequence $\mathbf{d}$ is 2-connected and simple, p. 27
$(2k+1)!!$	$(2k-1)(2k-3)\cdots 1$, the odd falling factorial
c	$2m/n$, the average degree
$D_j(\mathbf{d})$	$\lvert\{i : d_i = j\}\rvert$, the number of vertices of degree j
$\mathcal{D}(n,m)$	set of degree sequences $\mathbf{d} = (d_1, \ldots, d_n)$ such that $\sum_{i=1}^n d_i = 2m$ and $\min_{i=1}^n d_i \geq 2$
$\tilde{\mathcal{D}}$	used to define sets of 'typical' degree sequences, pp. 36, 42, 50
$\eta(\mathbf{d})$	$\sum_i d_i(d_i-1)/(2m)$
η_c	$\lambda_c e^{\lambda_c}/(e^{\lambda_c}-1)$
$f_k(\lambda)$	$e^\lambda - \sum_{i=0}^{k-1} \lambda^i/i!$
$\Phi(k)$	$(2k-1)!! = (2k)!/(2^k k!)$, the number of perfect matchings on $2k$ points
λ_c	the unique positive solution to $\lambda(e^\lambda - 1)/(e^\lambda - 1 - \lambda) = c$
$m'(\mathbf{d})$	$m - D_2(\mathbf{d})$, the number of edges in the kernel
(n,m)-graph	any graph on $[n]$ with m edges
p_c	$\lambda_c^2/(2(e^{\lambda_c}-1-\lambda_c))$, the probability that a truncated Poisson random variable $\mathrm{Po}(2,\lambda_c)$ has value 2
$Q(n,m)$	$\sum_{\mathbf{d}\in\mathcal{D}(n,m)} \prod_{j=1}^n 1/d_j! = f_2(\lambda_c)^n/\lambda_c^{2m} \cdot \mathbb{P}(\Sigma)$
r	$2m-2n$, an excess function
$R(\mathbf{d})$	$\sum_{i:d_i\geq 3} d_i$
Σ	used to denote the event that $\sum_{i=1}^n Y_i = 2m$ for independent truncated Poisson random variables with parameters $(2,\lambda_c)$
$T(\mathbf{d})$	number of 2-connected graphs with degree sequence $\mathbf{d}$
$T(n,m)$	number of 2-connected graphs on $[n]$ vertices and m edges
$U(\mathbf{d})$	probability that the random graph generated with pairing model with degree sequence $\mathbf{d}$ is simple, p. 49
$U'(\mathbf{d})$	probability that the random graph generated with pairing model with degree sequence $\mathbf{d}$ is 2-connected and simple, p. 49
$\mathbf{Y}$	used to denote a vector $(Y_1, \ldots, Y_n)$ of independent truncated Poisson random variables with parameters $(2,\lambda_c)$

Chapter 4

Asymptotic enumeration of sparse connected 3-uniform hypergraphs

The problem of counting connected graphs with given number of vertices and edges has been intensively studied throughout the years. As, we already mentioned in previous chapters, one of the best results is an asymptotic formula by Bender, Canfield and McKay [7] that works when $m - n \to \infty$ as $n \to \infty$, where $m = m(n)$ is the number of edges and n is the number of vertices. Pittel and Wormald [56] rederived this formula with improved error bounds for some ranges. Far less is known about connected hypergraphs. Karoński and Łuczak [38] derived an asymptotic formula for the number of connected k-uniform graphs on $[N]$ with M hyperedges for the case $M = N/(k-1) + o(\ln N/\ln\ln N)$, which is a range with small excess. This was later extended by Andriamampianina and Ravelomanana [3] for $M = N/(k-1) + o(N^{1/3})$, which still has very small excess. On the other direction, Behrisch, Coja-Oghlan and Kang [4] provided an asymptotic formula for the case $M = N/(k-1) + \Theta(N)$. Thus, there is a gap between the case $M - N/(k-1) = o(N^{1/3})$ and the linear case $M - N/(k-1) = \Omega(N)$ in which no asymptotic formulae were found. The case $M - N/(k-1) = \omega(N)$ is also open.

Behrisch, Coja-Oghlan and Kang [4] obtained their enumeration result by precisely estimating the joint distribution of the number of vertices and the number of edges in the giant component of the random hypergraph. We remark that the distribution of the number of vertices and edges has already been described by Bollobás and Riordan [14], but their result does not provide point probabilities, which would allow the enumeration result to be deduced.

In this chapter, we obtain results for 3-uniform hypergraphs. We obtain an asymptotic formula for the number of connected 3-uniform graphs with vertex set $[N]$ and M edges for $M = N/2 + R$ as long as R satisfies $R = o(N)$ and $R = \omega(N^{1/3}\ln^2 N)$. This leaves a gap from $M - N/2 = o(N^{1/3})$ and $M - N/2 = \omega(N^{1/3}\ln^2 N)$. Our technique is based in the approach that Pittel and Wormald [56] used to the enumerate connected graphs. The results in this chapter are joint work with N. Wormald.

4.1 Main result

A *hypergraph* is a pair $(V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a subset of nonempty sets in 2^V, which is the set of all subsets of V. The elements in V are called *vertices* and the elements in $\mathcal{E}$ are called *hyperedges*. For any integer $k \geq 2$, a *k-uniform hypergraph* is a hypergraph where each hyperedge has size k. For any hypergraph G, a *path* is a (finite) sequence $v_1 E_1 v_2 E_2 \dots v_k$, where $v_1, \dots, v_k$ are distinct vertices and $E_1, \dots, E_{k-1}$ are distinct hyperedges such that $v_i, v_{i+1} \in E_i$ for all $i \in [k-1]$. We say that a hypergraph is *connected* if, for any vertices u and v, there exists a path from u to v.

An (N, M, k)-hypergraph is a k-uniform hypergraph with $V = [N]$ and M edges. Let $C(N, M)$ denote the number of connected $(N, M, 3)$-hypergraphs. Our main result is an asymptotic formula for $C(N, M)$ for a sparse range of M. For $k \geq 0$, define $g_k(\lambda) = \exp(\lambda) + k$ and recall that $f_k(\lambda) = \exp(\lambda) - \sum_{i=0}^{k-1} \lambda^i / i!$.

Theorem 4.1.1. Let $M = M(N) = N/2 + R$ be such that $R = o(N)$ and $R = \omega(N^{1/3} \ln^2 N)$. Then

$$C(N, M) \sim \sqrt{\frac{3}{\pi N}} \exp\Big(N\phi(\check{n}^*) + N \ln N - N\Big),$$

where

$$\begin{aligned}\phi(x) = &- \frac{(1-x)}{2} \ln(1-x) + \frac{1-x}{2} \\ &+ \frac{2R}{N} \ln(N) - \big(\ln(2) + 2\big)\frac{R}{N} - \frac{1}{2}\ln(2)x \\ &+ \frac{R}{N} \ln\left(\frac{g_1(\lambda^{**})}{\lambda^{**} f_1(\lambda^{**})}\right) + \frac{1}{2} x \ln\left(\frac{f_1(\lambda^{**}) g_1(\lambda^{**})}{\lambda^{**}}\right),\end{aligned}$$

$$\check{n}^* = \frac{f_2(2\lambda^{**})}{f_1(\lambda^{**}) g_1(\lambda^{**})},$$

and λ^{**} is the unique positive solution of

$$\lambda \frac{e^{2\lambda} + e^{\lambda} + 1}{f_1(\lambda) g_1(\lambda)} = \frac{3M}{N}. \tag{4.1}$$

Our proof basically follows the same approach that Pittel and Wormald [56] use to the enumerate connected graphs in the sparser range. Pittel and Wormald [56] decompose a connected graph into two parts: a cyclic structure and an acyclic structure. The cyclic structure is a pre-kernel, which is a 2-core without isolated cycles. The acyclic structure is a rooted forest where the roots are the vertices of the pre-kernel. A rooted forest with roots $r_1, \dots, r_t$ (that are vertices in the forest) simply is a forest such that each component contains exactly one of the roots. The graph can then be obtained by 'gluing' these two structures together. Pittel and Wormald obtain an asymptotic formula for the number of the cyclic structures and combine it with a known formula for the acyclic parts to obtain an asymptotic formula for the number of connected graphs with given number of vertices and edges.

We will also decompose a connected 3-uniform hypergraph into two parts: a cyclic structure (which we will also call pre-kernel) and an acyclic structure (a forest rooted on the vertices of the pre-kernel). We will also obtain asymptotic formulae for these structures and then combine them to obtain an asymptotic formula for the number of connected $(N, M, 3)$-hypergraphs.

From now on, we will deal with 3-uniform hypergraphs most of the time. In this chapter, for convenience, we will use the word 'graph' to denote 3-uniform hypergraphs. When we want to refer to graphs in the usual sense, we will call them '2-uniform hypergraphs'. We will also use the word 'edge' instead of 'hyperedge'.

4.2 Relation to a known formula

As we mentioned before, Behrisch, Coja-Oghlan and Kang [4] provided an asymptotic formula for the number of connected (N, M, k)-hypergraphs for the range $M = N/(k-1) + \Omega(N)$. In this section, we show that, for $k = 3$, their formula is asymptotic to ours when $R = M - N = o(N)$. Behrisch, Coja-Oghlan and Kang obtained their result by computing the probability that the random hypergraph $H_k(N, M)$ with uniform distribution on all (N, M, k)-hypergraphs is connected.

Theorem 4.2.1 ([4, Theorem 5]). Let $k \geq 2$ be a fixed integer. For any compact set $\mathcal{J} \subset (k(k-1)^{-1}, \infty)$, and for any $\delta > 0$ there exists $N_0 > 0$ such that the following holds. Let $M = M(N)$ be a sequence of integers such that $\zeta = \zeta(N) = kM/N \in \mathcal{J}$ for all N. Then there exists a unique number $0 < r = r(N) < 1$ such that

$$r = \exp\left(-\zeta \frac{(1-r)(1-r^{k-1})}{1-r^k}\right). \tag{4.2}$$

Let $\Phi(\zeta) = r^{r/(1-r)}(1-r)^{1-\zeta}(1-r^k)^{\zeta/k}$. Furthermore, let

$$R_2(N, M) = \frac{1+r-\zeta r}{\sqrt{(1-r)^2 - 2\zeta r}} \exp\left(\frac{2\zeta r + \zeta^2 r}{2(1+r)}\right) \Phi(\zeta)^N, \quad \text{and set}$$

$$R_k(N, M) = \frac{1 - r^k - (1-r)\zeta(k-1)r^{k-1}}{\sqrt{(1 - r^k + \zeta(k-1)(r - r^{k-1}))(1-r^k) - \zeta k r (1-r^{k-1})^2}} \times \exp\left(\frac{\zeta(k-1)(r - 2r^k + r^{k-1})}{2(1-r^k)}\right) \Phi(\zeta)^N, \quad \text{if } k > 2.$$

For $N > N_0$, the probability that $H_k(N, M)$ is connected is in $((1-\delta)R_k(N, M), (1+\delta)R_k(N, M))$.

From this theorem, it is immediate that the number of connected (N, M, k)-graphs is asymptotic to

$$\binom{\binom{N}{k}}{M} R_k(N, M) =: D(N, M, k)$$

when $R = M - N/2 = \Omega(N)$. Next we assume $R/N = o(1)$ and do some simplifications in $D(N, M, 3)$. So suppose $R = M - N/2 = o(N)$. First we compare r in (4.2) with λ^{**} in (4.1):

$$r = \exp\left(-\frac{3M}{N}\frac{(1-r)(1-r^2)}{1-r^3}\right) \quad \text{and} \quad \lambda^{**}\frac{e^{2\lambda^{**}} + e^{\lambda^{**}} + 1}{e^{2\lambda^{**}} - 1} = \frac{3M}{N}.$$

By taking the logs in both sides of the definition of r, it is obvious that $r = \exp(-\lambda^{**})$. As we will see later, $\lambda^{**} \to 0$ and so $r \to 1$. We have that

$$\lim_{r\to 1} \frac{1 - r^3 - 2(1-r)\zeta r^2}{\sqrt{(1 - r^3 + 2\zeta(r - r^2))(1 - r^3) - 3\zeta r(1 - r^2)^2}} = \sqrt{3}$$

and

$$\lim_{r\to 1} \frac{\zeta(r - 2r^3 + r^2)}{1 - r^3} = 3/2.$$

(See Section A.1 for a Maple spreadsheet.) Thus,

$$\begin{aligned} D(N, M, k) &\sim \binom{\binom{N}{3}}{M} \sqrt{3}\exp(3/2)\Phi(3M/N)^N \\ &\sim \frac{\sqrt{2\pi\binom{N}{3}}\left(\frac{\binom{N}{3}}{e}\right)^{\binom{N}{3}}}{\sqrt{2\pi M}\left(\frac{M}{e}\right)^M \sqrt{2\pi\left(\binom{N}{3} - M\right)}\left(\frac{\binom{N}{3} - M}{e}\right)^{\binom{N}{3}-M}} \sqrt{3}\exp(3/2)\Phi(3M/N)^N, \end{aligned}$$

by Stirling's approximation. Thus, using $M = N/2 + R$ and $R = o(N)$,

$$\begin{aligned} &D(N, M, k) \\ &\sim \sqrt{\frac{3}{\pi N}}\exp\left(3/2 + \binom{N}{3}\ln\binom{N}{3} - M\ln M - \left(\binom{N}{3} - M\right)\ln\left(\binom{N}{3} - M\right) + N\Phi(3M/N)\right) \\ &= \sqrt{\frac{3}{\pi N}}\exp\left(3/2 - \left(\binom{N}{3} - M\right)\ln\left(1 - \frac{M}{\binom{N}{3}}\right) - M\ln M + M\ln\binom{N}{3} + N\Phi(3M/N)\right) \\ &= \sqrt{\frac{3}{\pi N}}\exp\left(3/2 - \left(\binom{N}{3} - M\right)\left(-\frac{M}{\binom{N}{3}} + O\left(\frac{M^2}{\binom{N}{3}^2}\right)\right) - M\ln M + M\ln\binom{N}{3} + N\Phi(3M/N)\right). \end{aligned}$$

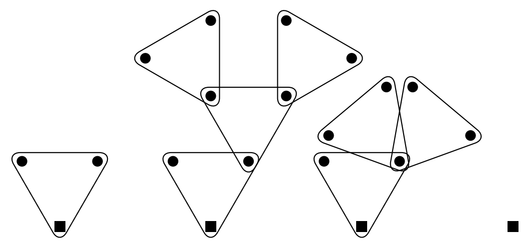

Figure 4.1: A rooted forest. The roots are the vertices represented by squares.

Thus,

$$D(N,M,k) = \sqrt{\frac{3}{\pi N}} \exp\left(3/2 + M - M\ln M + M\ln\binom{N}{3} + N\Phi(3M/N) + o(1)\right)$$
$$= \sqrt{\frac{3}{\pi N}} \exp\left(3/2 + M - M\ln M + M\ln\frac{N^3}{6} + M\ln\frac{N(N-1)(N-2)}{N^3} + N\Phi(3M/N) + o(1)\right)$$
$$= \sqrt{\frac{3}{\pi N}} \exp\left(3/2 + M - M\ln M + 3M\ln N - M\ln 6 + M\ln\left(1 - \frac{3N-2}{N^2}\right) + N\Phi(3M/N) + o(1)\right)$$
$$= \sqrt{\frac{3}{\pi N}} \exp\left(3/2 + M - M\ln M + 3M\ln N - M\ln 6 - M\frac{3N}{N^2} + N\Phi(3M/N) + o(1)\right)$$
$$\sim \sqrt{\frac{3}{\pi N}} \exp\left(M - M\ln M + 3M\ln N - M\ln 6 + N\Phi(3M/N)\right),$$

which is exactly the same as our formula in Theorem 4.1.1 after a series of simplifications. See Section A.1 for a Maple spreadsheet with these computations.

4.3 Basic definitions and results for hypergraphs

In this section, we present some basic definitions for hypergraphs and show how to decompose a hypergraph into a cyclic structure and an acyclic structure.

A *cycle* in a hypergraph $G = (V, \mathcal{E})$ is a (finite) sequence $(v_0, E_0, \ldots, v_k, E_k)$ such that $v_1, \ldots, v_k \in V$ are distinct vertices, $E_1, \ldots, E_k \in \mathcal{E}$ are distinct edges with $v_i \in E_i$ and $v_{i+1} \in E_i$ for every $0 \leq i \leq k$ (operations in the indices are in $\mathbb{Z}_{k+1}$). A *tree* is an acyclic connected hypergraph and a *forest* is an acyclic hypergraph. A *rooted forest* $G = (V, \mathcal{E})$ with set of roots $S \subseteq V$ is a forest such that each component of the forest has exactly one vertex in S. See Figure 4.1 for a rooted forest.

The *degree* of a vertex v in a hypergraph G is the number of edges in G containing v. Recall that we use the word ‘graph’ to denote 3-uniform hypergraphs. The *core* of a graph is its maximal

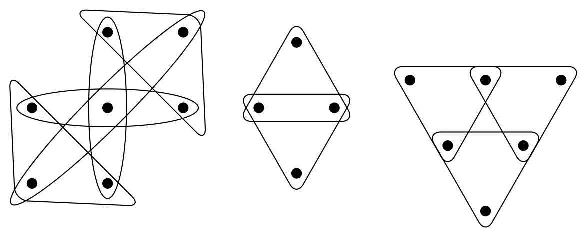

Figure 4.2: A core with two isolated cycles. The leftmost component is a pre-kernel.

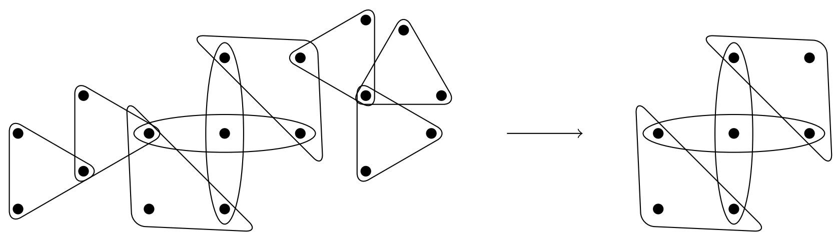

Figure 4.3: Obtaining the core.

induced subgraph such that every edge contains at least two distinct vertices of degree at least 2. To see that the core of a hypergraph is unique, it suffices to notice that the union of two cores would also be a core. We remark that the k-core of a graph is usually defined as the maximal subgraph such that every vertex has degree at least k. The core we defined contains the 2-core of the hypergraph and it allows some vertices of degree 1. We chose this definition of core since otherwise the structure we would have to combine with the 2-core would not necessarily be acyclic. We also say that a graph is a core, when its core is the graph itself. For an example of an core see Figure 4.2.

Every edge in a core has either one vertex of degree 1, or none. We say that an edge is a 2-*edge* if it has a vertex of degree 1 and that it is a 3-*edge* if it has no vertex of degree 1. It is easy to see that the core of graph can be obtained by iteratively removing edges that are not 2-edges nor 3-edges until all edges are 2-edges or 3-edges, and then deleting all vertices of degree 0. See Figure 4.3 for an example of this procedure.

We also define cycles as graphs. We say that a graph $G = (V, \mathcal{E})$ is a *cycle* if there is an ordering $(v_0, \dots, v_k)$ of a subset of V and an ordering $(E_0, \dots, E_k)$ of $\mathcal{E}$ such that $(v_0, E_0, \dots, v_k, E_k)$ is a cycle in G and every $v \in V$ is in some $E \in \mathcal{E}$. Note that, if a graph is a cycle, then all edges are actually 2-edges. An *isolated cycle* in a graph is a component that is a cycle. A *pre-kernel* is a core without isolated cycles (see Figure 4.2). So, every connected core that is not just a cycle is also a pre-kernel.

The following proposition explains how to decompose a graph into its core and a rooted forest.

Proposition 4.3.1. Let G be a connected graph with a nonempty core. The graph obtained from G by deleting the edges of the core of G and by setting all vertices in the core as roots is a rooted forest with $(N-n)/2$ edges, where N is the number of vertices in G and n is the number of vertices in the core. Moreover, the core of G is connected.

Proof. As we already mentioned, the core of G can be obtained by iteratively deleting edges that contain at most one vertex of degree at least 2. More precisely, start with $G' = G$ and while there is an edge in G' containing less than 2 vertices of degree at least 2 in G', redefine G' by deleting one such edge. When this procedure stops, G' is the core of G. Let F be the graph with vertex set $[N]$ with the deleted edges as its set of edges. Suppose for a contradiction that F has a cycle. Such a cycle is a cycle in G too. Let E be the first edge of the cycle that was deleted by the procedure described above. All other edges in the cycle were still present in the graph G' when E was deleted. Thus, since E was in the cycle, it had a least 2 vertices of degree at least 2. Hence, E could not have been deleted at this point, which shows that F has no cycles.

Suppose for a contradiction that the core of G is not connected. Then it has at least 2 components that are joined by a path in G with all edges in F since G is connected. The union of these 2-components and the path is a 2-core, which is a contradiction. Thus, the core is connected. This argument also shows that that each component of F has at most one vertex in the core. Every component of F must have one vertex in the core, otherwise it is disconnected from the core and so G would not be connected.

Now we determine the number of edges in F. As we discussed above, each component of F has exactly one vertex in the core. In the deletion procedure, for the initial G' (that is, G), every edge has at least one vertex of degree at least 2 since otherwise G would not be connected. We claim that the deletion procedure will only delete edges that contain exactly one vertex of degree 2 in the current G'. If not, let E be an edge that contained no vertex of degree at least 2 in G' in the moment it was deleted. Let v_0 be the vertex of the core in the same component of E in F. Then there is a path $(v_0, E_0, \dots, E_{k-1}v_k)$ in F, where $E_{k-1} = E$. The edge E_0 cannot be E since the vertex v_0 must have degree at least 2 the moment E_0 is deleted. A trivial induction proof then shows that the deletion procedure cannot delete any of the edges $E_0, \dots, E_{k-2}$ before deleting E_{k-1}, which shows that the moment E was deleted the vertex v_{k-1} still was in 2 edges: E and E_{k-2}. This is a contradiction. Thus, the moment any edge is deleted is has exactly one vertex of degree at least 2. This means that, for every deleted edge, we also delete exactly 2 vertices that are not in the core. Since there $N-n$ vertices to be deleted, the number of edges in F is $(N-n)/2$. □

For any graph G with N vertices and M edges such that its core has n vertices and m edges, we have that

$$m - n/2 = M - (N-n)/2 - n/2 = M - N/2 \tag{4.3}$$

since $m = M - (N-n)/2$ by Proposition 4.3.1. Intuitively speaking, this says that the 'excess' of edges $(M - N/2)$ in the graph is transfered to its core.

Let $g_{\text{forest}}(N, n)$ denote the number of forests with vertex set $[N]$ and $[n]$ as its set of roots. Let $g_{\text{pre}}(n, m)$ denote the number of connected pre-kernels with vertex set $[n]$ and m edges. Next, we show how to write $C(N, M)$ using g_{forest} and g_{pre}.

Proposition 4.3.2. For $M = M(N)$ such that $R := M - N/2 \to \infty$, we have that

$$C(N, M) = \sum_{\substack{1 \le n \le N \\ (N-n) \in 2\mathbb{Z}}} g_{\text{forest}}(N, n) g_{\text{pre}}(n, M - (N-n)/2), \tag{4.4}$$

for N sufficiently large.

Proof. In view of Proposition 4.3.1, it suffices to show that, for any connected graph G with N vertices and M edges, the core of G is a pre-kernel. If it is not, either the core is empty or it is a cycle. If the core is empty, then the graph G is a forest and so $M < N/2$, which is impossible since $M = N/2 + R$ with $R \to \infty$. If the core is a cycle, then $3m = 2(n-m) + m$ since each edge in the core has two vertices of degree 2 and one of degree 1. Thus, in this case, we have that $m = n/2$, which is impossible since $m - n/2 = M - N/2 = R \to \infty$ by (4.3). □

Basically, our approach to compute an asymptotic formula for $C(N, M)$ will be to analyse the summation in (4.4).

We will work with random graphs. More precisely, we will work with random multihypergraphs and then deduce results for simple graphs. A *k-uniform multihypergraph* is a triple $G = (V, \mathcal{E}, \Phi)$, where V and $\mathcal{E}$ are finite sets and $\Phi : \mathcal{E} \times [k] \to V$. We say that V is the *vertex set* of G and $\mathcal{E}$ is the *edge set* of G. From now on, we will use the word 'multigraph' to denote 3-uniform multihypergraphs.

Given a multigraph $G = (V, \mathcal{E}, \Phi)$, a *loop* is an edge $E \in \mathcal{E}$ such that there exist distinct $j, j' \in \{1, 2, 3\}$ such that $\Phi(E, j) = \Phi(E, j')$, a pair of *double edges* is a pair (E, E') of distinct edges in $\mathcal{E}$ such that the collection $\{\Phi(E, 1), \Phi(E, 2), \Phi(E, 3)\}$ is the same as the collection $\{\Phi(E', 1), \Phi(E', 2), \Phi(E', 3)\}$. A multigraph G with no loops nor double edges corresponds naturally to a graph because each edge corresponds to a unique subset of V of size 3. In this case we say that the multigraph is *simple*. Let $\mathcal{S}(n, m)$ denote the set of simple multigraphs with vertex set $[n]$ and edge set $[m]$. We have the following relation between simple multigraphs and graphs:

Lemma 4.3.3. For any $G = ([n], [m], \Phi) \in \mathcal{S}(n, m)$, let $s(G)$ be the graph with vertex set $[n]$ obtained by including one edge for each $i \in [m]$ incident to the vertices $\Phi(i, 1)$, $\Phi(i, 2)$ and $\Phi(i, 3)$. Let G' be a graph with vertex set $[n]$ with m edges. Then $|s^{-1}(G')| = m! 6^m$, that is, each graph corresponds to $m! 6^m$ simple multigraphs.

Proof. Let $G = ([n], [m], \Phi) \in \mathcal{S}(n, m)$ be such that $s(G) = G'$. For any permutation g of $[m]$, the multigraph $G_g := ([n], [m], \Phi')$ satisfies $s(G_g) = G'$, where $\Phi'(i, j) = \Phi(g(i), j)$ for each $i \in [m]$ and $j \in \{1, 2, 3\}$. (That is, any permutation of the label of the edges generates the same graph.) Moreover, for each $i \in [m]$ and permutation g_i of [3], the function $\Phi''(i, j) = \Phi''(i, g(j))$ satisfies $s([n], [m], \Phi'') = G'$. Since there are $m!$ permutations on $[m]$ and $3!$ permutations of [3], the number of graphs $G \in \mathcal{S}(n, m)$ with $s(G) = G'$ is $m!3!^m$. □

We extend the definitions of path and connectedness for multihypergraphs. For any multihypergraph $G = (V, \mathcal{E}, \Phi)$, a *path* is a (finite) sequence $v_1E_1v_2E_2 \dots v_k$, where $v_1, \dots, v_k$ are distinct vertices and $E_1, \dots, E_{k-1}$ are distinct hyperedges such that $v_i, v_{i+1} \in \mathrm{Im}(\Phi(E_i, \cdot))$ for all $i \in [k-1]$. We say that a multihypergraph is *connected* if, for any vertices u and v, there exists a path from u to v.

4.4 Overview of proof

In this section, we give an overview of our proof of the asymptotic formula for $C(N, M)$ in Theorem 4.1.1. Recall that $R = M - N/2 = o(N)$ and $R = \omega(N^{1/3} \log^2 N)$. Our approach is to analyse $C(N, M)$ by using (4.4), which shows how to obtain $C(N, M)$ from formulae for the number of rooted forests g_{forest} and the number of pre-kernels g_{pre}. The proof consists of the following steps.

1. We obtain an exact formula $g_{\text{forest}}(N, n)$ for the number of of rooted forests with set of roots $[n]$ and vertex set $[N]$. We show that, for even $N - n$,

$$g_{\text{forest}}(N, n) = \frac{n}{N} \cdot \frac{(N-n)! N^{(N-n)/2}}{((N-n)/2)! 2^{(N-n)/2}},$$

   and, for odd $N - n$, $g_{\text{forest}}(N, n) = 0$. The proof is in Section 4.5 and is a simple proof by induction.

2. Let $g_{\text{core}}(n, m)$ denote the number of (simple) cores on $[n]$ with m edges. We analyse g_{core} by writing it as follows:

$$g_{\text{core}}(n, m) = \sum_{n_1, \mathbf{d}} g_{\text{core}}(n, m, n_1, \mathbf{d}),$$

   where $g_{\text{core}}(n, m, n_1, \mathbf{d})$ is the number of (simple) cores with n vertices, $m = n/2 + R$ edges, n_1 vertices of degree 1, and degree sequence $\mathbf{d}$ for the vertices of degree at least 2. We use r

to denote R/n. We show that there is a constant α such that

$$g_{\text{core}}(n,m) \le \alpha n\sqrt{m}\cdot n!\exp(nf_{\text{core}}(\hat{n}_1^*)),\ \text{ for } R\to\infty,\ \text{ and} \tag{4.5}$$

$$g_{\text{core}}(n,m) \sim \frac{1}{2\pi n\sqrt{r}}\cdot n!\exp(nf_{\text{core}}(\hat{n}_1^*)),\ \text{ for } R\to\infty \text{ and } R=o(n), \tag{4.6}$$

where the function f_{core} is defined in Section 4.7 and λ^* is the unique positive solution for $\lambda f_1(\lambda)g_2(\lambda)/f_2(2\lambda) = 3m/n$ and $\hat{n}_1^* = 3m/(ng_2(\lambda^*))$. The proof is in Section 4.7.

3. We obtain an asymptotic formula for the number $g_{\text{pre}}(n,m)$ of simple connected pre-kernels with $n\to\infty$ vertices and $m = n/2 + rn$ edges when $R = o(n)$ and $R = \omega(n^{1/2}\log^{3/2} n)$. We show that

$$g_{\text{pre}}(n,m) \sim \frac{\sqrt{3}}{\pi n} n!\exp(nf_{\text{pre}}(\hat{x}^*)),$$

where f_{pre} is defined in Section 4.8 and $\hat{x}^*\in\mathbb{R}^4$ will be determined using λ^* as defined in the previous step.

4. We will at first work with cores since the function obtained for them is simpler than the formula for pre-kernels. We will find relations between the two formulae that justify why it is relevant to analyse the function for cores. We find a set I for n in which $n=\Theta(\sqrt{RN})$ such that

$$\sum_{n\in I}\binom{N}{n} g_{\text{forest}}(N,n)g_{\text{core}}(n,m) \sim \frac{\sqrt{3}}{\sqrt{\pi\lambda^{**}N}}\exp\left(Nt(\check{n}^*) + N\ln N - N\right), \tag{4.7}$$

where t is defined in Section 4.9, λ^{**} is the unique positive solution of the equation $\lambda(e^{2\lambda} + e^{\lambda}+1)/(f_1(\lambda)g_1(\lambda)) = 3M/N$ and $\check{n}^* = f_2(2\lambda^{**})(f_1(\lambda^{**})g_1(\lambda^{**}))$. We then show that the contribution to the summation for n outside I is insignificant by using (4.5):

$$\sum_{n\in[N]\setminus I}\binom{N}{n} g_{\text{forest}}(N,n)g_{\text{core}}(n,m) = o\left(\frac{1}{\sqrt{\pi N}}\exp\left(Nt(\check{n}^*) + N\ln N - N\right)\right). \tag{4.8}$$

5. We use Step 2, Step 3 and (4.7) to show that

$$\begin{aligned}\sum_{n\in I}\binom{N}{n} g_{\text{forest}}(N,n)g_{\text{pre}}(n,m) &\sim 2\sqrt{3r}\sum_{n\in I}\binom{N}{n} g_{\text{forest}}(N,n)g_{\text{core}}(n,m)\\ &\sim \sqrt{\frac{3}{\pi N}}\exp\left(Nt(\check{n}^*) + N\ln N - N\right)\end{aligned}$$

and using the relation $g_{\text{pre}}(n,m)\le g_{\text{core}}(n,m)$ (since every pre-kernel is a core) and (4.8),

$$\sum_{n\in[N]\setminus I}\binom{N}{n} g_{\text{forest}}(N,n)g_{\text{pre}}(n,m) = o\left(\sum_{n\in I}\binom{N}{n} g_{\text{forest}}(N,n)g_{\text{pre}}(n,m)\right).$$

6. The conclusion is then easily obtained by simplifying $t(\check{n}^*)$.

4.5 Counting forests

In this section we prove an exact formula for rooted forests. In this section we consider k-uniform hypergraphs, for any $k \geq 2$. We remark that this formula has also been proved in a note by Lavault [43] around the same time we obtained it. Lavault shows a one-to-one correspondence between rooted forests and a set of tuples whose size can be easily computed.

Recall that $g_{\text{forest}}(N, n)$ is the number of rooted forests on $[N]$ with set of roots $[n]$. (See Figure 4.1 for a rooted forest.)

Theorem 4.5.1. For integers $N \geq n \geq 0$ and any integer $k \geq 2$,

$$g_{\text{forest}}(N, n) = \begin{cases} \dfrac{n(N-n)!N^{m'-1}}{m'!(k-1)!^{m'}}, & \text{if } m' = \dfrac{N-n}{k-1} \text{ is a nonnegative integer;} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. A connected k-uniform hypergraph is a tree if and only if, by iteratively deleting edges that have at least $k-1$ vertices of degree 1, we delete all edges. It then is obvious that m' is the number of edges in the forest. We remark that the a tree can be seen as a 2-uniform hypergraph where each block is a clique on $[n-1]$ vertices (which is known as a clique tree).

The proof is by induction on N. We have that $g_{\text{forest}}(1, 1) = 1 = 1(1-1)!1^{0-1}/(0!(k-1)!^0) = 1$ and the formula also works for $g_{\text{forest}}(N, 0) = 0$. So assume that $N > 1$ and $n \geq 1$. We will show how to obtain a recurrence relation for $g_{\text{forest}}(N, n)$. Suppose that the vertex 1 is in j edges, where $0 \leq j \leq m'$. We choose $(k-1)j$ other vertices to be in these j edges. There are $\binom{N-n}{(k-1)j}$ ways to choose these vertices. The number of ways we can split the vertices into the edges is

$$\binom{(k-1)j}{k-1}\binom{(k-1)j-(k-1)}{k-1}\cdots\binom{k-1}{k-1}\frac{1}{j!} = \frac{\big((k-1)j\big)!}{(k-1)!^j j!}.$$

We can build the rooted forest by first choosing the edges containing 1 and then deleting 1 and considering the other $(k-1)j$ vertices in these edges as new roots. This gives us the following recurrence:

$$g_{\text{forest}}(N, n) = \sum_{j=0}^{m'} \binom{N-n}{(k-1)j} \frac{\big((k-1)j\big)!}{(k-1)!^j j!} g_{\text{forest}}\big(N-1, n-1+(k-1)j\big).$$

Note that $0 \leq n-1+(k-1)j \leq N-1$ since $j \in [0, m']$. The new number of edges is $m'' = \frac{1}{k-1}((N-1)-(n-1+(k-1)j)) = m'-j$. So, by induction hypothesis,

$$g_{\text{forest}}(N,n) = \sum_{j=0}^{m'} \binom{N-n}{(k-1)j} \frac{\big((k-1)j\big)!}{(k-1)!^j j!} \cdot \frac{(n-1+(k-1)j)(N-n-(k-1)j)!(N-1)^{m'-j-1}}{(m'-j)!(k-1)!^{m'-j}}$$

$$= \frac{(N-n)!}{(N-1)(k-1)!^{m'}} \sum_{j=0}^{m'} \frac{(n-1+(k-1)j)(N-1)^{m'-j}}{j!(m'-j)!}$$

$$= \frac{(N-n)!}{m'!(N-1)(k-1)!^{m'}} \sum_{j=0}^{m'} \binom{m'}{j} (n-1+(k-1)j)(N-1)^{m'-j}$$

$$= \frac{(N-n)!}{m'!(N-1)(k-1)!^{m'}} \left((n-1) \sum_{j=0}^{m'} \binom{m'}{j} (N-1)^{m'-j} + (k-1) \sum_{j=0}^{m'} \binom{m'}{j} j(N-1)^{m'-j} \right).$$

Using the Binomial Theorem,

$$\sum_{j=0}^{m'} \binom{m'}{j} (N-1)^{m'-j} = N^{m'}$$

and by differentiating both sides with respect to N,

$$\sum_{j=0}^{m'} \binom{m'}{j} (m'-j)(N-1)^{m'-j-1} = m'N^{m'-1},$$

and so

$$\begin{aligned}\sum_{j=0}^{m'} \binom{m'}{j} j(N-1)^{m'-j} &= m' \sum_{j=0}^{m'} \binom{m'}{j} (N-1)^{m'-j} - m'N^{m'-1}(N-1) \\ &= m'N^{m'} - m'N^{m'-1}(N-1) = m'N^{m'-1}.\end{aligned}$$

Hence,

$$\begin{aligned} g_{\text{forest}}(N,n) &= \frac{(N-n)!}{m'!(N-1)(k-1)!^{m'}} \left((n-1)N^{m'} + (k-1)m'N^{m'-1} \right) \\ &= \frac{(N-n)!N^{m'-1}}{m'!(N-1)(k-1)!^{m'}} \left(N(n-1) + N - n \right) \\ &= \frac{n(N-n)!N^{m'-1}}{m'!(k-1)!^{m'}}, \end{aligned}$$

and we are done. □

4.6 Tools

In this section, we will include some computations that will be used a number of times throughout the proofs.

Let k be a positive integer. Let $c : \mathbb{R} \to \mathbb{R}$ so that $c(y) > k$ for all $y \in \mathbb{R}$. Let $\lambda(y)$ be defined by

$$\frac{\lambda(y) f_{k-1}(\lambda(y))}{f_k(\lambda(y))} = c(y).$$

The existence and uniqueness of $\lambda(y)$ follow from Lemma 2.10.3 due to Pittel and Wormald. We compute the derivative λ' of $\lambda(y)$ by implicit differentiation. Assuming that c is differentiable with derivative c':

$$\lambda' \frac{f_{k-1}(\lambda(y))}{f_k(\lambda(y)))} \left(1 + \frac{\lambda(y) f_{k-2}(\lambda(y))}{f_{k-1}(\lambda(y))} - \frac{\lambda(y) f_{k-1}(\lambda(y))}{f_k(\lambda(y))} \right) = c'. \tag{4.9}$$

Let $T, t : \mathbb{R} \to \mathbb{R}$ be differentiable functions be such that $T(y)/t(y) > k$ for all $y \in \mathbb{R}$. Let t' and T' denote the derivatives of t and T, resp. We will compute the derivative of $t(y) \log f_k(\lambda(y)) - T(y) \log(\lambda(y))$. For $c(y) = T(y)/t(y) = \lambda(y) f_{k-1}(\lambda(y))/f_k(\lambda(y))$ and $\eta(y) = \lambda(y) f_{k-2}(\lambda(y))/f_{k-1}(\lambda(y))$, and using (4.9),

$$\begin{aligned}
&\frac{\mathrm{d}\Big(t(y) \log f_k(\lambda(y)) - T(y) \log(\lambda(y)) \Big)}{\mathrm{d}\, y} = \\
&= t' \log f_k(\lambda(y)) + \lambda' \frac{t(y) f_{k-1}(\lambda(y))}{f_k(\lambda(y))} - T' \log \lambda(y) - \lambda' \frac{T(y)}{\lambda(y)} \\
&= t' \log f_k(\lambda(y)) + t(\lambda) \frac{f_{k-1}(\lambda(y))}{f_k(\lambda(y))} \frac{\lambda(y) c'}{c(y)(1 + \eta(y) - c(y))} \\
&\quad - T' \log \lambda(y) - \frac{T(y)}{\lambda(y)} \frac{\lambda(y) c'}{c(y)(1 + \eta(y) - c(y))} \\
&= t' \log f_k(\lambda(y)) + \frac{t(y) c'}{1 + \eta(y) - c(y)} - T' \log \lambda(y) - \frac{t(y) c'}{1 + \eta(y) - c(y)} \\
&= t' \log f_k(\lambda(y)) - T' \log \lambda(y).
\end{aligned} \tag{4.10}$$

The following lemma is an application of standard results concerning Gaussian functions and the definition of Riemann integral.

Lemma 4.6.1. Let $\phi(n) \to 0$, $\psi(n) \to 0$, $T_n \to \infty$ and $s_n \to \infty$. Let $f_n = \exp(-\alpha x^2 + \beta x + \phi x^2 + \psi x)$ with constants $\alpha > 0$ and β. Let $\mathcal{P}_n = z + \mathbb{Z}$, where $z \in \mathbb{R}$. Then

$$\frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n / s_n \\ |x| \le T_n}} f_n(x) \sim \exp\left(\frac{\beta^2}{4\alpha} \right) \sqrt{\frac{\pi}{\alpha}}.$$

Proof. Let $\varepsilon \in (0, \min(\alpha, \beta))$ and let $f^+(x) = \exp(-\alpha x^2 + \beta x + \varepsilon x^2 + \varepsilon x)$ and $f^-(x) = \exp(-\alpha x^2 + \beta x - \varepsilon x^2 - \varepsilon x)$. Since $\phi = o(1)$ and $\psi = o(1)$, we may assume $f^-(x) \le f_n(x) \le f^+(x)$. We will show that

$$\frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n/s_n \\ |x| \le T_n}} f^+(x) \sim \exp\Big(\frac{(\beta+\varepsilon)^2}{4(\alpha+\varepsilon)}\Big)\sqrt{\frac{\pi}{\alpha+\varepsilon}} \tag{4.11}$$

and

$$\frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n/s_n \\ |x| \le T_n}} f^-(x) \sim \exp\Big(\frac{(\beta-\varepsilon)^2}{4(\alpha-\varepsilon)}\Big)\sqrt{\frac{\pi}{\alpha-\varepsilon}}. \tag{4.12}$$

Since we can choose ε arbitrarily close to zero, this proves the lemma. We will only show the proof for (4.11) since the proof for (4.12) is very similar. We have that

$$\int_{-\infty}^{\infty} f^+(x)dx = \lim_{C\to\infty} \int_{-C}^{C} f^+(x)dx$$

and

$$\lim_{-\infty}^{\infty} f^+(x)dx = e^{\frac{(\beta+\varepsilon)^2}{4(\alpha+\varepsilon)}}\sqrt{\frac{\pi}{\alpha+\varepsilon}}.$$

So it suffices to show that,

$$\left| \lim_{n\to\infty} \frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n/s_n \\ |x| \le T_n}} f^+(x) - \lim_{C\to\infty} \int_{-C}^{C} f^+(x)dx \right| = 0.$$

We have that

$$\left| \lim_{n\to\infty} \frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n/s_n \\ |x| \le T_n}} f^+(x) - \lim_{C\to\infty} \int_{-C}^{C} f^+(x)dx \right| \le$$

$$\left| \lim_{n\to\infty} \frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n/s_n \\ |x| \le T_n}} f^+(x) - \lim_{C\to\infty} \lim_{n\to\infty} \frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n/s_n \\ |x| \le C}} f^+(x) \right|$$

$$+ \left| \lim_{C\to\infty} \lim_{n\to\infty} \frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n/s_n \\ |x| \le C}} f^+(x) - \lim_{C\to\infty} \int_{-C}^{C} f^+(x)dx \right|$$

the last term goes to is zero by the definition of Riemann integral. It is known that the tail for Gaussian functions is very small. More precisely, for each $\varepsilon' > 0$ there exists n_0 such that, for each $n \geq n_0$,

$$\left| \frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n/s_n \\ |x| \leq T_n}} f^+(x) - \lim_{n\to\infty} \frac{1}{s_n} \sum_{\substack{x \in \mathcal{P}_n/s_n \\ |x| \leq T_{n_0}}} f^+(x) \right| \leq \varepsilon'.$$

Since $C \to \infty$, C is eventually bigger than T_{n_0}. And we are done since we can choose $\varepsilon' > 0$ arbitrarily small. □

4.7 Counting cores

In this section we obtain an asymptotic formula for the number of cores (not necessarily connected) with vertex set $[n]$ and $m = n/2 + R$ edges, when $R = \omega(\log n)$ and $R = o(n)$. We also obtain an upper bound for the number of such cores when $R \to \infty$. We remark that the asymptotics in this section are for $n \to \infty$. We will always use r to denote R/n.

For $n_1 \in \mathbb{R}$, define

$$\begin{aligned} n_2(n_1) &= n - n_1, \\ m_3(n_1) &= m - n_1, \\ Q_2(n_1) &= 3m - n_1, \\ c_2(n_1) &= Q_2(n_1)/n_2(n_1) = (3m - n_1)/(n - n_1). \end{aligned}$$

For any symbol y in this section, we use $\hat{y}$ to denote y/n.

We will use n_1 as the number of vertices of degree 1 in the core. Then $n_2(n_1)$ is the number of vertices of degree at least 2, $m_3(n_1)$ is the number of 3-edges, $Q_2(n_1)$ is the sum of degrees of vertices of degree at least 2, and $c_2(n_1)$ is the average degree of the vertices of degree at least 2. We omit the argument n_1 when it is obvious from the context.

Let J_m denote the set of reals n_1 such that $\max\{0, 2n - 3m\} \leq n_1 \leq \min\{n, m\}$. The lower bound $2n - 3m$ is used to ensure that $c_2(n_1) \geq 2$ for $n_1 \in J_m$. Let $\hat{J}_m = \{x/n : x \in J_m\}$, that is, $\hat{J}_m$ is a scaled version of J_m. For $n_1 \in J_m \setminus \{2n - 3m\}$, let λ_{n_1} be the unique positive solution of

$$\frac{\lambda f_1(\lambda)}{f_2(\lambda)} = c_2(n_1). \tag{4.13}$$

Such a solution exists and is unique since $c_2(n_1) = (3m - n_1)/(n - n_1)$ and $n_1 > 2n - 3m$ ensures that $3m - n_1 > 2(n - n_1)$ (see Lemma 2.10.3). By continuity reasons, we define $\lambda_{2n-3m} = 0$.

Let

$$\eta_2(n_1) = \frac{\lambda_{n_1} \exp(\lambda_{n_1})}{f_1(\lambda_{n_1})}. \tag{4.14}$$

Let $h_n(x) = x\ln(xn) - x$ and define, for $\hat{n}_1$ in the interior of $\hat{J}_m$,

$$\begin{aligned} f_{\text{core}}(\hat{n}_1) =& h_n(\hat{Q}_2(n_1)) - h_n(\hat{n}_2(n_1)) - h_n(\hat{n}_1) - h_n(\hat{m}_3(n_1)) \\ & - \hat{n}_1\ln(2) - \hat{m}_3(n_1)\ln(6) \\ & + \hat{n}_2(n_1)\ln(f_2(\lambda_{n_1})) - \hat{Q}_2(n_1)\ln(\lambda_{n_1}), \end{aligned} \tag{4.15}$$

We extend the definition f_{core} to $\hat{J}_m$ by setting the $f_{\text{core}}(\hat{n}_1)$ to be the limit of $f_{\text{core}}(x)$ as $x \to \hat{n}_1$, for the points $\hat{n}_1 \in \hat{J}_m \cap \{0, 1, \hat{m}, 2-3\hat{m}\}$. For all points in $\hat{J}_m \cap \{0, 1, \hat{m}, 2-3\hat{m}\}$ except $2-3\hat{m}$, this only means that $0\log 0$ should be interpreted as 1. For $\hat{n}_1 = 2-3\hat{m}$, as we already mentioned, $\lambda_{2n-3m} = 0$ by continuity reasons. But then $\hat{n}_2(n_1)\ln(f_2(\lambda_{n_1})) - \hat{Q}_2(n_1)\ln(\lambda_{n_1})$ is not defined (and note that $\hat{n}_2(n_1)\ln(f_2(\lambda_{n_1}))$ and $\hat{Q}_2(n_1)\ln(\lambda_{n_1})$ appear in the definition of f_{core}). For $\hat{n}_1 = 2-3\hat{m}$,

$$\begin{aligned} & \lim_{\lambda\to 0}\left(\hat{n}_2(n_1)\ln(f_2(\lambda)) - \hat{Q}_2(n_1)\ln(\lambda)\right) = \hat{n}_2(n_1)\lim_{\lambda\to 0}\left(\ln(f_2(\lambda)) - 2\ln(\lambda)\right) \\ & = \hat{n}_2(n_1)\lim_{\lambda\to 0}\left(\ln\left(\frac{\exp(\lambda) - 1 - \lambda}{\lambda^2}\right)\right) = \hat{n}_2(n_1)\ln\left(\frac{1}{2}\right). \end{aligned}$$

Thus,

$$f_{\text{core}}(2-3\hat{m}) = h_n(\hat{Q}_2) - h_n(\hat{n}_2) - h_n(\hat{n}_1) - h_n(\hat{m}_3) - \hat{n}_1\ln(2) - \hat{m}_3\ln(6) - \hat{n}_2\ln 2. \tag{4.16}$$

We will show that the $n!\exp(nf_{\text{core}}(\hat{n}_1))$ approximates the exponential part of the number of cores with n_1 vertices of degree 1. Recall that $g_{\text{core}}(n,m)$ is the number of cores with vertex set $[n]$ and m edges. We obtain the following result for g_{core}:

Theorem 4.7.1. Let $m(n) = n/2 + R$ with $R \to \infty$. There exists a constant α such that, for $n \geq 1$,

$$g_{\text{core}}(n,m) \leq \alpha n\sqrt{m}\cdot n!\exp\left(nf_{\text{core}}(\hat{n}_1^*)\right). \tag{4.17}$$

If $R = o(n)$ and $R = \omega(\log n)$, we have that

$$g_{\text{core}}(n,m) \sim \frac{1}{2\pi n\sqrt{r}}\cdot n!\exp\left(nf_{\text{core}}(\hat{n}_1^*)\right), \tag{4.18}$$

where $\hat{n}_1^* = 3m/(ng_2(\lambda^*))$ and λ^* is the unique positive solution of

$$\frac{\lambda f_1(\lambda)g_2(\lambda)}{f_2(2\lambda)} = \frac{3m}{n}. \tag{4.19}$$

We will show that the point $\hat{n}_1^*$ maximizes f_{core} in $\hat{J}_m$. The result in (4.18) will then obtained by expanding the summation around $\hat{n}_1^*$.

For all subsections of this section, let Σ_{n_1} denote the event that a random vector $\mathbf{Y} = (Y_1, \ldots, Y_{n-n_1})$ satisfies $\sum_i Y_i = 3m - n_1$, where the Y_i's are independent random variables with truncated Poisson distribution $\text{Po}(2, \lambda_{n_1})$. Also, whenever symbols y and $\hat{y}$ appear in the same computation, $\hat{y}$ denotes y/n.

4.7.1 Random cores

Recall that our aim in Section 4.7 is to find an asymptotic formula for $g_{\text{core}}(n,m)$. Note that up to this point there is no random graph involved in the problem. However, similarly to Chapter 3 (Section 3.2), we show how to reduce the asymptotic enumeration problem to approximating the expectation, in a probability space of random sequences $\mathbf{Y}$, of the probability that a certain type of random multigraph with given degree sequence $\mathbf{Y}$ is simple.

For integer $n_1 \in J_m$, let $\mathcal{D}_{n_1}$ be the set of all $\mathbf{d} \in (\mathbb{N} \setminus \{0,1\})^{n-n_1}$ with $\sum_{i=1}^{n-n_1} d_i = 3m - n_1$. For $n_1 \in J_m \cap \mathbb{Z}$ and $\mathbf{d} \in \mathcal{D}_{n_1}$, let $\mathcal{G}(n_1,\mathbf{d}) = \mathcal{G}_{n,m}(n_1,\mathbf{d})$ be the multigraph obtained by the following procedure. We will start by creating for each edge one bin/set with 3 points inside it. These bins are called *edge-bins*. We also create one bin for each vertex with the number of points inside it equal to the degree of the vertex. These bins are called *vertex-bins*. Each point in a vertex-bin will be matched to a point in an edge-bin with some constraints. The multigraph can then be obtained by creating one edge for each edge-bin i such that the vertices incident to the edge are the vertices with points matched to the edge-bin of i. We describe the procedure in detail now. In the following, in each step, every choice is made u.a.r. among all possible choices satisfying the stated constraints:

1. *(Edge-bins)* For each $i \in [m]$, create an edge-bin i with 3 points labelled $1, 2$ and 3.

2. *(Vertex-bins)* Choose a set V_1 of n_1 vertices in $[n]$ to be the vertices of degree 1. For each $v \in V_1$, create one vertex-bin v with one point inside each. Let $v_1 < \cdots < v_{n-n_1}$ be an enumeration of the vertices in $[n] \setminus V_1$. For each $i \in [n-n_1]$ create a vertex-bin v_i with d_i points.

3. *(Matching)* Match the points from the vertex-bins to the points in edge-bins so that each edge-bin has at most one point being matched to a point in a vertex-bin of size 1. This matching is called a *configuration*.

4. *(Multigraph)* $\mathcal{G}(n_1,\mathbf{d}) = ([n],[m],\Phi)$, where $\Phi(i,j) = v$, where v is the vertex-bin containing the point matched to j.

See Figure 4.4, for an example for the procedure described above.

Let $g_{\text{core}}(n,m,n_1)$ denote the number of cores with vertex-set $[n]$ with m edges and n_1 vertices of degree 1, and let $g_{\text{core}}(n,m,n_1,\mathbf{d})$ denote the number of such cores with the additional constraint that $\mathbf{d} \in \mathbb{N}^{n-n_1}$ is such that, given the set V_1 of vertices of degree 1 and an enumeration $v_1 < \ldots < v_{n-n_1}$ of the vertices in $[n] \setminus V_1$, the degree of v_i is d_i. We say that $\mathbf{d}$ is the degree sequence for the vertices of degree at least 2, although $\mathbf{d}$ is not indexed by the set of vertices of degree at least 2. The following proposition relates $g_{\text{core}}(n,m,n_1)$ and $g_{\text{core}}(n,m,n_1,\mathbf{d})$ to $\mathcal{G}_{n,m}(n_1,\mathbf{d})$ and $\mathbf{Y}$. Recall that $\mathcal{S}(n,m)$ is defined in Section 4.3 as the set of multigraphs with

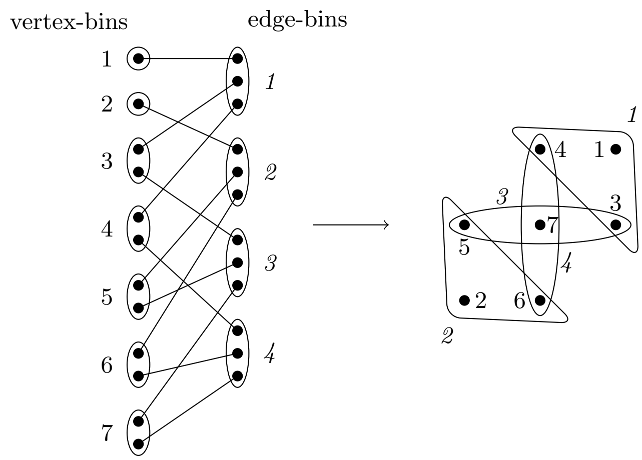

Figure 4.4: A core generated with vertex-bins and edge-bins

vertex set $[n]$ and m edges corresponding to simple graphs. Let $U(n_1, \mathbf{d})$ denote the probability that $\mathcal{G}_{n,m}(n_1, \mathbf{d}) \in \mathcal{S}(n, m)$.

Proposition 4.7.2. We have that, for any integer $n_1 \in J_m$

$$g_{\text{core}}(n, m, n_1, \mathbf{d}) = n! \frac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} 6^{m_3(n_1)}} \frac{1}{\prod_i d_i!} \mathbb{P}\left(U(n_1, \mathbf{d})\right), \tag{4.20}$$

and, for any integer $n_1 \in J_m \setminus \{2n - 3m\}$,

$$g_{\text{core}}(n, m, n_1) = n! \frac{Q_2(n_1)! f_2(\lambda_{n_1})^{n_2(n_1)}}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} 6^{m_3(n_1)} \lambda_{n_1}^{Q_2(n_1)}} \mathbb{E}\left(U(n_1, \mathbf{Y}) \Big| \Sigma_{n_1}\right) \mathbb{P}\left(\Sigma_{n_1}\right), \tag{4.21}$$

where Σ_{n_1} is the event that a random vector $\mathbf{Y} = (Y_1, \ldots, Y_{n-n_1})$ satisfies $\sum_i Y_i = 3m - n_1$ and the Y_i's are independent random variables with truncated Poisson distribution $\text{Po}(2, \lambda_{n_1})$.

Proof. First we compute the total number of configurations that can be generated. There are $\binom{n}{n_1}$ ways of choosing the vertices of degree 1 in Step 2. We can split Step 3 by first choosing the n_1 edge-bins and one point in each of these edge-bins to be matched to the points inside vertex-bins of size 1. There are $\binom{m}{n_1} 3^{n_1}$ possible choices for these edge-bins and the points inside them. There are $n_1!$ ways of matching these points to the points in vertex-bins of size 1 and there are $Q_2(n_1)!$ ways of matching the remaining points in the edge-bins to the vertex-bins of size at least 2. Thus, the total number of configurations is

$$\binom{n}{n_1} \binom{m}{n_1} 3^{n_1} n_1! Q_2(n_1)! =: \beta. \tag{4.22}$$

It is straightforward to see that every multigraph with degree sequence $\mathbf{d}$ for the vertices of degree at least 2 is generated by $\prod_{i=1}^{n-n_1} d_i!$ configurations. Together with Lemma 4.3.3, this implies that every graph with degree sequence $\mathbf{d}$ for the vertices of degree at least 2 is generated by

$$\alpha = m!6^m \prod_{i=1}^{n-n_1} d_i! \tag{4.23}$$

configurations. Thus, since each configuration is generated with the same probability,

$$g_{\text{core}}(n, m, n_1, \mathbf{d}) = \frac{\beta}{\alpha} U(n_1, \mathbf{d}). \tag{4.24}$$

Together with (4.22) and (4.23), and trivial simplifications, this implies (4.20).

We now prove (4.21). The proof is very similar to the proofs of Propositions 3.2.1 and 3.2.2 in Chapter 3 (which in turn are very similar to the proof of [55, Equation (13)]). Recall that $\mathcal{D}_{n_1}$ be the set of all $\mathbf{d} \in (\mathbb{N} \setminus \{0,1\})^{n_3(n_1)}$ with $\sum_i d_i = Q_2(n_1)$. We have that

$$\begin{aligned}
g_{\text{core}}(n, m, n_1) &:= \sum_{\mathbf{d} \in \mathcal{D}_{n_1}} g_{\text{core}}(n, m, n_1, \mathbf{d}) \\
&= n! \sum_{\mathbf{d} \in \mathcal{D}_{n_1}} \frac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} 6^{m_3(n_1)}} \frac{1}{\prod_{i=1}^{n_3(n_1)} d_i!} U(n_1, \mathbf{d}) \\
&= n! \frac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} 6^{m_3(n_1)}} \frac{f_2(\lambda_{n_1})^{n_2(n_1)}}{\lambda_{n_1}^{Q_2(n_1)}} \sum_{\mathbf{d} \in \mathcal{D}_{n_1}} U(n_1, \mathbf{d}) \prod_{i=1}^{n_3(n_1)} \frac{\lambda_{n_1}^{d_i}}{d_i! f_2(\lambda_{n_1})} \\
&= n! \frac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} 6^{m_3(n_1)}} \frac{f_2(\lambda_{n_1})^{n_2(n_1)}}{\lambda_{n_1}^{Q_2(n_1)}} \sum_{\mathbf{d} \in \mathcal{D}_{n_1}} U(n_1, \mathbf{d}) \, \mathbb{P}(\mathbf{Y} = \mathbf{d}) \\
&= n! \frac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} 6^{m_3(n_1)}} \frac{f_2(\lambda_{n_1})^{n_2(n_1)}}{\lambda_{n_1}^{Q_2(n_1)}} \mathbb{E}\left(U(n_1, \mathbf{Y}) | \Sigma_{n_1}\right) \mathbb{P}(\Sigma_{n_1}),
\end{aligned}$$

which proves (4.21). We remark that the only reason why the above proof does not work for $n_1 = 2n - 3m$ (and so for the whole $J_m \cap \mathbb{Z}$) is that $\lambda_{2n-3m} = 0$ (by continuity), which would cause a division by zero in (4.21). □

The next lemma gives conditions which are sufficient for the expectation in (4.24) to be asymptotic to 1.

Lemma 4.7.3. Let $m(n) = n/2 + R$ with $R \to \infty$ and $R = o(n)$. Uniformly for $n_1 = n_1(n) \in J_m \cap \mathbb{Z}$, we have that

$$\mathbb{E}\left(U(n_1, \mathbf{Y}) \Big| \Sigma_{n_1}\right) \sim 1.$$

Proof. Recall that $\mathcal{D}_{n_1}$ is the set of all $\mathbf{d} \in (\mathbb{N} \setminus \{0,1\})^{n_3(n_1)}$ with $\sum_i d_i = Q_2(n_1)$. For a constant $\varepsilon \in (0, 1/6)$, let $\mathcal{D}'_{n_1}$ be the subset of $\mathcal{D}_{n_1}$ such that $\mathbf{d} \in \mathcal{D}'_{n_1}$ if $d_i \leq n^\varepsilon$ for every i. We will show that $\mathbb{P}(\mathbf{Y} \in \mathcal{D}'_{n_1} \mid \Sigma_{n_1}) = 1 + o(1)$, that is, $\mathcal{D}'_{n_1}$ contains all 'typical' degree sequences. We then show that the degree sequences not in $\mathcal{D}'_{n_1}$ have no significant contribution to the expectation.

First we consider the case $n_1 = 2n - 3m = n/2 - 3R$. In this case the only degree sequence in $\mathcal{D}_{n_1}$ is the sequence of all 2's, which is in $\mathcal{D}'_{n_1}$. So suppose $n_1 > 2n - 3m$. Since $R = o(n)$ and $n_1 \leq m = n/2 + R$, we have that $3m - n_1 \sim 2(n - n_1)$ and so $\lambda_{n_1} \to 0$, Thus, for any $j \to \infty$, by computing the series of $f_2(\lambda_{n_1})$ with $\lambda_{n_1} \to 0$,

$$\mathbb{P}(Y_i \geq j) = \frac{\lambda_{n_1}^j}{j! f_2(\lambda)}(1 + O(\lambda_{n_1})) = \frac{2\lambda_{n_1}^{j-2}}{j!}(1 + O(\lambda_{n_1})) = o\Big(\exp\big(-\alpha j \log j \big) \Big),$$

for a positive constant α. Thus, by the union bound

$$\mathbb{P}(\max_i Y_i \geq n^\varepsilon) \leq n_2 \cdot o\Big(\exp\big(-\alpha \varepsilon n^\varepsilon \log n \big) \Big) = o\Big(\exp\big(-\alpha' n^\varepsilon \log n \big) \Big),$$

for a positive constant α'.

We estimate the probability of Σ_{n_1}. Let $R_2 = (3m - n_1) - 2(n - n_1) = n_1 - (2n - 3m)$. If $R_2 = o({n_2}^{1/3})$, then, by Theorem 2.10.8,

$$\mathbb{P}(\Sigma_{n_1}) \sim \frac{e^{-R_2} R_2^{R_2}}{R_2!} = \Omega\left(\frac{1}{\sqrt{R_2}} \right)$$

by Stirling's approximation (Lemma 2.5.1). If $R_2 \to \infty$, by Theorem 2.10.8,

$$\mathbb{P}(\Sigma_{n_1}) \sim \frac{1}{\sqrt{2\pi n_2 c_2 (1 + \eta_2 - c_2)}} = \Omega\left(\frac{1}{\sqrt{n_2}} \right)$$

where $\eta_2 = \lambda_{n_1} \exp(\lambda_{n_1}) / f_1(\lambda_{n_1})$ and since $c_2(1 + \eta_2 - c_2) \sim c_2 - 2$ by Lemma 2.10.7. Thus, for $\mathcal{D}''_{n_1} := \mathcal{D}_{n_1} \setminus \mathcal{D}'_{n_1}$.

$$\mathbb{P}(\mathbf{Y} \in \mathcal{D}''_{n_1} \mid \Sigma_{n_1}) \leq \frac{\mathbb{P}(\mathbf{Y} \in \mathcal{D}''_{n_1})}{\mathbb{P}(\Sigma_{n_1})} = O\left(\sqrt{n} \exp\big(-\alpha' n^\varepsilon \log n \big) \right) = o(1). \tag{4.25}$$

Now we show that

$$\mathbb{P}(\mathcal{G}(n_1, \mathbf{d}) \text{ simple}) = 1 + o(1), \tag{4.26}$$

for $\mathbf{d} \in \mathcal{D}_{n_1}$. We have to compute the probability that there are no loops and no double edges in $\mathcal{G}(n_1, \mathbf{d})$.

A loop arises from the edge-bins that have at least two points matched to points in the same vertex-bin. The expected number of loops is

$$\sum_i \binom{d_i}{2} \frac{(Q_2 - 2)!}{Q_2!} = O\left(\frac{m n^\varepsilon}{n^2} \right) = o(1)$$

since $Q_2 = 3m - n_1 \geq n/2 + 3R = \Omega(n)$ and $\mathbf{d} \in \mathcal{D}'_{n_1}$ (and $\varepsilon < 1/6$). Double edges arise from pairs edge-bins that the points in each of them are mapped to the same vertex-bins with the same multiplicities. Using that $\max_i d_i \leq n^\varepsilon$ for $\mathbf{d} \in \mathcal{D}'_{n_1}$, we have that the expected number of double edges (not involving loops) is at most

$$\binom{m_3}{2} n_2^3 (n^\varepsilon)^6 6! \frac{(Q_2 - 6)!}{Q_2!} = O\left(\frac{n^{5+6\varepsilon}}{n^6}\right) = o(1),$$

since $\varepsilon < 1/6$. (Note that 2-edges cannot be double edges because of the vertices of degree 1.) Thus, Markov's inequality implies (4.26).

Since $\mathcal{D}'_{n_1}$ is a finite set for each n and (4.26), by Lemma 2.7.1, we have that there exists $f(n) = 1 - o(1)$ such that $\mathbb{P}(\mathcal{G}(n_1, \mathbf{d}) \text{ simple}) \geq f$ for all $\mathbf{d} \in \mathcal{D}'_{n_1}$. Thus,

$$\begin{aligned}\mathbb{E}\left(U(\mathbf{Y})|\Sigma_{n_1}\right) &\leq \sum_{\mathbf{d} \in \mathcal{D}'_{n_1}} \mathbb{P}(\mathcal{G}(n_1, \mathbf{d}) \text{ simple }) \, \mathbb{P}(\mathbf{Y} = \mathbf{d}) + \mathbb{P}(\mathbf{Y} \in \mathcal{D}''_{n_1} \,|\, \Sigma_{n_1}) \\ &\geq \mathbb{P}(\mathbf{Y} \in \mathcal{D}'_{n_1} \,|\, \Sigma_{n_1}) f(n) \, \mathbb{P}(\mathbf{Y} \in \mathcal{D}''_{n_1} \,|\, \Sigma_{n_1}) = 1 - o(1),\end{aligned}$$

by (4.25). □

4.7.2 Proof of Theorem 4.7.1

In this section we present the proof of the asymptotic formula in Theorem 4.7.1 for the number $g_{\text{core}}(n, m)$ of cores (not necessarily connected) with vertex set $[n]$ with $m = n/2 + R$ edges, when $R = \omega(\log n)$ and $R = o(n)$:

$$g_{\text{core}}(n, m) \sim \frac{1}{2\pi n \sqrt{r}} \cdot n! \exp\Big(n f_{\text{core}}(\hat{n}_1^*)\Big), \tag{4.27}$$

where $\hat{n}_1^* = 3\hat{m}/g_2(\lambda^*)$ and λ^* is the unique positive solution for

$$\frac{\lambda f_1(\lambda) g_2(\lambda)}{f_2(2\lambda)} = \frac{3m}{n}. \tag{4.28}$$

We also show the upper bound in Theorem 4.7.1 for $g_{\text{core}}(n, m)$ that holds as long as $R \to \infty$. First we show that λ^* is well-defined.

Lemma 4.7.4. The equation $\lambda f_1(\lambda) g_2(\lambda)/f_2(2\lambda) = \alpha$ has a unique positive solution λ^*_α for any $\alpha > 3/2$. Moreover, for any positive constant ε, there exists a positive constant ε', such that, if $\alpha, \beta \in (0, \varepsilon)$, then $|\lambda^*_\alpha - \lambda^*_\beta| \leq \varepsilon' |\alpha - \beta|$.

Proof. It suffices to show that $f(\lambda) := \lambda f_1(\lambda) g_2(\lambda)/f_2(2\lambda)$ is a strictly increasing function of λ with $\lambda > 0$ and $\lim_{\lambda\to 0^+} f(\lambda) = 3/2$. See Section A.2 for a Maple spreadsheet. By computing the series of $f(\lambda)$ with $\lambda \to 0$, we obtain

$$f(\lambda) = \frac{3}{2} + \frac{\lambda}{4} + O(\lambda^2).$$

The derivative of f is

$$\frac{\mathrm{d}\, f(\lambda)}{\mathrm{d}\,\lambda} = \frac{2 + 2e^{2\lambda}\lambda - e^{\lambda}\lambda - 4e^{2\lambda}\lambda^2 - e^{3\lambda}\lambda - 2e^{\lambda}\lambda^2 + e^{4\lambda} + e^{3\lambda} - 3e^{2\lambda} - e^{\lambda}}{f_2(2\lambda)^2},$$

which we want to show that is positive for any $\lambda > 0$. Let $F(\lambda)$ denote the numerator in the above. It suffices to show that $F(\lambda)$ is positive for $\lambda > 0$. Let $F^{(0)} = F$. We will use the following strategy: starting with $i = 1$, we check that $F^{(i-1)}(0) \geq 0$ and compute the derivative $F^{(i)}$ of $F^{(i-1)}$. If for some i we can show that $F^{(i)}(\lambda) > 0$ for any $\lambda > 0$, then we obtain $F(\lambda) > 0$ for $\lambda > 0$. Otherwise, we try to simplify the derivative. If $\exp(\lambda)$ appears in every term of $F^{(i)}$, we redefine $F^{(i)}$ by dividing it by $\exp(\lambda)$. Eventually, we obtain

$$216e^{2\lambda} - 24\lambda e^{\lambda} - 44e^{\lambda} - 16\lambda - 52,$$

which is trivially positive since $\exp(2x) \geq \exp(x) \geq 1 + x$ for $x \geq 0$ and the sum of the coefficients of the negative terms is less than 216.

The proof of the second statement in the lemma follows trivially from the fact that the first derivative is always positive and, with $\lambda \to 0$,

$$\frac{\mathrm{d}\, f(\lambda)}{\mathrm{d}\,\lambda} = \frac{\lambda^4 + O(\lambda^5)}{4\lambda^4 + O(\lambda^5)} \to \frac{1}{4} > 0.$$

□

Since $3\hat{m} = 3/2 + 3r$, for $r = o(1)$ we have that λ^* is well-defined and $\lambda^* \to 0$, by Lemma 4.7.4. Let

$$w_{\text{core}}(n_1) = \begin{cases} n! \dfrac{Q_2(n_1)! f_2(\lambda_{n_1})^{n_2(n_1)}}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} 6^{m_3(n_1)} \lambda_{n_1}^{Q_2(n_1)}}, & \text{if } n_1 \in J_m \setminus \{2n - 3m\}; \\ n! \dfrac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^n 6^{m_3(n_1)}}, & \text{if } n_1 = 2n - 3m \in J_m. \end{cases} \tag{4.29}$$

Then, Proposition 4.7.2 implies that

$$\begin{aligned} g_{\text{core}}(n, m) = &\sum_{n_1 \in J_m \setminus \{2n-3m\}} w_{\text{core}}(n_1) \mathbb{E}\left(\mathcal{G}(n, m, n_1, \mathbf{Y}) \text{ simple} | \Sigma_{n_1}\right) \mathbb{P}(\Sigma_{n_1}) \\ &+ \mathbb{1}_{2n-3m \in J_m} w_{\text{core}}(2n - 3m)\, \mathbb{P}(\mathcal{G}(n, m, n_1, \mathbf{2}) \text{ simple}), \end{aligned}$$

where the last term comes from $n_1 = 2n - 3m$ and $\mathcal{D}_{2n-3m} = \{\mathbf{2}\}$.

Recall that $h_n(x) = x\ln(xn) - x$ and

$$\begin{aligned} f_{\text{core}}(\hat{n}_1) =& h_n(\hat{Q}_2) - h_n(\hat{n}_2) - h_n(\hat{n}_1) - h_n(\hat{m}_3) \\ & - \hat{n}_1 \ln(2) - \hat{m}_3 \ln(6) \\ & + \hat{n}_2 \ln(f_2(\lambda_{n_1})) - \hat{Q}_2 \ln(\lambda_{n_1}). \end{aligned}$$

The function $n!\exp(nf_{\text{core}}(\hat{n}_1))$ is an approximation for the exponential part of $w_{\text{core}}(n_1)$. We will analyse f_{core} and use it to draw conclusions about w_{core}. It will be useful to know the asymptotic values of $\hat{n}_1^*$ and some functions of it. In Equation (4.28), the RHS is $3m/n = 3/2 + 3r$ and so we can write r in terms of λ^*. Since $\hat{n}_1^*$ is defined as $3\hat{m}/g_2(\lambda^*)$, we can also write it in terms of λ^* and so we can write $Q_2(n_1^*)$, $n_2(n_1^*)$ and $m_3(n_1^*)$ in terms of λ^* (and n). As we have mentioned before, by Lemma 4.7.4, we have that $\lambda^* \to 0$. By computing the series with $\lambda^* \to 0$, we have that

$$\begin{aligned} &\lambda^* = 12r + O(r^2); \\ &\hat{n}_1^* = 1/2 - r + O(r^2); \\ &Q_2(n_1^*) = 3m - n_1^* = n + 4R + o(R); \\ &n_2(n_1^*) = n - n_1^* = n/2 + R + o(R); \\ &m_3(n_1^*) = m - n_1^* = 2R + o(R). \end{aligned} \tag{4.30}$$

Next, we state the main lemmas for the proof of Theorem 4.7.1. We defer their proofs to Section 4.7.3. First we show that $\hat{n}_1^*$ achieves the maximum value for f_{core} in $\hat{J}_m$.

Lemma 4.7.5. The point $\hat{n}_1^*$ is the unique maximum of the function $f_{\text{core}}(\hat{n}_1)$ for $\hat{n}_1 \in \hat{J}_m$. Moreover, we have that $f'_{\text{core}}(\hat{n}_1^*) = 0$, and $f'_{\text{core}}(\hat{n}_1) > 0$ for $\hat{n}_1 < \hat{n}_1^*$ and $f'_{\text{core}}(\hat{n}_1) < 0$ for $\hat{n}_1 > \hat{n}_1^*$.

Then we expand the summation around the maximum and approximate it by an integral, which we compute, obtaining the following:

Lemma 4.7.6. Suppose that $\delta = o(r)$ and $\delta^2 = \omega(r/n)$, with $r = o(1)$. We have that

$$\sum_{\substack{x \in [-\delta n, \delta n] \\ n_1^* + x \in \mathbb{Z}}} \exp(nf_{\text{core}}(\hat{n}_1^* + \hat{x}) \sim \sqrt{2\pi rn}\exp(nf_{\text{core}}(\hat{n}_1^*)).$$

Finally, we show that points far from the maximum do not contribute significantly to the summation:

Lemma 4.7.7. Suppose that $\delta = o(r)$ and $\delta^2 = \omega(r\log n/n)$ with $r = o(1)$. Then

$$\sum_{\substack{n_1 \in J_m \\ |n_1 - n_1^*| > \delta n}} w_{\text{core}}(n_1) = o\left(\frac{n!}{n\sqrt{r}}\exp(nf_{\text{core}}(\hat{n}_1^*))\right).$$

We are now ready to prove Theorem 4.7.1. First we prove (4.17). We discuss the relation of w_{core} and f_{core} more precisely here. The function $n!\exp(nf_{\mathrm{core}}(\hat{n}_1))$ can be obtained from the definition of $w_{\mathrm{core}}(n_1)$ in (4.29) as follows: replace $Q_2(n_1)!$ by $\exp(h_n(\hat{Q}_2))$, and do the same for $n_1!$, $n_2(n_1)!$, and $m_3(n_1)!$. That is, $n!\exp(nf_{\mathrm{core}}(\hat{n}_1))$ can be obtained from $w_{\mathrm{core}}(n_1)$ be replacing each factorial involving n_1 by its Stirling approximation (but ignoring the polynomials terms). By Stirling's approximation (Lemma 2.5.1), there exists constants α_1 and α_2 such that, for every $x \in \mathbb{N}$,

$$\alpha_1\sqrt{x}\left(\frac{x}{e}\right)^x \le x! \le \alpha_2\sqrt{x}\left(\frac{x}{e}\right)^x, \tag{4.31}$$

and so, there exists a constant α such that

$$w_{\mathrm{core}}(n_1) \le \alpha\sqrt{m}\exp(nf_{\mathrm{core}}(\hat{n}_1)).$$

Together with Lemma 4.7.5, this immediately implies (4.17).

Now we will prove (4.18). So assume that $R = o(n)$. In order to use Lemma 4.7.6 and Lemma 4.7.7, we need to choose $\delta = \delta(n)$ that satisfies $\delta = o(r)$ and $\delta^2 = \omega(r\log n/n)$. This is possible if and only if $r^2 = \omega(r\log n/n)$. That is, if and only if $R = \omega(\log n)$, which is one of the hypotheses of the theorem. Thus, let δ be such that $\delta = o(r)$ and $\delta^2 = \omega(r\log n/n)$.

Let $J(\delta) = [n_1^* - \delta n, n_1^* + \delta n] \cap \mathbb{Z}$. We have that $2n - 3m = n/2 - 3R$ is not in $J(\delta)$ because $n_1^* = n/2 - R + o(R)$ by (4.30) and $\delta n = o(R)$. By Proposition 4.7.2 and Lemma 4.7.3, for $n_1(n) \in J(\delta)$,

$$g_{\mathrm{core}}(n, m, n_1) \sim w_{\mathrm{core}}(n_1)\,\mathbb{P}(\Sigma_{n_1}). \tag{4.32}$$

For any $n_1(n) \in J(\delta)$, we have that $Q_2(n_1)$, $m_3(n_1)$, $n_2(n_1)$ are all $\Omega(R)$ by (4.30) and $\delta n = o(R)$. Thus, by (4.32), Stirling's approximation and the definition of f_{core}, for $n_1(n) \in J(\delta)$,

$$g_{\mathrm{core}}(n, m, n_1) = \frac{1}{2\pi}\sqrt{\frac{Q_2(n_1)}{n_2(n_1)n_1m_3(n_1)}} \cdot \mathbb{P}(\Sigma_{n_1}) \cdot n!\exp\Big(nf_{\mathrm{core}}(\hat{n}_1)\Big);$$

Using (4.30) and $\delta n = o(R)$, we obtain

$$\frac{1}{2\pi}\sqrt{\frac{Q_2(n_1)}{n_2(n_1)n_1m_3(n_1)}} = \frac{1}{2\pi}\sqrt{\frac{n + 4R + o(R)}{(n/2 + R + o(R))(n/2 - R + o(R))(2R + o(R))}} \sim \frac{1}{\pi n\sqrt{2r}} \tag{4.33}$$

and

$$\frac{Q_2(n_1)}{n_2(n_1)} = \frac{Q_2(n_1^*)}{n_2(n_1^*)}(1 + o(r)),$$

for $n_1(n) \in J(\delta)$. By Lemma 2.10.5, this implies that $\lambda_{n_1} \sim \lambda_{n_1^*}$ for $n_1 \in J(\delta)$. Moreover, we have that $Q_2(n_1) - 2n_2(n_1) = 2R + o(R) \to \infty$ by (4.30) and so, by Theorem 2.10.8,

$$\begin{aligned}
\mathbb{P}(\Sigma_{n_1}) &\sim (2\pi Q_2(n_1)(1+\eta_2(n_1)-c_2(n_1)))^{-1/2} \\
&= \left(2\pi(n+4R+o(R))\left(1+\lambda_{n_1}\frac{\exp(\lambda_{n_1})}{f_1(\lambda_{n_1})} - \lambda_{n_1}\frac{f_1(\lambda_{n_1})}{f_2(\lambda_{n_1})}\right)\right)^{-1/2} \\
&= \left(2\pi(n+4R+o(R))\left(1+\frac{1+\lambda_{n_1}+O(\lambda_{n_1}^2)}{1+\lambda_{n_1}/2+O(\lambda_{n_1}^2)} - \frac{1+\lambda_{n_1}/2+O(\lambda_{n_1}^2)}{1/2+\lambda_{n_1}/6+O(\lambda_{n_1}^2)}\right)\right)^{-1/2} \\
&= \left(2\pi(n+4R+o(R))\left(\frac{\lambda_{n_1}}{6}+O(\lambda_{n_1}^2)\right)\right)^{-1/2} \\
&= \big(2\pi(n+4R+o(R))\big(2r+O(r^2)\big)\big)^{-1/2} \sim \frac{1}{\sqrt{4\pi nr}},
\end{aligned} \tag{4.34}$$

for $n_1(n) \in J(\delta)$. Thus,

$$g_{\text{core}}(n,m,n_1) = (1+o(1))\frac{1}{\pi n\sqrt{2r}} \cdot \frac{1}{\sqrt{4\pi nr}} \cdot n! \exp\Big(nf_{\text{core}}(\hat{n}_1)\Big), \tag{4.35}$$

for $n_1(n) \in J(\delta)$. Since $J(\delta)$ is a finite set for each n, by Lemma 2.7.1 there exists a function $q(n) = o(1)$ such that the $o(1)$ in (4.35) is bounded in absolute value by $q(n)$. Thus, by Lemma 4.7.6,

$$\begin{aligned}
g_{\text{core}}(n,m) = \sum_{n_1 \in J(\delta)} g_{\text{core}}(n,m,n_1) &\sim \frac{1}{\pi n\sqrt{2r}} \cdot \frac{1}{\sqrt{4\pi nr}} \sum_{n_1\in J(\delta)} n! \exp\Big(nf_{\text{core}}(\hat{n}_1)\Big) \\
&\sim \frac{1}{\pi n\sqrt{2r}} \cdot \frac{1}{\sqrt{4\pi nr}} \cdot \sqrt{2\pi rn}\exp(nf_{\text{core}}(\hat{n}_1^*)) \\
&\sim \frac{1}{2\pi n\sqrt{r}}\exp(nf_{\text{core}}(\hat{n}_1^*)),
\end{aligned}$$

which together with Lemma 4.7.7 finishes the proof of Theorem 4.7.1.

4.7.3 Proof of lemmas in Section 4.7.2

In this section, we prove Lemmas 4.7.5, 4.7.6, and 4.7.7. See Section A.3 for a Maple spreadsheet with some computations in this section.

Using (4.10) with $T = \hat{Q}_2$ and $t = \hat{n}_2$, the derivative of $f_{\text{core}}(\hat{n}_1)$ is

$$-\ln(\hat{Q}_2) + \ln(\hat{n}_2) - \ln(\hat{n}_1) + \ln(\hat{m}_3) + \ln(3) - \ln f_2(\lambda) + \ln\lambda. \tag{4.36}$$

The second derivative is

$$\frac{1}{\hat{Q}_2} - \frac{1}{\hat{n}_2} - \frac{1}{\hat{n}_1} - \frac{1}{\hat{m}_3} - \frac{(1-c_2)^2}{\hat{Q}_2(1+\eta_2-c_2)} < 0, \tag{4.37}$$

because $1/\hat{Q}_2 < 1/\hat{m}_3$. The third derivative is

$$\begin{aligned} &\frac{1}{\hat{Q}_2^2} - \frac{1}{\hat{n}_2^2} + \frac{1}{\hat{n}_1^2} - \frac{1}{\hat{m}_3^2} \\ &- \frac{\mathrm{d}\,\frac{(1-c_2)^2}{\hat{Q}_2}}{\mathrm{d}\,\hat{n}_1}\frac{1}{(1+\eta_2-c_2)} + \frac{(1-c_2)^2}{\hat{Q}_2(1+\eta_2-c_2)^2}\frac{\mathrm{d}(1+\eta_2-c_2)}{\mathrm{d}\,\hat{n}_1}. \end{aligned} \tag{4.38}$$

In order to approximate the value of f_{core} around the maximum by using Taylor's approximation, we will bound the third derivative.

Lemma 4.7.8. Let $\delta = o(r)$ and $\hat{n}_1 \in [\hat{n}_1^* - \delta, \hat{n}_1^* + \delta]$. Then the third derivative of f_{core} at $\hat{n}_1$ is $O(1/r^2)$.

Proof. We will bound each term in (4.38). By (4.30),

$$\begin{aligned} \frac{1}{\hat{Q}_2^2} &= \frac{1}{(1+4r+o(r))^2} \sim 1 \\ \frac{1}{\hat{n}_2^2} &= \frac{1}{(1/2+O(r))^2} \sim 4 \\ \frac{1}{\hat{n}_1^2} &= \frac{1}{(1/2+O(r))^2} \sim 4 \\ \frac{1}{\hat{m}_3^2} &= \frac{1}{(2r+o(r))^2} \sim \frac{1}{4r^2}. \end{aligned}$$

We have that $\lambda := \lambda_{n_1} = \lambda^* + o(r) = o(1)$ by Lemma 2.10.5, and so $1+\eta_2-c_2 = \lambda/6 + O(\lambda^2) \sim \lambda^*/6 \sim 2r$. Thus, by (4.30),

$$\frac{\mathrm{d}\,\frac{(1-c_2)^2}{c_2}}{\mathrm{d}\,\hat{n}_1}\frac{1}{(1+\eta_2-c_2)} = \frac{(3\hat{m}-1)^2(6\hat{m}-3\hat{n}_1+1)}{\hat{n}_2^3\hat{Q}_2^2}\frac{1}{(1+\eta_2-c_2)} = \Theta\left(\frac{1}{r}\right).$$

We now bound the last term in the third derivative. The previous computations imply

$$\frac{(1-c_2)^2}{\hat{Q}_2(1+\eta_2-c_2)^2} \sim \frac{1}{4r^2},$$

since $1 - c_2 = \hat{Q}_2/\hat{n}_2 \sim 2$ by (4.30). So we need to bound

$$\frac{\mathrm{d}(1+\eta_2 - c_2)}{\mathrm{d}\,\hat{n}_1}.$$

We have that

$$\frac{\mathrm{d}\,c_2}{\mathrm{d}\,\hat{n}_1} = \frac{3\hat{m}-1}{(1-\hat{n}_1)^2} = \Theta(1)$$

and, by using (4.9),

$$\frac{\mathrm{d}\,\eta_2}{\mathrm{d}\,\hat{n}_1} = \lambda \frac{\mathrm{d}\,c_2}{\mathrm{d}\,\hat{n}_1} \frac{1}{c_2(1+\eta_2-c_2)} \left(\frac{\exp(\lambda)}{f_1(\lambda)} + \frac{\lambda\exp(\lambda)}{f_1(\lambda)} - \frac{\lambda\exp(\lambda)^2}{f_1(\lambda)^2} \right) = \Theta(1),$$

because

$$\lambda \left(\frac{\exp(\lambda)}{f_1(\lambda)} + \frac{\lambda\exp(\lambda)}{f_1(\lambda)} - \frac{\lambda\exp(\lambda)^2}{f_1(\lambda)^2} \right) = \frac{\lambda}{2} + O(\lambda^2).$$

Thus the last term has contribution $O(1/r^2)$ to the third derivative. □

We now present the proofs for Lemmas 4.7.5, 4.7.6, and 4.7.7.

Proof of Lemma 4.7.5. By setting the derivative of f_{core} in (4.36) to 0 and using the definition of λ_{n_1} in (4.13), we obtain the Equation (4.19), which has a unique positive solution λ^* by Lemma 4.7.4 The second derivative computation in (4.37) implies that f_{core} is strictly concave and so λ^* is the unique maximum. □

Proof of Lemma 4.7.6. Using Taylor's approximation and Lemma 4.7.8, for any $\hat{n}_1 \in [\hat{n}_1^* - \delta, \hat{n}_1^* + \delta]$,

$$\exp\left(n f_{\text{core}}(\hat{n}_1)\right) = \exp\left(n f_{\text{core}}(\hat{n}_1^*) + \frac{n f''_{\text{core}}(\hat{n}_1^*)|\hat{n}_1^* - \hat{n}_1|^2}{2} + O(\delta^3/r^2) \right). \tag{4.39}$$

Since $\delta^3/r^2 = o(r^3/r^2) = o(1)$, this implies that

$$\sum_{\substack{x\in[-\delta n,\delta n]\\ \hat{n}_1^*+x\in\mathbb{Z}}} \exp\left(n f_{\text{core}}(\hat{n}_1^* + \hat{x}) \right) \sim \sum_{\substack{x\in[-\delta n,\delta n]\\ \hat{n}_1^*+x\in\mathbb{Z}}} \exp\left(n f_{\text{core}}(\hat{n}_1^*) + \frac{f''_{\text{core}}(\hat{n}_1^*)x^2}{2n} \right).$$

By changing the variable in summation below to $y = \sqrt{\frac{|f''_{\text{core}}(\hat{n}_1^*)|}{n}}x$,

$$\sum_{\substack{x\in[-\delta n,\delta n]\\ \hat{n}_1^*+x\in\mathbb{Z}}} \exp\left(\frac{f''_{\text{core}}(\hat{n}_1^*)x^2}{2n} \right) = \sum_{\substack{y\in[-T_n,T_n]\\ y\in\mathcal{P}_n/s_n}} \exp\left(-\frac{y^2}{2} \right),$$

where $T_n := \delta\sqrt{n|f''_{\mathrm{core}}(\hat{n}_1^*)|}$, $\mathcal{P}_n := -n_1^* + \mathbb{Z}$, and $s_n := \sqrt{n/|f_{\mathrm{core}}(\hat{n}_1^*)|}$. By using (4.37) and (4.30), we approximated $f''_{\mathrm{core}}(\hat{n}_1^*)$:

$$f''_{\mathrm{core}}(\hat{n}_1^*) \sim -\frac{1}{r}, \tag{4.40}$$

and, since $\delta^2 = \omega(r/n)$ and $rn = R \to \infty$, we have that $T_n \to \infty$ and $s_n \to \infty$. Thus, by Lemma 4.6.1,

$$\sum_{\substack{y\in[-T_n,T_n]\\ y\in\mathcal{P}_n/s_n}} \exp\left(-\frac{y^2}{2}\right) \sim \sqrt{2\pi}s_n \sim \sqrt{2\pi}\sqrt{rn},$$

which finished the proof of Lemma 4.7.6. □

Proof of Lemma 4.7.7. Instead of working directly with w_{core}, we will prove an upper bound for w_{core} using f_{core} and then bound the summation using this upper bound.

By Stirling's approximation (Lemma 2.5.1) and the definitions of w_{core} and f_{core}, for $n_1 \in J_m$,

$$w_{\mathrm{core}}(n_1) \le n^\beta n! \exp\big(n f_{\mathrm{core}}(\hat{n}_1)\big), \tag{4.41}$$

for some constant β. First we bound the summation for the tail $n_1 \le n_1^* - \delta n$. By Lemma 4.7.5, (4.39) and (4.40)

$$\begin{aligned}
\sum_{\substack{n_1\le n_1^*-\delta n\\ n_1\in\mathbb{Z}}} w_{\mathrm{core}}(n_1) &\le n^{\beta+1} n! \exp\Big(n f_{\mathrm{core}}(\hat{n}_1^* - \delta)\Big)\\
&\le n! \exp\Big(n f_{\mathrm{core}}(\hat{n}_1^*)\Big) \exp\Big(n f''_{\mathrm{core}}(\hat{n}_1^*)\delta^2/2 + (\beta+1)\ln n + o(1)\Big)\\
&= O\left(\frac{n! \exp(n f_{\mathrm{core}}(\hat{n}_1^*))}{\exp(n\delta^2/(2r) - (\beta+1)\ln n + o(1))}\right)
\end{aligned}$$

and we are done since $n\delta^2/r = \omega(\ln n)$. The proof for $\hat{n}_1 \ge \hat{n}_1^* + \delta$ is similar. □

4.8 Counting pre-kernels

In this section we obtain an asymptotic formula for the number of pre-kernels with vertex set $[n]$ with $m = n/2 + R$ edges, when $R = \omega(n^{1/2}\log^{3/2} n)$ and $R = o(n)$. We remark that the asymptotics in this section are for $n \to \infty$. We will always use r to denote R/n.

For $x = (n_1, k_0, k_1, k_2) \in \mathbb{R}^4$, let

$$\begin{aligned}
n_2(x) &= k_0 + k_1 + k_2, \\
n_3(x) &= n - n_1 - n_2(x) = n - n_1 - k_0 - k_1 - k_2, \\
m_2(x) &= n_1, \\
m_2^-(x) &= n_1 - k_0, \\
P_2(x) &= 2m_2^-(x) = 2n_1 - 2k_0, \\
m_3(x) &= m - n_1, \\
P_3(x) &= 3m_3(x) = 3m - 3n_1, \\
Q_3(x) &= 3m - n_1 - 2n_2(x) = 3m - n_1 - 2k_0 - 2k_1 - 2k_2, \\
T_3(x) &= P_3(x) - k_1 - 2k_2 = 3m - 3n_1 - k_1 - 2k_2, \\
T_2(x) &= P_2(x) - k_1 = 2n_1 - 2k_0 - k_1,
\end{aligned} \tag{4.42}$$

For any symbol y in this section (and following subsections), we use $\hat{y}$ to denote y/n.

We will have n_1 as the number of vertices of degree 1, k_0 as the number of vertices of degree 2 such that the two edges incident to it are 2-edges, k_2 as the number of vertices of degree 2 such that the two edges incident to it are 3-edges and k_1 as the remaining vertices of degree 2. Then it is clear that n_2 is the number of vertices of degree 2, n_3 is the number of vertices of degree at least 3, Q_3 is the sum of degrees of vertices of degree at least 3, m_3 is the number of 3-edges, m_2 is the number of 2-edges, and m_2^- is the number of 2-edges that contain exactly two vertices of degree 2. We omit the argument x when it is obvious from the context.

For $x \in \mathbb{R}^4$, let

$$c_3(x) = \frac{Q_3(x)}{n_3(x)} = \frac{3m - n_1 - 2n_2(x)}{n - n_1 - n_2(x)},$$

that is, c_3 is the average degree of the vertices of degree at least 3. Note that $c_3(x) = \hat{Q}_3(x)/\hat{n}_3(x) = \hat{c}_3(x)$. For $x \in \mathbb{R}^4$ such that $\hat{Q}_3(x) > 3\hat{n}_3(x) > 0$, let $\lambda = \lambda(x)$ be the unique positive solution of

$$\frac{\lambda f_2(\lambda)}{f_3(\lambda)} = c_3(x). \tag{4.43}$$

Such $\lambda(x)$ always exists and is unique by Lemma 2.10.3. By continuity reasons, we define $\lambda(x) = 0$ when $c_3(x) = 3$.

Let S_m be the region of $\mathbb{R}^4$ such that $x = (n_1, k_0, k_1, k_2) \in S_m$ if all of the following conditions hold:

- $n_1, k_0, k_1, k_2 \in [0, n]$;
- $Q_3(x) \geq 3n_3(x) \geq 0$, and $Q_3(x) = 0$ whenever $n_3(x) = 0$;

- $m_3(x), m_2(x), m_2^-(x), T_3(x), T_2(x) \geq 0$.

We will work with pre-kernels with n_1 vertices of degree 1 and k_i vertices of degree 2 incident to exactly i 3-edges, for $i = 0, 1, 2$. We say that such pre-kernels have parameters (n_1, k_0, k_1, k_2). The region S_m is defined so that all tuples (n_1, k_0, k_1, k_2) for which it there exists a pre-kernel with such parameters are included. Let $\hat{S}_m = \{x/n : x \in S_m\}$ denote the scaled version of S_m. The set S_m is not closed because $Q_3(x) = 0$ whenever $n_3(x) = 0$. This constraint is added because $Q_3(x)$ should be the sum of the degrees of vertices of degree at least 3 and $n_3(x)$ should be the number of vertices of degree at least 3.

For $\hat{x} = (\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2)$ in the interior of $\hat{S}_m$, define

$$\begin{aligned} f_{\text{pre}}(\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2) = {} & h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(m_2) \\ & - h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{n}_3) - h_n(\hat{m}_3) \\ & - h_n(\hat{P}_3 - \hat{k}_1 - 2\hat{k}_2) - h_n(\hat{P}_2 - \hat{k}_1) - 2h_n(\hat{m}_2^-) \\ & - \hat{k}_2 \ln 2 - \hat{m}_2^- \ln 2 - \hat{m}_3 \ln 6 \\ & + \hat{n}_3 \ln f_3(\lambda) - \hat{Q}_3 \ln \lambda, \end{aligned} \tag{4.44}$$

where $\lambda = \lambda(x)$. As we will see later, for $x = (n_1, k_0, k_1, k_2) \in S_m \cap Z^4$, we have that $n! \exp(n f_{\text{pre}}(\hat{x}))$ approximates the exponential part of the number of pre-kernels with parameters (n_1, k_0, k_1, k_2).

We extend the definition of f_{pre} for points $\hat{x} \in \hat{S}_m$ that are in the boundary of $\hat{S}_m$ as the limit of $f_{\text{pre}}(x^{(i)})$ on any sequence of points $(x^{(i)})_{i \in \mathbb{N}}$ in the interior of $\hat{S}_m$ with $x^{(i)} \to \hat{x}$. One of the reasons the points x with $Q_3(x) > c_3(x) = 0$ are not allowed is that $f_{\text{pre}}(x_i)$ does not necessarily converge on a sequence of points $(x^{(i)})_{i \in \mathbb{N}}$ converging to x. For the points in the boundary where $Q_3(x) > c_3(x)$, this only means that $0 \log 0$ should be interpreted as 1. For $\hat{x} \in \hat{S}_m$ such that $\hat{Q}_3(\hat{x}) = 3\hat{n}_3(\hat{x})$, we have that $\lambda(x) = 0$. This means that $\hat{n}_3(x) \ln f_3(\lambda(x)) - \hat{Q}_3(x) \ln \lambda(x)$ is not defined (and note that $\hat{n}_3(x) \ln f_3(\lambda(x))$ and $-\hat{Q}_3(x) \ln \lambda(x)$ are the last two terms in the definition of $f_{\text{pre}}(\hat{x})$). We compute $\lim_{\lambda \to 0}(\hat{n}_3(x) \ln f_3(\lambda) - \hat{Q}_3(x) \ln \lambda)$. We have that $\lim_{\lambda \to 0} (\ln f_3(\lambda) - 3 \ln \lambda) = -\ln 6$. Thus, $\lim_{\lambda \to 0}(\hat{n}_3(x) \ln f_3(\lambda) - \hat{Q}_3(x) \ln \lambda) = -\hat{n}_3(x) \ln 6$ and

$$\begin{aligned} f_{\text{pre}}(\hat{x}) = {} & h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(m_2) \\ & - h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{n}_3) - h_n(\hat{m}_3) \\ & - h_n(\hat{P}_3 - \hat{k}_1 - 2\hat{k}_2) - h_n(\hat{P}_2 - \hat{k}_1) - 2h_n(\hat{m}_2^-) \\ & - \hat{k}_2 \ln 2 - \hat{m}_2^- \ln 2 - \hat{m}_3 \ln 6 \\ & - \hat{n}_3 \ln 6. \end{aligned} \tag{4.45}$$

We obtain the following asymptotic formula for the number of pre-kernels with n vertices and $m = m(n)$ edges.

Theorem 4.8.1. Let $m = m(n) = n/2 + R$ such that $R = o(n)$ and $R = \omega(n^{1/2} \log^{3/2} n)$. Then

$$g_{\text{pre}}(n, m) \sim \frac{\sqrt{3}}{\pi n} n! \exp(n f_{\text{pre}}(\hat{x}^*)),$$

where $\hat{x}^*$ is defined as $(\hat{n}_1^*, \hat{k}_0^*, \hat{k}_1^*, \hat{k}_2^*)$ with

$$\begin{aligned} \hat{n}_1^* &= \frac{3\hat{m}}{g_2(\lambda^*)}, \\ \hat{k}_0^* &= \frac{3\hat{m}}{g_2(\lambda^*)} \frac{2\lambda^*}{f_1(\lambda^*) g_1(\lambda^*)}, \\ \hat{k}_1^* &= \frac{3\hat{m}}{g_2(\lambda^*)} \frac{2\lambda^*}{g_1(\lambda^*)}, \\ \hat{k}_2^* &= \frac{3\hat{m}}{g_2(\lambda^*)} \frac{\lambda^* f_1(\lambda^*)}{2 g_1(\lambda^*)}, \end{aligned} \tag{4.46}$$

and $\lambda^* = \lambda^*(n)$ is the unique nonnegative solution for the equation

$$\frac{\lambda f_1(\lambda) g_2(\lambda)}{f_2(2\lambda)} = 3\hat{m}. \tag{4.47}$$

We discussed the existence and uniqueness of λ^* in Section 4.7. Also, note that (4.47), implies

$$r = \frac{1}{3} \frac{\lambda^* f_1(\lambda^*) g_2(\lambda^*)}{f_2(2\lambda^*)} - \frac{1}{2}. \tag{4.48}$$

We will show that the point $\hat{x}^*$ maximizes f_{pre} in a region that contains all points $(\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2)$ for which there exists a pre-kernel with parameters (n_1, k_0, k_1, k_2). The result is then obtained basically by expanding the summation around $\hat{x}^*$ in a region such that each term in (4.42) are nonnegative and $c_3 \geq 3$. This approach is similar to the one in Section 4.7 in which we analyse cores, but it will require much more work since we are now dealing with a 4-dimensional space. We remark that $\lambda^* = \lambda(x^*)$, that is, $\lambda^* f_2(\lambda^*)/f_3(\lambda^*) = c_3(x^*)$.

Similarly to Section 4.7 that deals with cores, it will be useful to know approximations for some parameters at the point $\hat{x}^* = (\hat{n}_1^*, \hat{k}_0^*, \hat{k}_1^*, \hat{k}_2^*)$ that achieves the maximum. For $r = o(1)$, we proved in Lemma 4.7.5 that $\lambda^* = o(1)$. From (4.47), we can write r in terms of λ^* and so we can write $\hat{n}_1^*$, $\hat{k}_0^*$, $\hat{k}_1^*$ and $\hat{k}_2^*$ in terms of λ^*. Thus, using (4.46), and computing the series of each function in (4.42) as $\lambda^* \to 0$, we have

$$
\begin{aligned}
r &= \tfrac{1}{12}\lambda^* + \tfrac{1}{36}(\lambda^*)^2 + O((\lambda^*)^3) & \hat{Q}_3^* &= \tfrac{1}{2}\lambda^* + \tfrac{1}{12}(\lambda^*)^2 + O((\lambda^*)^3)\\
\hat{n}_1^* &= \tfrac{1}{2} - \tfrac{1}{12}\lambda^* - \tfrac{1}{36}(\lambda^*)^2 + O((\lambda^*)^3) & \hat{m}_3^* &= \tfrac{1}{6}\lambda^* + \tfrac{1}{18}(\lambda^*)^2 + O((\lambda^*)^3)\\
\hat{k}_0^* &= \tfrac{1}{2} - \tfrac{7}{12}\lambda^* + \tfrac{2}{9}(\lambda^*)^2 + O((\lambda^*)^3) & \hat{m}_2^{-*} &= \tfrac{1}{2}\lambda^* - \tfrac{1}{4}(\lambda^*)^2 + O((\lambda^*)^3)\\
\hat{k}_1^* &= \tfrac{1}{2}\lambda^* - \tfrac{1}{3}(\lambda^*)^2 + O((\lambda^*)^3) & \hat{T}_2^* &= \tfrac{1}{2}\lambda^* - \tfrac{1}{6}(\lambda^*)^2 + O((\lambda^*)^3)\\
\hat{k}_2^* &= \tfrac{1}{8}(\lambda^*)^2 + O((\lambda^*)^3) & \hat{T}_3^* &= \tfrac{1}{4}(\lambda^*)^2 + O((\lambda^*)^3)\\
\hat{n}_3^* &= \tfrac{1}{6}\lambda^* + \tfrac{1}{72}(\lambda^*)^2 + O((\lambda^*)^3).
\end{aligned}
\tag{4.49}
$$

In the following subsections, we will use $\mathbf{Y} = (Y_1, \ldots, Y_{n_3})$ to denote a vector of independent random variables $Y_1, \ldots, Y_{n_3}$ such that each Y_i has truncated Poisson distribution with parameters $(3, \lambda(x))$ and $\Sigma(x)$ to denote the event $\sum_i Y_i = Q_3$.

4.8.1 Kernels

In this section, we define the notion of kernels of pre-kernels, which will be useful to study properties of pre-kernels and to generate random pre-kernels.

Recall that the pre-kernel is a core with no isolated cycles. Let the *kernel* of a pre-kernel G be the multihypergraph obtained as follows. Start by obtaining G' from G by deleting all vertices of degree 1 and replacing each edge containing a vertex of degree 1 by a new edge of size 2 incident to the other two vertices (and note that the multihypergraph is not necessarily uniform anymore). While there is a vertex v of degree 2 in G' such that the two edges incident to v have size 2, update G' by deleting both edges, and adding a new edge of size 2 containing the vertices other than v that were in the deleted edges. When this procedure is finished, delete all vertices of degree less than 2. The final multihypergraph is the kernel of G. This procedure obviously produces a unique multihypergraph (disregarding edge labels). See Figure 4.5 for an example of the procedure above.

This procedure is similar to the one for obtaining kernels of 2-uniform hypergraphs described by Pittel and Wormald [56]: the kernel of a pre-kernel is obtained by repeatedly replacing edges uv and vw, where v is a vertex of degree 2, by a new edge uw until no vertex of degree 2 remains, and then deleting all isolated vertices. Note that in our procedure there may be vertices of degree 2 in the kernel while there is no vertex of degree 2 in the kernel of a 2-uniform hypergraph.

In the kernel all edges have size 2 or 3. We call these edges 2-edges and 3-edges in the kernel, resp. It is trivial from the description above that in the kernel every degree 2 vertex is contained in at least one 3-edge. We say that any multihypergraph in which all edges have size 2 or 3, there are no vertices of degree 1, and every vertex of degree 2 is in at least one edge of size 3, is a kernel. The reason for this is that given such a multihypergraph, one can create a pre-kernel following the

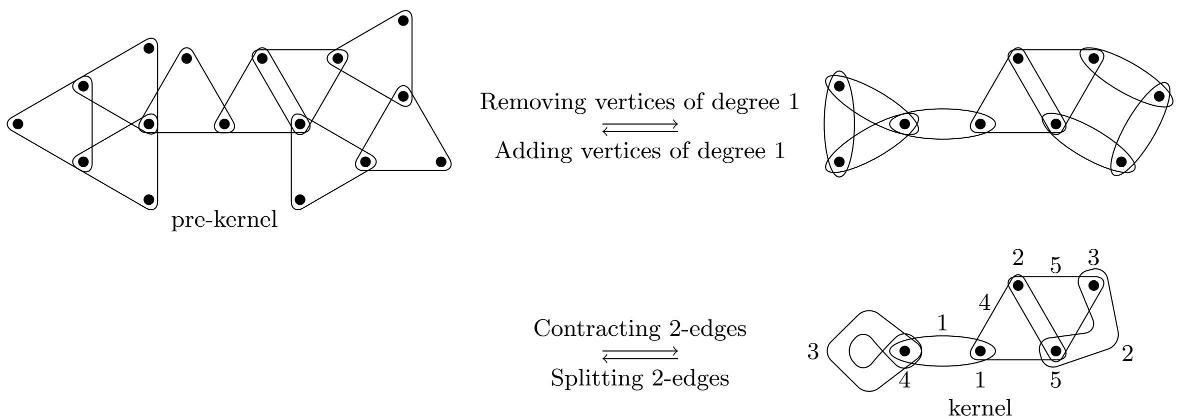

Figure 4.5: Obtaining a kernel from a pre-kernel, and vice-versa.

procedure we discuss next. Consider the following operation: we *split* one 2-edge with vertices u and v by deleting the edge and adding a new vertex w and two new 2-edges, one containing u and w and the other containing v and w. Given the kernel of a pre-kernel, one can split edges from the kernel in a way that it reverses the steps of the procedure for finding the kernel. After including the vertices of degree 1 in the 2-edges, the resulting graph is the pre-kernel. Note that, by replacing 2-edges in the kernel by splitting 2-edges and adding new vertices (of degree 1) to all 2-edges, the resulting multigraph does not have any isolated cycle. Thus, whenever the resulting multigraph is simple, it is a pre-kernel.

4.8.2 Random kernels and pre-kernels

Recall that our aim in Section 4.8 is to find an asymptotic formula for $g_{\text{pre}}(n,m)$, the number of connected pre-kernels with vertex set $[n]$ and m edges. Similarly to Section 4.7.1 about random cores, we show how to reduce the enumeration problem for pre-kernels to approximating the expectation, in a probability space of random degree sequences, of the probability that a random graph with given degree sequence is connected and simple.

We will describe a procedure to generate pre-kernels. For $x = (n_1, k_0, k_1, k_2) \in S_m \cap \mathbb{Z}^4$, let $\mathcal{D}(x) \subseteq \mathbb{N}^{n_3(x)}$ be such that $\mathbf{d} \in \mathcal{D}(x)$ if $d_i \geq 3$ for all i and $\sum_{i=1}^{n_3(x)} d_i = Q_3(x)$. Our strategy to generate a random pre-kernel is the following. We start by choosing the vertices and 3-edges that will be in the kernel. We then generate a random kernel with degree sequence $\mathbf{d}$ for the vertices of degree at least 3, k_1+k_2 vertices of degree 2, m_2^- 2-edges and m_3 3-edges so that k_i vertices of degree 2 are contained in exactly i 3-edges for $i = 1, 2$. The pre-kernel is then obtained by splitting 2-edges k_0 times and assigning the vertices of degree 1.

The kernel is generated in a way similar to the random cores in Section 4.7 but with different restrictions. For each vertex, we create a bin/set with the number of points inside it equal to the degree of the vertex. These bins are called *vertex-bins*. For each edge, we create one bin/set with 2 or 3 points inside it, depending on whether it is a 2-edge or a 3-edge. These bins are called *edge-bins*. Each point in a vertex-bin will be matched to a point in an edge-bin with some constraints. The kernel can then be obtained by creating one edge for each edge-bin i such that the vertices incident to it are the vertices with points matched to point in the edge-bin i. We describe how to generate a random kernel $\mathcal{K}(V, M_3, k_1, k_2, \mathbf{d})$ where $V \subseteq [n]$ is a set of size $k_1 + k_2 + n_3$ and $M_3 \subseteq [m]$ is a set of size m_3. In each step, every choice is made u.a.r. among all possible choices satisfying the stated conditions:

1. *(Vertex-bins)* Choose a set V_3 of n_3 vertices in V to be the vertices of degree at least 3. Let $v_1 < \cdots < v_{n_3}$ be an enumeration of V_3. For each $i \in [n_3]$, create a vertex-bin v_i with points labelled $1, \ldots, d_i$ inside it. For each $v \in V \setminus V_3$, create a vertex-bin v with points labelled 1 and 2 inside it.
2. *(Edge-bins)* For each $i \in M_3$, create one edge-bin with points labelled 1, 2, and 3 inside it. Let $M_2 = \{(i, 0) : i \in [m_2^-]\}$. For each $i \in M_2$, create one edge-bin with points labelled 1 and 2 inside it.
3. *(Matching)* Match the points from the vertex-bins to the points in edge-bins so that, for $i = 1, 2$, k_i vertex-bins with two points have exactly i points being matched to an edge-bin of size 3. This matching is called a *kernel-configuration* with parameters $(V, M_3, k_1, k_2, \mathbf{d})$.
4. *(Kernel)* The kernel $\mathcal{K}(V, M_2, k_1, k_2) = (V, M_2 \cup M_3, \Phi)$ is the multihypergraph such that for each $E \in M_2 \cup M_3$, we have that $\Phi(E, i) = v$, where v is the vertex corresponding to the vertex-bin containing j and j is the point matched to point i in the edge-bin E in the previous step.

See Figure 4.6 for an example of this procedure. The constraints in Step 3 ensures that each vertex of degree 2 is contained by at least one 3-edge and so the procedure above always generates a kernel. It is also trivial that all kernels (with edges $M_3 \cup M_2$) can be generated by this procedure.

We now describe the pre-kernel model precisely. For $x = (n_1, k_0, k_1, k_2) \in S_m \cap \mathbb{Z}^4$ and $\mathbf{d} \in \mathcal{D}(x)$, let $\mathcal{P}(x, \mathbf{d}) = \mathcal{P}_{n,m}(x, \mathbf{d})$ be the random graph generated as follows. In each step, every choice is made u.a.r. among all possible choices satisfying the stated conditions:

1. *(Kernel)* Let V be a subset of $[n]$ of size $n - n_1 - k_0$ and M_3 be a subset of $[m]$ of size $m_3(x)$. Let $\mathcal{K} = (V, M_{\mathcal{K}}, \Phi_{\mathcal{K}})$ be the random kernel $\mathcal{K}(V, M_3, k_1, k_2, \mathbf{d})$.
2. *(Splitting edges)*Let V_{k_0} be a subset of $[n]\setminus$ of size k_0. This set will be the set of vertices of degree at 2 contained by two 2-edges. Let $v_1, \ldots, v_{k_0}$ be an enumeration of the vertices

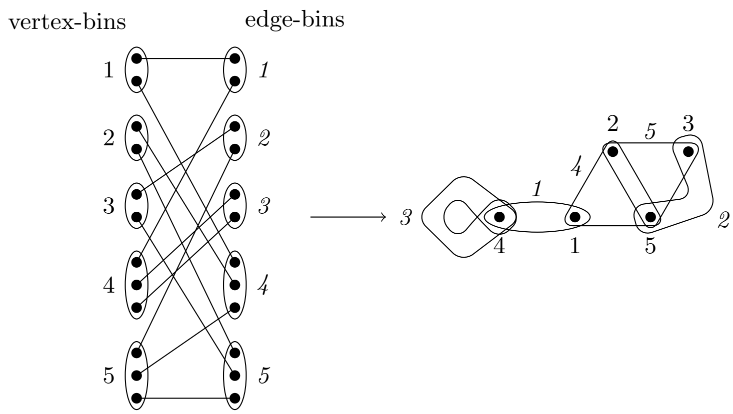

Figure 4.6: A kernel generated with vertex and edge-bins

in V_{k_0}. Let $P = \mathcal{K}$. For $i = 1$ to k_0, do the following operation: split a 2-edge of P with new vertex v_i and update P.

3. *(Assigning* 2*-edges and vertices of degree* 1*)* Let V_1 be a subset of size n_1 in $[n] \setminus V$. These will be the vertices of degree 1 in the multigraph. Assign for each 2-edge E of P a (unique) edge E' from $[m] \setminus M_3$ and a (unique) vertex u in V_1. Place a perfect matching $M_{E'}$ between the collection $\{\Phi_{\mathcal{K}}(E,1), \Phi_{\mathcal{K}}(E,2), u\}$ and $\{1,2,3\}$. We call this matching together with the sequence of splittings in the previous step a *splitting-configuration*.

4. *(Pre-kernel)* Let $\mathcal{P}(n_1, k_0, k_1, k_2, \mathbf{d}) = ([n], [m], \Phi)$, where $\Phi(E,\cdot) = \Phi_{\mathcal{K}}(E,\cdot)$ if $E \in M_3$ and, otherwise, $\Phi(E,i) = v$, where v is the vertex matched to i in M_E.

When the procedure above results in a (simple) graph, it is a pre-kernel since it is obtained by splitting the 2-edges of a kernel and assigning vertices of degree 1 to the 2-edges. It is trivial all pre-kernels are generated since all kernels and the ways of splitting the edges are considered.

For $(n_1, k_0, k_1, k_2) \in S_m$, let $g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2)$ denote the number of connected (simple) pre-kernels with vertex set $[n]$ and m edges such that n_1 vertices have degree 1, and $k_0 + k_1 + k_2$ vertices have degree 2 so that k_i of the degree 2 vertices are incident to exactly i 3-edges for $i = 0, 1, 2$. For $\mathbf{d} \in \mathcal{D}(n_1, k_0, k_1, k_2)$, let $g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2, \mathbf{d})$ denote the number of such pre-kernels with the additional constraint that $\mathbf{d}$ is the degree sequence of the vertices of degree at least 3.

In order to analyse $g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2, \mathbf{d})$ it will be useful to know the number of kernel-configurations.

Lemma 4.8.2. Let $x = (n_1, k_0, k_1, k_2) \in S_m \cap \mathbb{Z}^4$ and $\mathbf{d} \in \mathcal{D}(x)$. The number of kernel-configurations with parameters $(V, M_3, k_1, k_2, \mathbf{d})$, where V is a set of size $k_1 + k_2 + n_3$ and M_3 is a set of size m_3, is

$$\binom{k_1+k_2+n_3}{n_3}\binom{k_1+k_2}{k_1}2^{k_1}\binom{P_3}{k_1+2k_2}(k_1+2k_2)!\binom{P_2}{k_1}k_1!Q_3! = \frac{(k_1+k_2+n_3)!P_3!P_2!Q_3!2^{k_1}}{n_3!k_1!k_2!T_3!T_2!}.$$

Moreover, each kernel with parameters $(V, M_3, k_1, k_2, \mathbf{d})$ is generated by exactly $2^{k_1+k_2}\prod_{i=1}^{n_3} d_i!$ kernel-configurations.

Proof. There are $\binom{k_1+k_2+n_3}{n_3}$ ways of choosing the vertices of degree at least 3 in the first step. The step where the kernel-configuration is created can be described in the following more detailed way:

1. Choose k_1 vertex-bins of size 2. Let U be a set containing exactly one point of each of these vertex-bins and let D be the set consisting of all points in vertex-bins of size 2 that are not in U.
2. Choose $k_1 + 2k_2$ points inside edges-bins of size 3 and match them to points in D.
3. Choose k_1 points inside edges-bins of size 2 and match them to points in U.
4. Match the remaining unmatched Q_3 points from the vertex-bins to the unmatched points in the edge-bins.

In Step 1, there are $\binom{k_1+k_2}{k_1}$ choices for the vertex-bins of size 2 and 2^{k_1} choices for U. There are $\binom{P_3}{k_1+2k_2}(k_1+2k_2)!$ choices for Step 2, $\binom{P_2}{k_1}k_1!$ for Step 3 and $Q_3!$ choices for Step 4. The first part of the lemma then follows trivially.

Each kernel with parameters $(V, M_3, k_1, k_2, \mathbf{d})$ is generated by $2^{k_1+k_2}\prod_{i=1}^{n_3} d_i!$ distinct kernel-configurations, because any permutation of the points inside vertex-bins can be done without changing the resulting kernel. □

The following proposition relates $g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2, \mathbf{d})$ and $g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2)$ to the random pre-kernels $\mathcal{P}(x, \mathbf{d})$ and random degree sequences. The proof is similar to the proof of Proposition 4.7.2. We include it here for completeness.

Proposition 4.8.3. For $x = (n_1, k_0, k_1, k_2) \in S_m \cap \mathbb{Z}^4$ and $\mathbf{d} \in \mathcal{D}(x)$,

$$\begin{aligned} &g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2, \mathbf{d}) \\ &= n!\frac{P_3!P_2!Q_3!(m_2-1)!\,\mathbb{P}(\mathcal{P}(n_1, k_0, k_1, k_2, \mathbf{d}) \text{ simple and connected})}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2^- - 1)!m_2^-!2^{k_2}2^{m_2^-}6^{m_3}\prod_i d_i!} \end{aligned}$$

and, if $Q_3(x) > n_3(x)$, then

$$\begin{aligned} &g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2) \\ &= n! \frac{P_3! P_2! Q_3! (m_2 - 1)!}{k_0! k_1! k_2! n_3! m_3! (P_3 - k_1 - 2k_2)! (P_2 - k_1)! (m_2^- - 1)! m_2^-! 2^{k_2} 2^{m_2^-} 6^{m_3}} \\ &\quad \cdot \frac{f_3(\lambda)^{n_3}}{\lambda^{Q_3}} \mathbb{E}\left(\mathbb{P}\left(\mathcal{P}(n_1, k_0, k_1, k_2, \mathbf{Y}) \text{ simple and connected}\right) \Big| \Sigma(x) \right) \mathbb{P}\left(\Sigma(x)\right), \end{aligned} \tag{4.50}$$

where $\mathbf{Y} = (Y_1, \ldots, Y_{n_3})$ is a vector of independent random variables $Y_1, \ldots, Y_{n_3}$ such that each Y_i has truncated Poisson distribution with parameters $(3, \lambda(x))$ and $\Sigma(x)$ denotes the event $\sum_i Y_i = Q_3$.

Proof. Any multigraph obtained by the process for $\mathcal{P}(x, \mathbf{d})$ is generated by $2^{k_1+k_2}(\prod_{i=1}^{n_3} d_i!) m_2^-! 2^{m_2^-}$ combinations of kernel-configurations and splitting-configuration. This is because each kernel is generated by $2^{k_1+k_2} \prod_{i=1}^{n_3} d_i!$ kernel-configurations by Lemma 4.8.2 and permuting the labels and points inside each of the 2-edges in the kernel do not change the resulting multigraph. Thus, by Lemma 4.3.3, each pre-kernel with parameters $(x, \mathbf{d})$ is generated by

$$\alpha := 2^{k_1+k_2} \Big(\prod_{i=1}^{n_3} d_i! \Big) m_2^-! 2^{m_2^-} m! 6^m \tag{4.51}$$

combinations of kernel-configurations and splitting-configurations. Next we compute the total number of such combinations. In Step 1 in which we generate the kernel, there are $\binom{n}{k_1+k_2+n_3}$ ways of choosing V and $\binom{m}{m_3}$ ways of choosing M_3. The number of ways of generating the kernel is

$$\frac{(k_1 + k_2 + n_3)! P_3! P_2! Q_3! 2^{k_1}}{n_3! k_1! k_2! T_3! T_2!}$$

by Lemma 4.8.2. In Step 2, there are $\binom{n_1+k_0}{k_0}$ ways of choosing V_{k_0} and $m_2^-(m_2^- + 1) \cdots (m_2^- + k_0 - 1) = (m_2 - 1)!/(m_2^- - 1)!$ ways of splitting the edges. In Step 3, that are $(m_2!)^2$ ways of assigning the 2-edges and vertices of degree 1 and 6^{m_2} ways of placing the matchings. Thus, the total number of combinations of kernel-configurations and splitting-configurations is

$$\binom{n}{k_1 + k_2 + n_3} \binom{m}{m_3} \frac{(k_1 + k_2 + n_3)! P_3! P_2! Q_3! 2^{k_1}}{n_3! k_1! k_2! T_3! T_2!} \binom{n_1 + k_0}{k_0} \frac{(m_2 - 1)!}{(m_2^- - 1)!} (m_2!)^2 6^{m_2} =: \beta.$$

Hence, since each combination is generated with the same probability, we have that

$$g_{\text{pre}}(n_1, k_0, k_1, k_2, \mathbf{d}) = \frac{\beta}{\alpha} \mathbb{P}(\mathcal{P}(n_1, k_0, k_1, k_2, \mathbf{d}) \text{ simple and connected}), \tag{4.52}$$

where β is the total number of configurations. which together with (4.52) and trivial simplifications implies (4.8.3).

We now prove (4.50). Again, the proof is very similar to the proofs of Proposition 3.2.1 and Proposition 3.2.2 in Chapter 3. For $x = (n_1, k_0, k_1, k_2)$, let $U(x, \mathbf{d})$ denote the probability that $\mathcal{P}_{n,m}(n_1, k_0, k_1, k_2, \mathbf{d})$ is simple and connected. For $x = (n_1, k_0, k_1, k_2)$,

$$
\begin{aligned}
g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2) &:= \sum_{\mathbf{d}\in\mathcal{D}(x)} g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2, \mathbf{d}) \\
&= n! \sum_{\mathbf{d}\in\mathcal{D}(x)} \frac{P_3!P_2!Q_3!(m_2-1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2^- - 1)!m_2^-!2^{k_2}2^{m_2^-}6^{m_3}\prod_i d_i!} U(n_1, k_0, k_1, k_2, \mathbf{d}) \\
&= n! \frac{P_3!P_2!Q_3!(m_2-1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2^- - 1)!m_2^-!2^{k_2}2^{m_2^-}6^{m_3}} \frac{f_3(\lambda(x))^{n_3}}{\lambda(x)^{Q_3}} \sum_{\mathbf{d}\in\mathcal{D}(x)} \prod_i \frac{\lambda(x)^{d_i}}{d_i! f_3(\lambda(x))} U(x, \mathbf{d}) \\
&= n! \frac{P_3!P_2!Q_3!(m_2-1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2^- - 1)!m_2^-!2^{k_2}2^{m_2^-}6^{m_3}} \frac{f_3(\lambda(x))^{n_3}}{\lambda(x)^{Q_3}} \sum_{\mathbf{d}\in\mathcal{D}(x)} U(x, \mathbf{d})\, \mathbb{P}(\mathbf{Y} = \mathbf{d}) \\
&= n! \frac{P_3!P_2!Q_3!(m_2-1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2^- - 1)!m_2^-!2^{k_2}2^{m_2^-}6^{m_3}} \frac{f_3(\lambda(x))^{n_3}}{\lambda(x)^{Q_3}} \mathbb{E}\left(U(x, \mathbf{Y}) | \Sigma(x)\right) \mathbb{P}(\Sigma(x))
\end{aligned}
$$

which proves (4.50). □

The goal of the next lemmas is to show that the expectation in (4.50) goes to 1 for points x close to x^*. For $x \in S_m \cap \mathbb{Z}^4$ and $\phi = \phi(n) > 0$, let

$$\tilde{\mathcal{D}}_\phi(x) = \left\{\mathbf{d} \in \mathcal{D}(x) : |\eta(\mathbf{d}) - \mathbb{E}\left(\eta(\mathbf{Y})\right)| \le R\phi\right\}$$

where $\eta(\mathbf{d}) := \sum_{i=1}^{n_3} d_i(d_i - 1)/(2m)$, and recall that $R = m - n/2$. We will show that, for some function $\phi = o(1)$, conditioned upon $\Sigma(x)$, the probability that $\mathbf{Y}$ is in $\tilde{\mathcal{D}}_\phi(x)$ goes to 1. Intuitively, this means that the set $\tilde{\mathcal{D}}_\phi(x)$ contains all 'typical' degree sequences for points $x \in S$ that are close to x^*. For $\psi = \psi(n) = o(1)$, let

$$
\begin{aligned}
S^*_\psi = \Bigg\{ x = (n_1, k_0, k_1, k_2) \in S : &\left|\hat{n}_1 - \frac{1}{2}\right| \le \psi r; \\
&\left|\hat{k}_0 - \frac{1}{2}\right| \le \psi r;\ \left|\hat{k}_1 - 6r\right| \le \psi r; \\
&\left|\hat{k}_2 - 18r^2\right| \le \psi r^2;\ \left|\hat{n}_3 - 2r\right| \le \psi r; \\
&\left|\hat{Q}_3 - 6r\right| \le \psi r;\ \left|\hat{m}_3 - 2r\right| \le \psi r; \\
&\left|\hat{m}_2^- - 6r\right| \le \psi r;\ \left|\hat{T}_2 - 6r\right| \le \psi r; \\
&\left|\hat{T}_3 - 36r^2\right| \le \psi r^2 \Bigg\}.
\end{aligned}
\tag{4.53}
$$

We define S_ψ^* this way so that all points in it are close to x^*, where we are using (4.49) to find around which values each of the functions in the definition of S_ψ^* should be concentrated. The idea is to define ψ later in a way that it is small enough so that we can approximate the summation of $n!\exp(nf_{\text{pre}}(\hat{x}))$ in the integer points x in S_ψ^*, but large enough so that what is not included do not significant effect in the summation $\sum_{x\in S_m\cap\mathbb{Z}^4} n!\exp(nf_{\text{pre}}(\hat{x}))$.

Next we show that for points in $x \in S_\psi^*$ with $\psi = o(1)$ the set $\tilde{\mathcal{D}}_\phi(x)$ (for some $\phi = o(1)$) is a set of 'typical' degree sequences.

Lemma 4.8.4. Let $\psi = o(1)$. There exists $\phi = o(1)$ such that, for every integer point $x \in S_\psi^*$, we have that $\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}_{\phi(x)} \,|\, \Sigma(x)) = 1 - o(1)$.

We then show that for $x = x(n) \in S_\psi^* \cap \mathbb{Z}^4$ and $\mathbf{d} \in \tilde{\mathcal{D}}_{\phi(x)}$, the random pre-kernel $\mathcal{P}(x, \mathbf{d})$ is connected and simple a.a.s.

Lemma 4.8.5. Assume $R = o(n)$. Let $\psi, \phi = o(1)$. Let $x = x(n) \in S_\psi^*$ be an integer point and $\mathbf{d} = \mathbf{d}(n) \in \tilde{\mathcal{D}}_\phi(x)$. Then $\mathcal{P}(x, \mathbf{d})$ is simple a.a.s.

Lemma 4.8.6. Assume $R = o(n)$. Let $\psi, \phi = o(1)$. Let $x \in S_\psi^*$ be an integer point and $\mathbf{d} = \mathbf{d}(n) \in \tilde{\mathcal{D}}_\phi(x)$. Then $\mathcal{P}(x, \mathbf{d})$ is connected a.a.s.

The proofs for Lemmas 4.8.4, 4.8.5, and 4.8.6 are presented in Sections 4.8.3, 4.8.4, and 4.8.5, respectively. We now show how to prove that the expectation in (4.50) goes to 1 assuming Lemmas 4.8.4, 4.8.5, and 4.8.6.

Corollary 4.8.7. Let $\psi = o(1)$ and let $x = (n_1, k_0, k_1, k_2) \in S_\psi^* \cap \mathbb{Z}^4$. Then

$$\mathbb{E}\left(\mathbb{P}\left(\mathcal{P}(n_1, k_0, k_1, k_2, \mathbf{Y}) \text{ simple and connected}\right)\Big|\Sigma(x)\right) \sim 1.$$

Proof. Let $U(\mathbf{Y})$ denote the probability that $\mathcal{P}(x, \mathbf{Y})$ is connected and simple. Let $\phi = o(1)$ be given by Lemma 4.8.4. We have that

$$\mathbb{E}\left(U(\mathbf{Y})\Big|\Sigma(x)\right) \geq \sum_{\mathbf{d}\in\tilde{\mathcal{D}}_\phi(x)} \mathbb{P}(U(\mathbf{d}))\,\mathbb{P}(\mathbf{Y} = \mathbf{d} \,|\, \Sigma(x)).$$

By Lemmas 4.8.5 and 4.8.6, we have that $\mathbb{P}(U(\mathbf{d})) = 1 - o(1)$ for every $\mathbf{d} = \mathbf{d}(n) \in \tilde{\mathcal{D}}_\phi(x)$. Since $\tilde{\mathcal{D}}_\phi(x)$ is a finite set for each n, this implies that there exists a function $q(n) = o(1)$ such that $\mathbb{P}(U(\mathbf{d})) \geq 1 - q(n)$ for every $\mathbf{d} \in \tilde{\mathcal{D}}_\phi(x)$ by Lemma 2.7.1. Thus,

$$\mathbb{E}\left(u(\mathbf{Y})\Big|\Sigma(x)\right) \geq (1 - q(n))\,\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}_\phi(x)) = 1 - o(1).$$

by Lemma 4.8.4. □

4.8.3 Typical degree sequences

In this section, given an integer point $x \in S_m$ 'close' to the point x^* (more precisely $x \in S^*_\psi$ and $\psi = o(1)$), we show that, for a random vector of $\mathbf{Y} = (Y_1, \ldots, Y_{n_3(x)})$ of independent truncated Poisson random variables with parameters $(3, \lambda(x))$ conditioned upon the event $\Sigma(x)$ that $\sum_{i=1}^{n_3(x)} Y_i = Q_3(x)$, the value of $\sum_{i=1}^{n_3(x)} \binom{Y_i}{2}$ is concentrated around its expected value. More specifically, we present the proof for Lemma 4.8.4. Recall that

$$\tilde{\mathcal{D}}_\phi(x) = \left\{ \mathbf{d} \in \mathcal{D}(x) : |\eta(\mathbf{d}) - \mathbb{E}\left(\eta(\mathbf{Y})\right)| \leq R\phi \right\}$$

where $\eta(\mathbf{d}) = \sum_{i=1}^{n_3} d_i(d_i - 1)/(2m)$. We want to show that, given $x \in S^*_\psi \cap \mathbb{Z}^4$ with $\psi = o(1)$, there exists $\phi = o(1)$ such that $\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}_{\phi(x)} \,|\, \Sigma(x)) > 1 - \phi$, where $\mathbf{Y} = (Y_1, \ldots, Y_{n_3})$ is a vector of independent random variables with distribution $\mathrm{Po}(3, \lambda(x))$.

Recall that $n_3 \sim 2rn = 2R \to \infty$, and $Q_3/n_3 \sim 6r/(2r) = 3$ for $x \in S^*_\psi$. Thus, by the definition of $\lambda(x)$ (in (4.43)) and Lemma 2.10.3, we must have $\lambda(x) = o(1)$. Then by Lemma 2.10.7, $\mathrm{Var}(Y_i(Y_i - 1)) = \Theta(\lambda)$. Thus, by Chebyshev's inequality,

$$\mathbb{P}\left(|\eta(\mathbf{Y}) - \mathbb{E}\left(\eta(\mathbf{Y})\right)| \geq R\phi \right) \leq \frac{\mathrm{Var}(\eta(\mathbf{Y}))}{R^2\phi^2} = \frac{n_3\Theta(\lambda)}{R^2\phi^2} = o\left(\frac{n_3}{R^2\phi^2} \right).$$

If $R_3 := Q_3 - 3n_3 \leq \log n_3$, by Theorem 2.10.8 and Stirling's approximation (Lemma 2.5.1)

$$\mathbb{P}(\Sigma(x)) = (1 + o(1))e^{-R_3}\frac{R_3^{R_3}}{R_3!} = \Omega\left(\frac{1}{\sqrt{R_3}} \right) = \Omega\left(\frac{1}{\sqrt{\log n_3}} \right).$$

If $Q_3 - 3n_3 \geq \log n_3$, by Theorem 2.10.8,

$$\mathbb{P}(\Sigma(x)) \sim \frac{1}{\sqrt{2\pi n_3 c_3(1 + \eta_3 - c_3)}} = \Omega\left(\frac{1}{\sqrt{n_3}} \right),$$

where $c_3 = Q_3/n_3$ and $\eta_3 = \lambda(x) f_1(\lambda(x))/f_2(\lambda(x))$, and we used Lemma 2.10.7. Thus,

$$\mathbb{P}\left(|\eta(\mathbf{Y}) - \mathbb{E}\left(\eta(\mathbf{Y})\right)| \geq R\phi | \Sigma \right) = O\left(\frac{n_3}{R^2\phi^2}\sqrt{n_3} \right) = O\left(\frac{1}{R^{1/2}\phi^2} \right)$$

since $n_3 \sim 2R$ and so it is suffices to choose $\phi^2 = \omega(\sqrt{1/R})$. This finishes the proof of Lemma 4.8.4.

4.8.4 Simple pre-kernels

In this section, given an integer point $x \in S_m$ 'close' to the point x^* and $\mathbf{d} \in \mathbb{N}^{n_3}$ with some constraints (more precisely $x \in S^*_\psi$ and $\mathbf{d} \in \tilde{\mathcal{D}}_\phi(x)$ with $\psi, \phi = o(1)$), we show that the random multigraph $\mathcal{P}(x, \mathbf{d})$ defined in Section 4.8.2 is simple a.a.s., thus proving Lemma 4.8.5. Recall that a multigraph is simple if it has no loops and no double edges (as defined in Section 4.3). Any loop (or double edge) involving only 3-edges in the kernel remains a loop (or double edge) in the pre-kernel. Any double edge involving 2-edges in the kernel will not be a double edge in the pre-kernel, because each 2-edge will be assigned a unique vertex of degree 1 in the procedure that creates the pre-kernel from the kernel. A loop in the kernel that is an 2-edge will cease to be a loop in the pre-kernel if it is split at least once. Note that, if a 2-edge that is a loop in the kernel is split exactly once, the two 2-edges created will not form a double edge in the final multigraph since the assignment of vertices of degree 1 to the 2-edges eliminates all double edges involving 2-edges. It is clear that no other loops or double edges can be created. We rewrite these conditions for the kernel-configuration: the pre-kernel $\mathcal{P} = \mathcal{P}(x, \mathbf{d})$ is simple if and only if

(A) (*No loops in* 3-*edges*) No edge-bin of size 3 has at least 2 points matched to points from the same vertex-bin.

(B) (*No double* 3-*edges*) Assuming no loops in 3-edges, no pair of edges-bins of size 3 has their points matched to points in the same 3 vertices.

(C) (*No loops in* 2-*edges*) For every edge-bin of size 2, its points are matched to points from distinct vertex-bins or the 2-edge corresponding to this edge-bin is split at least once in the process that obtains the pre-kernel from the kernel.

We will show that, for $x \in S^*_\psi$ and $\mathbf{d} \in \tilde{\mathcal{D}}_\phi(x)$ with $\psi, \phi = o(1)$, the random multigraph $\mathcal{P}(x, \mathbf{d})$ is simple a.a.s., which proves Lemma 4.8.5. We need to show that each of the conditions (A), (B) and (C) holds a.a.s. We will use the detailed procedure for obtaining kernel-configurations described in the proof of Lemma 4.8.2. We work in the probability space conditioned upon the vertices of degree 3 and the points in U being already chosen, since the particular choices of these vertices and points do not affect the probability of loops or double edges in the kernel.

First we prove (A) holds a.a.s. Consider the case that the loop is on a vertex of degree 2. There are k_2 possible choices for the vertex-bin. There are m_3 choices for the edge-bin of size 3 and $3 \cdot 2$ choices for the points inside of the edge-bin to be matched to the points in the vertex-bin of size 2. Thus, we have $6k_2m_3$ choices. Following the proof of Lemma 4.8.2, after the vertices of degree 3 and U are chosen, there are

$$\binom{P_3}{k_1 + 2k_2}(k_1 + 2k_2)!\binom{P_2}{k_1}k_1!Q_3! \tag{4.54}$$

ways of completing the kernel-configuration. The number of completions of kernel-configurations containing a given matching that matches 2 points in a vertex-bin of size 2 to 2 points in an edge-bin of size 3 is then

$$\binom{P_3 - 2}{k_1 + 2k_2 - 2}(k_1 + 2k_2 - 2)!\binom{P_2}{k_1}k_1!Q_3!$$

Thus, using the definition of S^*_ϕ, the probability that there is a loop on a vertex of degree 2 in a 3-edge is at most

$$6k_2m_3\frac{1}{P_3(P_3-1)} = O\left(\frac{k_2m_3}{P_3^2}\right) = O\left(\frac{(r^2n)(rn)}{(rn)^2}\right) = O(r) = o(1).$$

Now consider the case that the loop is on a vertex of degree at least 3. There are $\sum_{i=1}^{n_3}\binom{d_i}{2} = \eta(\mathbf{d})$ possible choices for the vertex-bin and 2 points inside it. Since $\mathbf{d} \in \mathcal{D}(x)$ and $\mathbb{E}\left(\eta(\mathbf{Y})\right) = n_3\mathbb{E}\left(Y_1(Y_1-1)\right) \sim 6n_3 = \Theta(R)$,

$$\eta(\mathbf{d}) = \Theta(n_3).$$

There are m_3 choices for the edge-bin of size 3 and $3\cdot 2$ choices for the points inside of the edge-bin to be matched to the chosen points in the vertex-bin. Thus, we have $O(n_3m_3)$ choices. The number of completions of kernel-configurations containing one given matching that matches 2 points in a vertex-bin of size at least 3 and 2 points in a edge-bin of size 3 is

$$\binom{P_3-2}{k_1+2k_2}(k_1+2k_2)!\binom{P_2}{k_1}k_1!(Q_3-2)!$$

Thus, using (4.54), the probability that there is a loop on a vertex-bin of size at least 3 in edge-bin of size 3 is

$$O\left(n_3m_3\cdot\frac{(P_3-2)!}{P_3!}\frac{(Q_3-2)!}{Q_3!}\frac{(T_3)!}{(T_3-2)!}\right) = O\left(\frac{n_3m_3T_3^2}{P_3^2Q_3^2}\right) = O\left(\frac{(rn)(rn)(r^2n)^2}{(rn)^2(rn)^2}\right) = O(r^2) = o(1).$$

This finishes the proof that Condition (A) holds a.a.s. Now we prove that Condition (B) holds a.a.s. We consider 4 cases:

(B1) The edge-bins corresponding to the double edge have their points matched to points in 3 vertex-bins all of size 2.

(B2) The edge-bins corresponding to the double edge have their points matched to points in 2 vertex-bins of size 2 and 1 vertex-bin of size at least 3.

(B3) The edge-bins corresponding to the double edge have their points matched to points in 1 vertex-bin of size 2 and 2 vertex-bins of size at least 3.

(B4) The edge-bins corresponding to the double edge have none of their points matched to points in vertex-bins of size 2.

Let us start with (B1). We have $O(k_2^3m_3^2)$ choices for the 3 vertex-bins and 2 edge-bins involved. There are $O(1)$ matchings between the points 6 in these vertex-bins and the 6 points in these edge-bins that creates a double edge. The number of completions for the kernel-configurations containing a giving matching creating such a double edge is

$$\binom{P_3-6}{k_1+2k_2-6}(k_1+2k_2-6)!\binom{P_2}{k_1}k_1!Q_3!,$$

where we are following the proof of Lemma 4.8.2, after the vertices of degree 3 and U are chosen. Thus, using (4.54), the expected number of double edges as in (B1) is at most

$$O\left(k_2^3m_3^2\right)\frac{(P_3-6)!}{P_3!}=O\left(\frac{k_2^3m_3^2}{P_3^6}\right)=O\left(\frac{(r^2n)^3(rn)^2}{(rn)^6}\right)=O\left(\frac{r^2}{n}\right).$$

Now let us consider (B2). We have $O(k_2^2\eta(\mathbf{d})m_3^2)$ choices for the 2 vertex-bins of size 2 and the points inside them, the vertex-bin of size at least 3 and the points inside them, and the 2 edge-bins of size 3 involved in the double edge. We match 4 points from the 2 vertex-bins of size 2 to the 4 points in the edge-bins of size 3 and 2 points from the vertex-bin of size at least 3 to 2 points in the edges-bins of size 3. The number of completions for the kernel-configurations containing a giving matching creating such a double edge is

$$\binom{P_3-6}{k_1+2k_2-4}(k_1+2k_2-4)!\binom{P_2}{k_1}k_1!(Q_3-2)!$$

Thus, using (4.54) and the definition of S_ψ^*, the expected number of double edges as in (B2) is at most

$$\begin{aligned}O\left(k_2^2n_3m_3^2\right)\frac{(P_3-6)!}{P_3!}\frac{(Q_3-2)!}{Q_3!}\frac{T_3!}{(T_3-2)!}&=O\left(\frac{k_2^2n_3m_3^2T_3^2}{P_3^6Q_3^2}\right)\\&=O\left(\frac{(r^2n)^2(rn)(rn)^2(r^2n)^2}{(rn)^6(rn)^2}\right)=O\left(\frac{r^3}{n}\right).\end{aligned}$$

We analyse (B3) now. There are 2 vertex-bins of size at least 3 involved. We have $O(k_2\eta(\mathbf{d})^2m_3^2)$ choices for the vertex-bin of size 2, the 2 vertex-bins of size at least 3 and the points inside them, and the 2 edge-bins involved. There are $O(1)$ matchings between the 6 points in the vertex-bins (2 in the vertex-bin of size 2 and 4 in the other vertex-bins) and the 6 points in the edge-bins creating a double edge. The number of completions for the kernel-configurations containing a giving matching creating such a double edge is

$$\binom{P_3-6}{k_1+2k_2-2}(k_1+2k_2-2)!\binom{P_2}{k_1}k_1!(Q_3-4)!$$

Thus, using (4.54) and the definition of S_ψ^*, the expected number of double edges as in (B3) is at most

$$\begin{aligned}O\left(k_2n_3^2m_3^2\right)\frac{(P_3-6)!}{P_3!}\frac{(Q_3-4)!}{Q_3!}\frac{T_3!}{(T_3-4)!}&=O\left(\frac{k_2n_3^2m_3^2T_3^4}{P_3^6Q_3^4}\right)\\&=O\left(\frac{(r^2n)(rn)^2(rn)^2(r^2n)^4}{(rn)^6(rn)^4}\right)=O\left(\frac{r^4}{n}\right).\end{aligned}$$

We analyse (B4) now. We have $O(\eta(\mathbf{d})^3 m_3^2)$ choices for the 3 vertex-bins of size at least 3 and the points inside them and the 2 edge-bins involved. There are $O(1)$ matchings between the 6 points in the vertex-bins and the 6 points in the edge-bins creating a double edge. The number of completions for the kernel-configurations containing a giving matching creating such a double edge is

$$\binom{P_3-6}{k_1+2k_2}(k_1+2k_2)!\binom{P_2}{k_1}k_1!(Q_3-6)!$$

Thus, using (4.54) and the definition of S_ψ^*, the expected number of double edges as in (B4) is at most

$$\begin{aligned} O\left(n_3^3 m_3^2\right)\frac{(P_3-6)!}{P_3!}\frac{(Q_3-6)!}{Q_3!}\frac{T_3!}{(T_3-6)!} &= O\left(\frac{n_3^3 m_3^2 T_3^6}{P_3^6 Q_3^6}\right) \\ &= O\left(\frac{(rn)^3(rn)^2(r^2n)^6}{(rn)^6(rn)^6}\right) = O\left(\frac{r^5}{n}\right). \end{aligned}$$

This finishes the proof of that Condition (B) holds a.a.s.

Now consider the event in case (C). First we will bound the expected number of edge-bins of size 2 with points matched to points from the same vertex-bin (and so corresponding to loops in the kernel). Since every vertex-bin of size 2 has at least one point being matched to a point in an edge-bin of size 3, if an edge-bin of size 2 has points matched to the same vertex-bin, such vertex-bin must have size at least 3. Thus, we have $\eta(\mathbf{d}) = \Theta(n_3)$ choices for such vertex-bin and the two points inside it that will be matched to the points in the 2-edge, and m_2^- choices for the edge-bin of size 2 (and 2 choices for the matching of these points). The number of completions for the kernel-configurations containing a giving matching creating such a loop is

$$\binom{P_3}{k_1+2k_2}(k_1+2k_2)!\binom{P_2-2}{k_1}k_1!(Q_3-2)!$$

Thus, using (4.54) and the definition of S_ψ^*, the expected number of loops as in (C) is at most

$$\begin{aligned} \frac{(P_2-2)!}{P_2!}\frac{(Q_3-2)!}{Q_3!}\frac{T_2!}{(T_2-2)!}O\left(n_3 m_2^-\right) &= O\left(\frac{n_3 m_2^- T_2^2}{P_2^2 Q_3^2}\right) \\ &= O\left(\frac{(rn)(rn)(rn)^2}{(rn)^2(rn)^2}\right) = O(1).(C) \end{aligned}$$

So let $\alpha(n) \to \infty$ such that $\alpha r \to 0$. Then the number of edge-bins corresponding to 2-edges that are loops in the kernel is less than α a.a.s. For any 2-edge in the kernel, let A_i be the event that it is not split by the i-th splitting operation performed when creating the pre-kernel from the kernel.

Then

$$\mathbb{P}\left(\bigcap_{i=1}^{k_0} A_i\right) = \prod_{i=1}^{k_0} \mathbb{P}\left(A_i \,\middle|\, \bigcap_{j=1}^{j-1} A_j\right) = \frac{m_2^- - 1}{m_2^-}\frac{m_2^-}{m_2^- + 1} \cdot \dots \cdot \frac{m_2^- + k_0 - 2}{m_2^- + k_0 - 1}$$
$$= \frac{n_1 - k_0 - 1}{n_1 - 1} \sim \frac{6rn}{(1/2)n} \sim 12r.$$

This together with the fact the expected number of 2-edges that are loops in the kernel is less than α a.a.s. implies that the probability there is a 2-edge that is a loop in the pre-kernel is $O(\alpha r) + o(1) = o(1)$. This finishes the proof of Lemma 4.8.5.

4.8.5 Connected pre-kernels

In this section, we analyse the probability that the random multigraph $\mathcal{P}(x, \mathbf{d})$ is connected for x 'close' to x^* and $\mathbf{d} \in \mathbb{N}^{n_3}$ with some constraints (more precisely $x \in S_\psi^*$ and $\mathbf{d} \in \tilde{\mathcal{D}}_\phi(x)$ with $\psi, \phi = o(1)$). We will show that $\mathcal{P}(x, \mathbf{d})$ is connected a.a.s., proving Lemma 4.8.6. Our strategy has some similarities with the proof by Łuczak[46] for connected random 2-uniform hypergraphs with given degree sequence and minimum degree at least 3. The main difference is that, in our case, we have some vertices of degree 2 and the matching on the set of points in the bins has some constraints because of these vertices. This makes it more difficult to compute the probability of connectedness.

A pre-kernel is connected if and only of its kernel is connected, since the pre-kernel is obtained by splitting 2-edges of the kernel and assigning vertices of degree 1. Thus, we only need to analyse the connectivity of the kernel. Let $\mathbf{d}$ denote the degree sequence of the vertices of degree at least 3, k_i the number of vertices of degree 2 that are in exactly i 3-edges (for $i = 1, 2$), m_2^- the number of 2-edges and m_3 the number of 3-edges.

We say that a kernel-configuration is *connected* if the 2-uniform multigraph described as follows is connected: contract each vertex-bin and each edge-bin into a single vertex and add one edge uv for each edge ij of the matching in the kernel-configuration such that i is in the bin corresponding to u and j is in the bin corresponding to v. Given a kernel-configuration with matching M, perform the following operations:

1. For each vertex-bin v with more than 6 points, partition the points of v into new vertex-bins so that each of the new vertex-bins has 3, 4 or 5 points. Delete v and keep M unchanged. See Figure 4.7.

2. For each edge-bin e of size 2 such that exactly one of its points, say p_e, is matched to a point, say p_v, in a vertex-bin v of size 2, do the following. Let p'_e be the point in e other than p_e and let p'_v be the point in v other than p_v. Let i be the point matched to p'_e in M

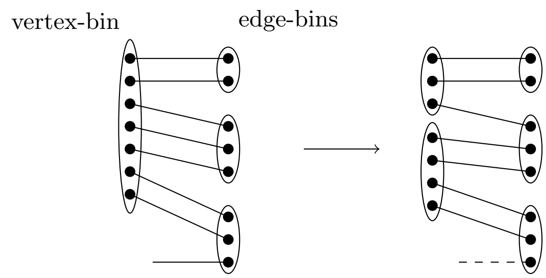

Figure 4.7: Breaking a vertex-bin into smaller pieces.

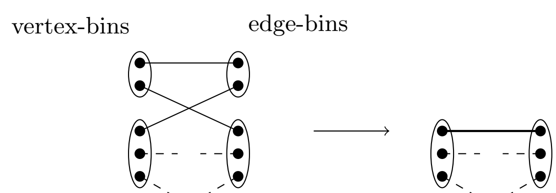

Figure 4.8: Transforming an edge-bin of size 2 matched to a vertex-bin of size 2 into an edge of the matching

and let j be the point matched to p'_v in M. Delete v and e from the kernel-configuration. Add a new edge to M connecting i and j. See Figure 4.8.

3. For each edge-bin e of size 2 such that both of its points p_e and p'_e are matched to points p_v and p_w in vertex-bins v and w of size 2, do the following. Let p'_v be the point in v other than p_v and let p'_w be the point in w other than p_w. Let i be the point matched to p'_v in M and let j be the point matched to p'_w in M. Delete v, w and e from the kernel-configuration. Create a new vertex-bin of size 2 with points p'_v and p'_w and add the edges $p'_v i$ and $p'_w j$ to M. See Figure 4.9.

See Figure 4.10 for an example of the procedure. If the kernel-configuration created in Step 1 is connected, the original kernel-configuration was also connected, since splitting vertex-bins cannot turn a disconnected kernel-configuration into a connected one. We say that the structures in Step 2 and Step 3 are connected if the 2-uniform hypergraph obtained by contracting each bin into a single vertex is connected. It is trivial that, if the structure obtained is connected, then the original kernel-configuration was connected.

Recall that M is chosen u.a.r. from all possible matchings when generating a random kernel as described in Section 4.8.2. This implies that, in the structure obtained after Step 3, the resulting matching has uniform distribution among the perfect matchings on the set of points in the bins such that each point in an edge-bin of size 2 is matched to a point in a vertex-bin of size at least 3,

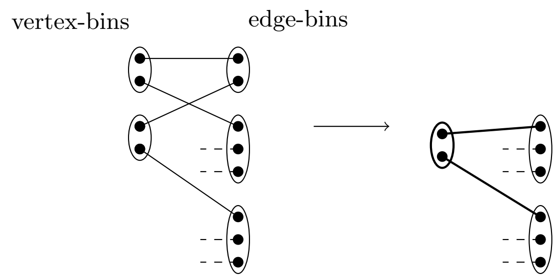

Figure 4.9: Transforming an edge-bin of size 2 matched to two vertex-bins of size 2 into a vertex-bin of size 2

each point in a vertex-bin of size 2 is matched to a point in an edge-bin of size 3, each point in a vertex-bin of size at least 3 is matched to a point in an edge-bin, and each point in an edge-bin of size 3 is matched to a point in a vertex-bin.

Here we describe a new model to generate structures as the one obtained by the process above. Let and $\mathbf{t} \in \{3,4,5\}^N$ and let $\mathbf{t}' \in \{3,4,5\}^{N'}$. Let $L \leq \sum_i t_i/2$ and $L' \leq \sum_i t'_i/2$ be such that $\sum_i t_i - 2L = \sum_i t'_i - 2L' =: K$. Let $B(\mathbf{t}, \mathbf{t}', L, L')$ be generated as follows. In each step, every choice is made u.a.r.:

1. (*Left-bins*) For each $i \in [N]$, create one bin/set with t_i points in it. We call these bins *left-bins*.
2. (*Right-bins*) For each $i \in [N']$, create one bin/set with t'_i points in it. We call these bins *right-bins*.
3. (*Left-connectors*) Create L bins with 2 points inside each. We call these bins *left-connectors*.
4. (*Right-connectors*) Create L' bins with 2 points inside each. We call these bins *right-connectors*.
5. (*Matching*) Choose a perfect matching such that each point in a left-connector is matched to a point in a left-bin, each point in a right-connector is matched to a point in a right-bin, each point in a left-bin is either matched to a point in a left-connector or in a right-bin, and each point in a right-bin is either matched to a point in a right-connector or in a left-bin. The edges in the matching from points in right-bins to points in left-bins are called *across-edges*.

In the structure we obtained from the kernel-configuration, vertex-bins of size at least 3 have the same role as the left-bins, edge-bins of size 3 have the same role as the right-bins, vertex-bins of size 2 have the same role as the right-connectors, and edge-bins of size 2 have the same role as the left-connectors. See Figure 4.10.

We will prove that $B(\mathbf{t}, \mathbf{t}', L, L')$ with $K \to \infty$ is connected a.a.s.

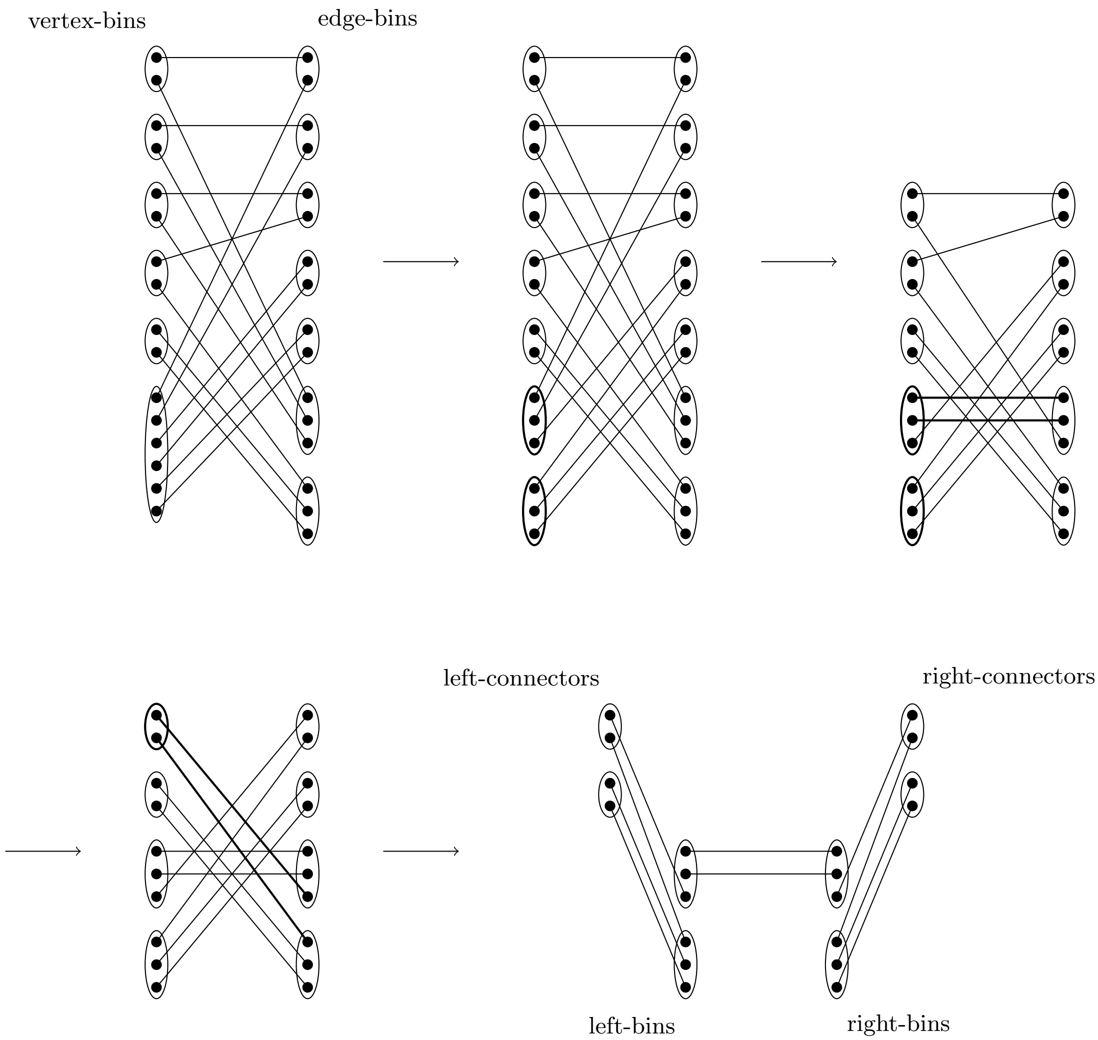

Figure 4.10: Modifying a kernel-configuration

Lemma 4.8.8. Let and $\mathbf{t} \in \{3,4,5\}^N$ and let $\mathbf{t}' \in \{3,4,5\}^{N'}$. Let $L \le \sum_i t_i/2$ and $L' \le \sum_i t_i'/2$ be such that $\sum_i t_i - 2L = \sum_i t_i - 2L' =: K$. If $K \to \infty$, then $B(\mathbf{t}, \mathbf{t}', L, L')$ is connected a.a.s.

Before presenting the proof for this lemma, we explain how to prove Lemma 4.8.6 assuming Lemma 4.8.8 holds. In the structure obtained from the kernel-configuration, the number of points from vertex-bins of size at least 3 (which corresponds to left-bins) that are matched to points in edge-bins of size 3 (which corresponds to right-bins) is $T_3 + m_2^-(1)$, where $m_2^-(1)$ is the number of edge-bins as described in Step 2 of the procedure. In order to use Lemma 4.8.8 to conclude that the kernel-configuration is connected a.a.s. (and thus proving Lemma 4.8.6), it suffices to show that $m_2^-(1) \to \infty$ a.a.s. (which ensures that the condition $K \to \infty$ is satisfied).

Let U be the set of points in vertex-bins of size 2 that will be matched to points in edge-bins of size 2. (See Step 3 in the proof of Lemma 4.8.2.) There are

$$\binom{2m_2^-}{k_1} k_1!$$

ways of matching the points in U to points in edge-bins of size 2. For every edge-bin i of size 2, let X_i be the indicator random for the event that i has both of its points matched to points in U. For $x \in S_\psi^*$, we have that $m_2^- \sim k_1$ and so

$$\mathbb{P}(X_i = 1) = \frac{\binom{k_1}{2} 2! \binom{2m_2^- - 2}{k_1 - 2} (k_1 - 2)!}{\binom{2m_2^-}{k_1} k_1!} \sim \frac{1}{4},$$

$$\mathbb{P}(X_i = 1, X_j = 1) = \frac{\binom{k_1}{4} 4! \binom{2m_2^- - 4}{k_1 - 4} (k_1 - 4)!}{\binom{2m_2^-}{k_1} k_1!} \sim \frac{1}{16}, \text{ for } i \ne j,$$

and so $\mathbb{E}\left(\sum_i X_i\right) \sim m_2^-/4$ and $\mathrm{Var}(\sum_i X_i) = o(\mathbb{E}\left(\sum_i X_i\right)^2)$. Thus, by Chebyshev's inequality,

$$\mathbb{P}\left(\left| \sum_i X_i - \mathbb{E}\left(\sum_i X_i\right) \right| \ge t\mathbb{E}\left(\sum_i X_i\right) \right) = \frac{o(1)}{t^2}$$

and so we can choose t going to 0 sufficiently slowly so that $m_2^-(2) = \sum_i X_i \sim m_2^-/4$ a.a.s. Similarly, $m_2^-(0) = \sum_i X_i \sim m_2^-/4$ a.a.s. Thus,

$$m_2^-(1) \ge (1 + o(1)) \frac{m_2^-}{2} \to \infty$$

since $x \in S_\psi^*$.

We finish this section by presenting the proof for Lemma 4.8.8.

Proof of Lemma 4.8.8. Let $Q = \sum_i t_i$ and let $Q' = \sum_i t'_i$. The number of choices for the matching in Step 5 is

$$\binom{Q}{2L}(2L)!\binom{Q'}{2L'}(2L')!K! = \frac{Q!Q'!}{K!}.$$

Let A be a set of left-bins with P points of which S points are matched to a set of left-connectors (covering all points in these left-connectors). Similarly, let A' be a set of right-bins with P' points of which S' points are matched to a set of right-connectors. Note that S and S' must be even numbers. We compute the number of configurations such that A, A' form a connected component with $r := P - S = P' - S'$ across-edges:

$$\begin{aligned}&\left(\binom{L}{S/2}\binom{P}{S}S!\binom{Q-P}{2L-S}(2L-S)!\right)\\ &\times r!(K-r)!\\ &\times\left(\binom{L'}{S'/2}\binom{P'}{S'}S'!\binom{Q'-P'}{2L'-S'}(2L'-S')!\right)\end{aligned}$$

Thus, the probability that A, A' form a connected component (with parameters S, S') is exactly

$$\frac{\binom{L}{S/2}\binom{L'}{S'/2}\binom{K}{r}}{\binom{Q}{P}\binom{Q'}{P'}}.$$

So we want to bound the summation:

$$\sum_{\substack{(P,S,n)\\(P',S',n')}}\sum_{(A,A')}\frac{\binom{L}{S/2}\binom{L'}{S'/2}\binom{K}{r}}{\binom{Q}{P}\binom{Q'}{P'}} \tag{4.55}$$

where the second summation is over the pairs (A, A') where A is a set of n left-bins with P points and S points matched to left-connectors and A' is a set of n' right-bins with P' points and S' points matched to right-connectors; and $r = P - S = P' - S'$. Let C be an integer constant to be determined later.

First consider the case where

$$\begin{gathered}P \le C \text{ and } P' \le C,\\ \text{or}\\ Q - P \le C \text{ and } Q' - P' \le C.\end{gathered}$$

We only need to check one of the options above because if $A \cup A'$ is disconnected from the rest of the graph the same is true for the $\overline{A} \cup \overline{A}'$ where $\overline{A}$ is the complement of A in the set of left-bins and $\overline{A'}$ is the complement of A' in the set of right-bins. So let us assume $P \le C$ and $P' \le C$. Then the number of choices for (P, S, n) and (P', S', n') is $O(1)$. Moreover, there are at most $\binom{N}{n}$ choices for A and $\binom{N'}{n'}$ choices for A', where N is the number of left-bins and N' is the number of right-bins. Then the summation in (4.55) for this case is at most

$$\frac{\binom{L}{S/2}\binom{L'}{S'/2}\binom{K}{r}\binom{N}{n}\binom{N'}{n'}}{\binom{Q}{P}\binom{Q'}{P'}} = O\left(\frac{L^{S/2}(L')^{S'/2}K^r N^n (N')^{n'}}{Q^P (Q')^{P'}}\right)$$
$$= O\left(\frac{1}{Q^{P-S/2-r/2-n}(Q')^{P'-S'/2-r/2-n'}}\right)$$
$$= O\left(\frac{1}{Q^{P/6}(Q')^{P'/6}}\right) = o(1),$$

since $P - S/2 - r/2 - n \ge P - S/2 - (P-S)/2 - P/3 = P/6$ (and similarly for $P' - S'/2 - r/2 - n'$) and P or P' is at least 1.

Now consider the case where

$$P \le C \text{ and } Q' - P' \le C,$$
$$\text{or}$$
$$Q - P \le C \text{ and } P' \le C.$$

If $P \le C$ and $Q' - P' \le C$. Then $r = P - S \le C$ and $r = P' - S' \ge P' - 2L' \ge Q - C - 2L' = K - C$, which is impossible since $K \to \infty$ and $C = O(1)$.

Finally consider the case

$$P \ge C \text{ and } P' \ge C,$$
$$\text{or}$$
$$Q - P \ge C \text{ and } Q' - P' \ge C.$$

Using Stirling's approximation (Lemma 2.5.1), there is a positive constant α such that

$$\frac{\binom{K}{r}}{\binom{\lceil K/2 \rceil}{\lceil r/2 \rceil}\binom{\lfloor K/2 \rfloor}{\lfloor r/2 \rfloor}} \le \alpha\sqrt{K}.$$

Thus, for P and P' in this range,

$$\sum_{\substack{(P,S,n)\\(P',S',n')}} \sum_{(A,A')} \frac{\binom{L}{S/2}\binom{L'}{S'/2}\binom{K}{r}}{\binom{Q}{P}\binom{Q'}{P'}} \le \alpha \sum_{\substack{(P,S,n)\\(P',S',n')}} \sum_{(A,A')} \frac{\binom{L}{S/2}\binom{L'}{S'/2}\sqrt{K}\binom{\lceil K/2\rceil}{\lceil r/2\rceil}\binom{\lfloor K/2\rfloor}{\lfloor r/2\rfloor}}{\binom{Q}{P}\binom{Q'}{P'}}$$

$$\begin{aligned}
&\le \alpha \sum_{\substack{(P,S,n)\\(P',S',n')}} \frac{\binom{N}{n}\binom{N'}{n'}\binom{L}{S/2}\binom{L'}{S'/2}\sqrt{K}\binom{\lceil K/2\rceil}{\lceil r/2\rceil}\binom{\lfloor K/2\rfloor}{\lfloor r/2\rfloor}}{\binom{Q}{P}\binom{Q'}{P'}} \\
&\le \alpha \sum_{\substack{(P,S,n)\\(P',S',n')}} \frac{\binom{N}{n}\binom{N'}{n'}\sqrt{K}}{\binom{Q-L-\lceil K/2\rceil}{P-S/2-\lceil r/2\rceil}\binom{Q'-L'-\lfloor K/2\rfloor}{P'-S'/2-\lfloor r/2\rfloor}} \\
&= \alpha \sum_{\substack{(P,S,n)\\(P',S',n')}} \frac{\binom{N}{n}\binom{N'}{n'}\sqrt{K}}{\binom{Q/2-u(K)}{P/2-u(r)}\binom{Q'/2-d(K)}{P'/2-d(r)}},
\end{aligned}$$

where $u(x) := \lceil x/2\rceil - x/2$ and $d(x) := x/2 - \lfloor x/2\rfloor$. Note that, for $P' \le Q'/2$,

$$\frac{\binom{N'}{n'}}{\binom{Q'/2-d(K)}{P'/2-d(r)}} \le \frac{\binom{Q'/3}{P'/3}}{\binom{Q'/2-d(K)}{P'/2-d(r)}} \le \frac{1}{\binom{Q'/6-d(K)}{P'/6-d(r)}} \le 1,$$

and for $P' \ge Q'/2$

$$\begin{aligned}
\frac{\binom{N'}{n'}}{\binom{Q'/2-d(K)}{P'/2-d(r)}} &= \frac{\binom{N'}{N'-n'}}{\binom{Q'/2-d(K)}{Q'/2-P'/2-d(K)+d(r)}} \le \frac{\binom{Q'/3}{Q'/3-P'/3}}{\binom{Q'/2-d(K)}{Q'/2-P'/2-d(K)+d(r)}} \\
&\le \frac{1}{\binom{Q'/6-d(K)}{Q'/6-P'/6-d(K)+d(r)}} \le 1,
\end{aligned}$$

Thus, for $P' \leq Q'$,

$$\frac{\binom{N'}{n'}}{\binom{Q'/2 - d(K)}{P'/2 - d(r)}} \leq 1. \tag{4.56}$$

For $C \leq P \leq \beta \log Q$,

$$\frac{\binom{N}{n}}{\binom{Q/2 - u(K)}{P/2 - d(r)}} \leq \frac{\binom{Q/3}{P/3}}{\binom{Q/2 - u(K)}{P/2 - u(r)}} \leq \binom{Q/6 - u(K)}{P/6 - u(r)}^{-1} = O\left(\frac{Q}{\beta \log Q}\right)^{-P/6 + u(r)}$$

and so by choosing C big enough and using (4.56)

$$\sum_{\substack{(P,S,n) \\ (P',S',n') \\ C \leq P \leq \beta \log Q}} \frac{\binom{N}{n}\binom{N'}{n'}\sqrt{K}}{\binom{Q/2 - u(K)}{P/2 - u(r)}\binom{Q'/2 - d(K)}{P'/2 - d(r)}} \leq Q^{11/2} \log Q \cdot O\left(\frac{\beta \log Q}{Q}\right)^6 = o(1).$$

The range $Q - \beta \log Q \leq P \leq Q - C$ can be treated similarly.

There exists a constant $\gamma > 0$ such that, for $\beta \log Q \leq P \leq Q/2$,

$$\frac{\binom{N}{n}}{\binom{Q/2 - u(K)}{P/2 - u(r)}} \leq \frac{\binom{Q/3}{P/3}}{\binom{Q/2 - u(K)}{P/2 - u(r)}} \leq \binom{Q/6 - u(K)}{P/6 - u(r)}^{-1} = O(\gamma^{P/6 - u(r)}),$$

and so, by (4.56),

$$\sum_{\substack{(P,S,n) \\ (P',S',n') \\ \beta \log Q \leq P \leq Q/2}} \frac{\binom{N}{n}\binom{N'}{n'}\sqrt{K}}{\binom{Q/2 - u(K)}{P/2 - u(r)}\binom{Q'/2 - d(K)}{P'/2 - d(r)}} \leq Q^{13/2} \cdot O\left(\gamma^{\beta \log N}\right) = o(1),$$

for sufficiently large constant β. The range $Q/2 \leq P \leq Q - \beta \log Q$ can be treated similarly. The same argument works for (P', S', n') and Q'. We are done because $P \leq Q - C$ or $P' \leq Q' - C$ (otherwise, it falls in a case that has already been treated). □

4.8.6 Proof of Theorem 4.8.1

In this section we obtain an asymptotic formula for the number of connected pre-kernels with vertex set $[n]$ and $m = n/2 + R$ edges, when $R = \omega(n^{1/2}\log^{3/2} n)$ and $R = o(n)$. The complete proof is contained in this section together with Sections 4.8.7, 4.8.8 and 4.8.9, in which we prove some lemmas we state in this section. This proves Theorem 4.8.1.

We rewrite the conditions defining $S_m \subseteq \mathbb{R}^4$. We have that $(n_1, k_0, k_1, k_2) \in S_m$ if all of the following conditions are satisfied:

(C1) $n_1, k_0, k_1, k_2 \geq 0$;

(C2) $T_2 \geq 0$ (equivalently, $2n_1 - 2k_0 - k_1 \geq 0$);

(C3) $T_3 \geq 0$; (equivalently, $3n_1 + k_1 + 2k_2 \leq 3m$);

(C4) $Q_3 \geq 3n_3 \geq 0$ (equivalently, $k_0 - k_1 - k_2 \leq 3m - n$ and $n_1 - k_0 - k_1 - k_2 \leq n$);

(C5) $Q_3 = 0$ whenever $n_3 = 0$.

For $x = (n_1, k_0, k_1, k_2) \in S_m$, let

$$w_{\mathrm{pre}}(x) = \begin{cases} \dfrac{P_3!P_2!Q_3!(m_2-1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2^- - 1)!m_2^-!2^{k_2}2^{m_2^-}6^{m_3}} \dfrac{f_3(\lambda)^{n_3}}{\lambda^{Q_3}}, & \text{if } Q_3 > 3n_3; \\ \dfrac{P_3!P_2!Q_3!(m_2-1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2^- - 1)!m_2^-!2^{k_2}2^{m_2^-}6^{m_3}} \dfrac{1}{6^{n_3}}, & \text{otherwise.} \end{cases} \tag{4.57}$$

Recall that $\hat{x}^* = (\hat{n}_1^*, \hat{k}_0^*, \hat{k}_1^*, \hat{k}_2^*)$ is defined as

$$\hat{n}_1^* = \frac{3\hat{m}}{g_2(\lambda^*)}, \qquad \hat{k}_0^* = \frac{3\hat{m}}{g_2(\lambda^*)}\frac{2\lambda^*}{f_1(\lambda^*)g_1(\lambda^*)},$$
$$\hat{k}_1^* = \frac{3\hat{m}}{g_2(\lambda^*)}\frac{2\lambda^*}{g_1(\lambda^*)}, \qquad \hat{k}_2^* = \frac{3\hat{m}}{g_2(\lambda^*)}\frac{\lambda^* f_1(\lambda^*)}{2g_1(\lambda^*)},$$

where $\lambda^* = \lambda^*(n)$ is the unique nonnegative solution of the equation

$$\frac{\lambda f_1(\lambda)g_2(\lambda)}{f_2(2\lambda)} = 3\hat{m}.$$

The existence and uniqueness of λ^* was discussed in Lemma 4.7.4.

We will show that $\hat{x}^*$ is the unique point achieving the maximum for f_{pre} in the set $\hat{S}_m$ and then we will expand the summation around $\hat{x}^*$. To determine the region where the summation will be expanded we will analyse the Hessian of f_{pre}. Let

$$H_0 = \frac{1}{36}\begin{pmatrix} 33 & 12 & 15 & 18 \\ 12 & 6 & 6 & 6 \\ 15 & 6 & 7 & 8 \\ 18 & 6 & 8 & 12 \end{pmatrix} \quad \text{and} \quad T = \frac{1}{30}\begin{pmatrix} -47 & -16 & -11 & -6 \\ -16 & 22 & 12 & 2 \\ -11 & 12 & 31/3 & -4/3 \\ -6 & 2 & -4/3 & -4/3 \end{pmatrix} \tag{4.58}$$

Later we will see that the Hessian of f_{pre} at $\hat{x}^*$ is $(-1/r^2)H_0 - (1/r)T + O(J)$, where J denotes the 4×4 matrix of all 1's. For two matrices A, B of same dimensions, we say that a matrix $A = O(B)$ if $A_{ij} = O(B_{ij})$ for all i, j.

Let $z_1 = (1, 1, -3, 0)$. Then z_1 is an eigenvector of H_0 with eigenvalue 0. Let $e_i \in \mathbb{R}^4$ be the vector such that the i-th coordinate is 1 and all the others are 0. Let

$$B := \Big\{x \in \mathbb{R}^4 : x = \gamma_1 z_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4,\ |\gamma_1| \le \delta_1 n \text{ and } |\gamma_i| \le \delta n \text{ for } i = 2, 3, 4\Big\},$$

and let $\hat{B} = \{(n_1/n, k_0/n, k_1/n, k_2/n) : (n_1, k_0, k_1, k_2) \in B\}$, that is, $\hat{B}$ is a scaled version of B. We will choose δ_1 and δ later. The set $x^* + B$ (this is the Minkowski sum of $\{x^*\}$ and B) is the region where we will approximate $\sum_x n! \exp(n f_{\text{pre}}(\hat{x}))$ by using Taylor's approximation. For this, we show that, for an appropriate choice for δ_1 and δ, the set $x^* + B$ is contained in S_m.

Lemma 4.8.9. Suppose that $\delta_1 = o(r)$ and that $\delta = o(r^2)$. Let $x \in B$. For any function F among $n_1(x + x^*)$, $k_i(x^* + x)$ for $i = 0, 1, 2$, $Q_3(x^* + x) - 3n_3(x^* + x)$, and the linear functions defined in (4.42), we have that $F(x^* + x) \sim F(x^*)$. Moreover, $\lambda(x) \sim \lambda(x^*)$.

Proof. Write x as $x = \gamma_1 z_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4$ with $|\gamma_1| \le \delta_1$ and $|\gamma_i| \le \delta$ for $i = 2, 3, 4$. We will show that $F(\gamma_1 z_1) = o(F(x^*))$ and $F(\gamma e_i) = o(F(x^*))$ for $i = 2, 3, 4$. Since F is a linear function, this implies that $F(x^* + x) = F(x^*) + F(x) = F(x^*) + o(F(x^*))$, proving the first statement in the lemma.

Using (4.49), we have that $F(x^*) = \Omega(r^2 n)$ for all the functions F under consideration and so, for $i = 2, 3, 4$, we have that $F(\gamma_i e_i) = o(r^2 n) = o(F(x^*))$ since $|\gamma_i| \le \delta n = o(r^2 n)$.

Using (4.49), we have that $F(x^*) = \Omega(rn)$ for all F under consideration except k_2, T_3 and $Q_3 - 3n_3$. Since $|\gamma_1| \le \delta_1 n = o(rn)$, we have that $F(\gamma_1 z_1) = o(rn) = o(F(x^*))$ for all F under consideration, except k_2, T_3 and $Q_3 - 3n_3$. So let F be one of the functions k_2, T_3 or $Q_3 - 3n_3$. Then, using $z_1 = (1, 1, -3, 0)$, we have that $F(z_1) = 0$ and so $F(x^* + x) = F(x^*)$, finishing the proof of the first statement in the lemma.

Since $Q_3(x^* + x) \sim Q_3(x^*)$ and $n_3(x^* + x) \sim n_3(x^*)$, we have that $c_3(x + x^*) = Q_3(x^* + x)/n_3(x^* + x) \sim c_3(x^*)$. Thus, since $\lambda(y)$ is defined as the unique solution of $\lambda f_2(\lambda)/f_3(\lambda) = c_3(y)$, we have that $\lambda(x) \sim \lambda(x^*)$ by Lemma 2.10.5. $\square$

Corollary 4.8.10. Suppose that $\delta_1 = o(r)$ and that $\delta = o(r^2)$. Let $x \in B$. Then there exists $\psi = o(1)$ such that $x^* + x \in S^*_\psi$ and $x^* + x$ is in the interior of S_m.

Proof. Recall that S^*_ψ is defined in (4.53). Lemma 4.8.9 and the definition of S^*_ψ immediately imply the first part of the conclusion.

We check whether $x^* + x$ satisfies the conditions (C1)–(C5) strictly. We have that x^* satisfies the constraints (C1)–(C4) with slack $\Omega(r^2 n)$ by (4.49) and recall that $r^2 n \to \infty$. By Lemma 4.8.9, we have that $x^* + x$ also satisfies all the constraints (C1)–(C4) with slack $\Omega(r^2 n)$.

It remains to check (C5). We have that $n_3(x^* + x) \sim n_3(x^*) = \Omega(rn) = \omega(1)$ and so (C5) is satisfied strictly. We conclude that $x^* + x$ is in the interior of S_m. □

The following lemmas are the main steps in the proof of Theorem 4.8.1. We show that $\hat{x}^*$ is the unique maximum for f_{pre} in $\hat{S}$ and compute a bound for any other local maximum.

Lemma 4.8.11. The point $\hat{x}^* = (\hat{n}^*_1, \hat{k}^*_0, \hat{k}^*_1, \hat{k}^*_2)$ is the unique maximum for f_{pre} in $\hat{S}_m$ and

$$f_{\text{pre}}(\hat{x}^*) = 2r \ln n - 4r \ln r + \left(-\frac{2}{3}\ln(2) - \frac{1}{3}\ln(3) + \frac{1}{3}\right)\lambda^* + \left(-\frac{2}{9}\ln(2) - \frac{1}{9}\ln(3) + \frac{7}{36}\right)(\lambda^*)^2 + O((\lambda^*)^3).$$

Moreover, there exists a constant $\beta < -(2/9)\ln(2) - (1/9)\ln(3) + (7/36)$ such that any other local maximum in $\hat{S}_m$ has value at most

$$2r \ln n - 4r \ln r + \left(-\frac{2}{3}\ln(2) - \frac{1}{3}\ln(3) + \frac{1}{3}\right)\lambda^* + \beta(\lambda^*)^2.$$

We then estimate the summation of $\exp(nf_{\text{pre}}(\hat{x} + \hat{x}^*))$ over points $x \in B$ such that $x + x^*$ is integer.

Lemma 4.8.12. Suppose that $\delta_1^3 = o(r/n)$ and $\delta_1^2 = \omega(r/n)$, and $\delta^3 = o(r^4/n)$ and $\delta^2 = \omega(r^2/n)$. Then

$$\sum_{\substack{x \in B \\ x + x^* \in \mathbb{Z}^4}} \exp\left(nf_{\text{pre}}(\hat{x} + \hat{x}^*)\right) \sim 144\sqrt{3}\pi^2 n^2 r^{7/2} \exp(nf_{\text{pre}}(\hat{x}^*)).$$

Finally, we bound the contribution from points far from the maximum.

Lemma 4.8.13. Suppose that $\delta_1^3 = o(r/n)$ and $\delta_1^2 = \omega(r \ln n/n)$, and $\delta^3 = o(r^4/n)$ and $\delta^2 = \omega(r^2 \ln n/n)$. We have that

$$\sum_{\substack{x \in S \setminus (x^* + B) \\ x \in \mathbb{Z}^4}} w_{\text{pre}}(x) = o\left(n! \exp(nf_{\text{pre}}(\hat{x}^*))\right).$$

The proof of Lemma 4.8.11 is deferred to Section 4.8.8. The proofs of Lemmas 4.8.12 and 4.8.13 are presented in Section 4.8.9. We are now ready to prove Theorem 4.8.1.

In order to use Lemmas 4.8.12 and 4.8.13, we need to check if there exists δ_1 such that $\delta_1^3 = o(r/n)$ and $\delta_1^2 = \omega(r \ln n/n)$, and δ such that $\delta^3 = o(r^4/n)$ and $\delta^2 = \omega(r^2 \ln n/n)$. There exists such δ_1 if and only if $(r/n)^2 = \omega((r \ln n/n)^3)$, which is true if and only if $n/r = \omega(\ln^3 n)$, which is true since $r = o(1)$. There exists such δ if and only if $(r^4/n)^2 = \omega((r^2 \ln n/n)^3)$, which is true if and only if $r^2 = \omega(\ln^3 n/n)$, which is one of the hypotheses of the theorem.

By Proposition 4.8.3 and Lemma 4.8.9, we have that, for $x \in (x^* + B)$,

$$g_{\text{pre}}(x) = w_{\text{pre}}(x)\mathbb{E}\left(\mathbb{P}\left(\mathcal{P}(x, \mathbf{Y}) \text{ simple and connected}\right) \Big| \Sigma(x)\right) \mathbb{P}\left(\Sigma(x)\right),$$

where $\Sigma(x)$ is the event that a random vector $\mathbf{Y} = (Y_1, \ldots, Y_{n_3(x)})$ of independent truncated Poisson random variables with parameters $(3, \lambda(x))$ satisfy $\sum_{i=1}^{n_3(x)} = Q_3(x)$. By Corollary 4.8.7 and Lemma 4.8.9,

$$\mathbb{E}\left(\mathbb{P}\left(\mathcal{P}(x, \mathbf{Y}) \text{ simple and connected}\right) \Big| \Sigma(x)\right) \sim 1. \tag{4.59}$$

By Stirling's approximation, the definition of f_{pre} (in (4.44) and (4.45)), and definition of w_{pre} (in 4.57) , we have that

$$w_{\text{pre}}(x) \sim n! \frac{1}{(2\pi n)^{5/2}} \left(\frac{\hat{P}_3 \hat{P}_2 \hat{Q}_3}{\hat{k}_0 \hat{k}_1 \hat{k}_2 \hat{n}_3 \hat{m}_3 \hat{T}_3 \hat{T}_2 \hat{m}_2} \right)^{1/2} \exp(n f_{\text{pre}}(\hat{x})).$$

Since $x \in (x^* + B)$, by Lemma 4.8.9, we have that

$$\frac{\hat{P}_3 \hat{P}_2 \hat{Q}_3}{\hat{k}_0 \hat{k}_1 \hat{k}_2 \hat{n}_3 \hat{m}_3 \hat{T}_3 \hat{T}_2 \hat{m}_2} \sim \frac{\hat{P}_3^* \hat{P}_2^* \hat{Q}_3^*}{\hat{k}_0^* \hat{k}_1^* \hat{k}_2^* \hat{n}_3^* \hat{m}_3^* \hat{T}_3^* \hat{T}_2^* \hat{m}_2^*} \sim \frac{1}{r^{5/2} 4\sqrt{6}}.$$

Next we estimate $\mathbb{P}(\Sigma(x))$. We will use Theorem 2.10.8, applied with n_3 as the parameter n in Theorem 2.10.8 and $c_3 = Q_3/n_3$ as c in Theorem 2.10.8. By Lemma 4.8.9 and (4.49), we have that $Q_3(x) - 3n_3(x) \sim (Q_3(x^*) - n_3(x^*)) \sim 12R^2/n = \omega \ln(n)$. Thus, by Theorem 2.10.8,

$$\mathbb{P}(\Sigma(x)) \sim \frac{1}{\sqrt{2\pi Q_3(x)(1 + \eta_3(x) - c_3(x))}},$$

where $\eta_3(x) = \lambda(x) f_1(\lambda(x))/f_2(\lambda(x))$ and $c_3(x) = Q_3(x)/n_3(x) = \lambda(x) f_2(\lambda(x))/f_3(\lambda(x))$. Since $Q_3(x)/n_3(x) \sim \hat{Q}_3(\hat{x}^*)/\hat{n}_3(\hat{x}^*)$, Lemma 2.10.5 implies that $\lambda(x) \sim \lambda^* \to 0$ and so (omitting the

(x) in the following)

$$
\begin{aligned}
1+\eta_3-c_3 &= \frac{f_2(\lambda)f_3(\lambda)+\lambda f_1(\lambda)f_3(\lambda)-\lambda f_2(\lambda)^2}{f_2(\lambda)f_3(\lambda)} \\
&= \frac{\left(\frac{\lambda^2}{2}+\frac{\lambda^3}{6}\right)\left(\frac{\lambda^3}{6}+\frac{\lambda^4}{24}\right)+\lambda\left(\lambda+\frac{\lambda^2}{2}\right)\left(\frac{\lambda^3}{6}+\frac{\lambda^4}{24}\right)+\lambda\left(\frac{\lambda^2}{2}+\frac{\lambda^3}{6}\right)^2+O(\lambda^7)}{\left(\frac{\lambda^2}{2}+\frac{\lambda^3}{6}\right)\left(\frac{\lambda^3}{6}+\frac{\lambda^4}{24}\right)+O(\lambda^7)} \\
&= \frac{\lambda^6/144}{\lambda^5/12}\big(1+O(\lambda)\big) \sim \frac{\lambda}{12} \sim \frac{\lambda^*}{12} \sim r,
\end{aligned}
$$

by Lemma 4.8.9 and (4.49). Moreover, $Q_3 \sim 6R$ by (4.49). Hence,

$$
\mathbb{P}(\Sigma) \sim \frac{1}{\sqrt{2\pi(6R)(1+\eta_3-c_3)}} \sim \frac{1}{r\sqrt{12\pi n}}.
$$

Thus,

$$
g_{\text{pre}}(x) = n!\frac{1}{144(\pi n)^3 r^{7/2}} \sum_{x\in B} \exp(nf_{\text{pre}}(x))(1+o(1)), \tag{4.60}
$$

for all $x \in (x^*+B)$. Since $(x^*+B)\cap\mathbb{Z}^4$ is a finite set for each n, Lemma 2.7.1 implies that there is a function $q(n)=o(1)$ such that the error in (4.60) is bounded by $q(n)$ uniformly for all $x \in (x^*+B)\cap\mathbb{Z}^4$. Thus,

$$
\begin{aligned}
\sum_{x\in(x^*+B)\cap\mathbb{Z}^4} g_{\text{pre}}(x) &\sim n!\frac{1}{144(\pi n)^3 r^{7/2}} \sum_{x\in(x^*+B)} \exp(nf_{\text{pre}}(x)) \\
&\sim n!\frac{1}{144(\pi n)^3 r^{7/2}} \cdot 144\sqrt{3}\pi^2 n^2 r^{7/2} \exp(nf_{\text{pre}}(\hat{x}^*)),
\end{aligned}
$$

by Lemma 4.8.12. Thus,

$$
\sum_{x\in(x^*+B)\cap\mathbb{Z}^4} g_{\text{pre}}(x) \sim n!\frac{\sqrt{3}}{\pi n}\exp(nf_{\text{pre}}(\hat{x}^*)).
$$

Together with Lemma 4.8.13, this finishes the proof of Theorem 4.8.1.

4.8.7 Partial derivatives

In this section, we will analyse the first, second, and third partial derivatives of f_{pre}. This will be used in the proof that $\hat{x}^*$ achieves the maximum for f_{pre} (Lemma 4.8.11) and also to approximate the summation around $\hat{x}^*$ (Lemma 4.8.12). See Section A.4 for a Maple spreadsheet.

Recall that $h_n(y) = y\ln(yn) - y$ and, for $\hat{x} = (\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2)$,

$$\begin{aligned} f_{\text{pre}}(\hat{x}) = {} & h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(m_2) \\ & - h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{n}_3) - h_n(\hat{m}_3) \\ & - h_n(\hat{T}_3) - h_n(\hat{T}_2) - 2h_n(\hat{m}_2^-) \\ & - \hat{k}_2 \ln 2 - \hat{m}_2^- \ln 2 - \hat{m}_3 \ln 6 \\ & + \hat{n}_3 \ln f_3(\lambda(x)) - \hat{Q}_3 \ln \lambda(x), \end{aligned}$$

where $\lambda(x)$ is the unique positive solution to $\lambda f_2(\lambda)/f_3(\lambda) = c_3$, where $c_3 = \hat{Q}_3/\hat{n}_3$.

Using (4.10) to compute the partial derivatives of $\hat{n}_3 \ln f_3(\lambda(x)) - \hat{Q}_3 \ln \lambda(x)$ (w.r.t. $\hat{n}_1$, $\hat{k}_0$, $\hat{k}_1$ and $\hat{k}_2$), we obtain

$$\exp\left(\frac{\mathrm{d}\, f_{\text{pre}}(x)}{\mathrm{d}\, \hat{n}_1}\right) = \frac{4\hat{T}_3^3 \hat{n}_3 \hat{n}_1 \lambda}{9\hat{m}_3^2 \hat{Q}_3 \hat{T}_2^2 f_3 \lambda}; \tag{4.61}$$

$$\exp\left(\frac{\mathrm{d}\, f_{\text{pre}}(x)}{\mathrm{d}\, \hat{k}_0}\right) = \frac{\hat{n}_3 \hat{T}_2^2 \lambda^2}{2\hat{Q}_3^2 \hat{k}_0 f_3(\lambda)}; \tag{4.62}$$

$$\exp\left(\frac{\mathrm{d}\, f_{\text{pre}}(x)}{\mathrm{d}\, \hat{k}_1}\right) = \frac{\hat{T}_3 \hat{n}_3 \hat{T}_2 \lambda^2}{\hat{k}_1 \hat{Q}_3^2 f_3(\lambda)}; \tag{4.63}$$

$$\exp\left(\frac{\mathrm{d}\, f_{\text{pre}}(x)}{\mathrm{d}\, \hat{k}_2}\right) = \frac{\hat{T}_3^2 \hat{n}_3 \lambda^2}{2\hat{k}_2 \hat{Q}_3^2 f_3(\lambda)}; \tag{4.64}$$

For the second partial derivatives, we need to compute

$$\frac{\partial^2(\hat{n}_3 \ln f_3(\lambda(x)) - \hat{Q}_3 \ln \lambda(x))}{\partial a \partial b},$$

for any $a, b \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}$. Using (4.10), this is

$$\begin{aligned} \frac{\partial}{\partial a}\left(\frac{\partial \hat{n}_3}{\partial b} \ln f_3(\lambda) - \frac{\partial \hat{Q}_3}{\partial b} \ln \lambda\right) &= \frac{\partial}{\partial a}\left(-\ln f_3(\lambda) - \frac{\partial \hat{Q}_3}{\partial b} \ln \lambda\right) = \frac{\partial \lambda}{\partial a}\left(-\frac{f_2(\lambda)}{f_3(\lambda)} - \frac{\partial \hat{Q}_3}{\partial b}\frac{1}{\lambda}\right) \\ &= \frac{\partial c_3}{\partial a}\frac{\lambda}{c_3(1+\eta_3-c_3)}\left(-\frac{f_2(\lambda)}{f_3(\lambda)} - \frac{\partial \hat{Q}_3}{\partial b}\frac{1}{\lambda}\right), \\ &= \left(\frac{\partial \hat{Q}_3}{\partial a}\frac{1}{\hat{n}_3} - \frac{\partial \hat{n}_3}{\partial a}\frac{\hat{Q}_3}{\hat{n}_3^2}\right)\frac{1}{c_3(1+\eta_3-c_3)}\left(-c_3 - \frac{\partial \hat{Q}_3}{\partial b}\right) \\ &= -\left(c_3 + \frac{\partial \hat{Q}_3}{\partial a}\right)\left(c_3 + \frac{\partial \hat{Q}_3}{\partial b}\right)\frac{1}{\hat{Q}_3(1+\eta_3-c_3)}. \end{aligned} \tag{4.65}$$

The second partial derivatives now are

$$
\begin{aligned}
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{n}_1 \partial \hat{n}_1} &= \frac{9}{P_3} + \frac{4}{P_2} - \frac{9}{T_3} + \frac{1}{Q_3} - \frac{1}{n_3} - \frac{1}{m_3} - \frac{4}{T_2} - \frac{2}{m_2^-} + \frac{1}{n_1} + D_1 \\
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{n}_1 \partial \hat{k}_0} &= -\frac{4}{P_2} + \frac{2}{Q_3} - \frac{1}{n_3} + \frac{4}{T_2} + \frac{2}{m_2^-} + D_k \\
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{n}_1 \partial \hat{k}_1} &= -\frac{3}{T_3} + \frac{2}{Q_3} - \frac{1}{n_3} + \frac{2}{T_2} + D_k \\
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{n}_1 \partial \hat{k}_2} &= -\frac{6}{T_3} + \frac{2}{Q_3} - \frac{1}{n_3} + D_k \\
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_0 \partial \hat{k}_0} &= \frac{4}{P_2} + \frac{4}{Q_3} - \frac{1}{n_3} - \frac{4}{T_2} - \frac{2}{m_2^-} - \frac{1}{k_0} + D_{kk} \\
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_0 \partial \hat{k}_1} &= \frac{4}{Q_3} - \frac{1}{n_3} - \frac{2}{T_2} + D_{kk} \\
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_0 \partial \hat{k}_2} &= \frac{4}{Q_3} - \frac{1}{n_3} + D_{kk} \\
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_1 \partial \hat{k}_1} &= -\frac{1}{k_1} - \frac{1}{T_3} + \frac{4}{Q_3} - \frac{1}{n_3} - \frac{1}{T_2} + D_{kk} \\
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_1 \partial \hat{k}_2} &= -\frac{2}{T_3} + \frac{4}{Q_3} - \frac{1}{n_3} + D_{kk} \\
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_2 \partial \hat{k}_2} &= -\frac{1}{k_2} - \frac{4}{T_3} + \frac{4}{Q_3} - \frac{1}{n_3} + D_{kk},
\end{aligned}
\tag{4.66}
$$

where

$$
\begin{aligned}
D_1 &= -\frac{(c_3-1)^2}{(1+\eta_3-c_3)\hat{Q}_3}; \\
D_k &= -\frac{(c_3-1)(c_3-2)}{(1+\eta_3-c_3)\hat{Q}_3}; \\
D_{kk} &= -\frac{(c_3-2)^2}{(1+\eta_3-c_3)\hat{Q}_3}.
\end{aligned}
$$

In the next lemma, we find an approximation for the Hessian f_{pre} at $\hat{x}^*$. It follows immediately by computing the series of each partial second derivative with $\lambda \to 0$. See Section A.4 for a Maple spreadsheet.

Lemma 4.8.14. The Hessian of f_{pre} at $\hat{x}^*$ is $(-1/r^2)H_0 - (1/r)T + O(J)$, where H_0 and T are defined in (4.58) and J is a 4×4 matrix with all entries equal to 1.

We will bound the third partial derivatives for points close to x^*.

Lemma 4.8.15. Suppose that $\delta_1^3 = o(r/n)$ and $\delta^3 = o(r^4/n)$. Then for any $x \in B$ we have that

$$n\frac{\partial f_{\text{pre}}(\hat{x}^* + \hat{x})}{\partial t_1 \partial t_2 \partial t_3} t_1(\hat{x}) t_2(\hat{x}) t_3(\hat{x}) = o(1),$$

for any $t_1, t_2, t_3 \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}$.

Proof. Let $x \in B$. Then $x = \alpha z_1 + b$, where $|\alpha| \le \delta_1$ and $b = (0, b_2, b_3, b_4)$ and $|b_i| \le \delta$ and $b^T z_1 = 0$. Recall that $z_1 = (1, 1, -3, 0)$ and so $x = (\alpha, \alpha + b_2, -3\alpha + b_3, b_4)$. Then, by using (4.66), we may compute each partial derivative $\frac{\partial f_{\text{pre}}}{\partial t_1 \partial t_2 \partial t_3} t_1(\hat{x}) t_2(\hat{x})(t_3(\hat{x}))$ exactly. We omit the lengthy computations here. (See Section A.4 for a Maple spreadsheet.) The third derivative is the sum of the part involving λ and the part that does not involve λ. The part not involving λ can be written as

$$\sum_{\substack{a=(a_1,a_2,a_3,a_4)\in\mathbb{N}^4,\\ a_1+a_2+a_3+a_4=3}} T(a)\alpha^{a_1}(\alpha + b_2)^{a_2}(-3\alpha + b_3)^{a_3} b_4^{a_4},$$

where each $T(a)$ is a sum of terms in the format $1/z^2$, where

$$z \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2, \hat{n}_3, \hat{P}_2, \hat{P}_3, \hat{Q}_3, \hat{T}_2, \hat{T}_3\}.$$

This can be expanded so that it is

$$\sum_{\substack{f=(f_1,f_2,f_3,f_4)\in\{0,1,2,3\}\times\{0,1\}^3,\\ f_1+f_2+f_3+f_4=3}} T_2(f)\alpha^{f_1} b_2^{f_2} b_3^{f_3} b_4^{f_4},$$

where each $T_2(f)$ is also a sum of terms in the format $1/z^2$, where

$$z \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2, \hat{n}_3, \hat{P}_2, \hat{P}_3, \hat{Q}_3, \hat{T}_2, \hat{T}_3\}.$$

Since $\delta_1^3 = o(r/n)$ and $\delta^3 = o(r^4/n)$ and $R^3 = \omega(N)$, we have that $\delta_1 = o(r)$ and $\delta = o(r^2)$. Thus, by Lemma 4.8.9, we have that $z \sim z(x^*)$. Using this fact and computing the series of each term with $r \to 0$, we obtain $T_2(f) = O(1/r^{4-f_1})$, and so $\delta = o(r^4/n)$ and $\delta_1 = o(1/\sqrt{n})$ ensure $|\alpha^{f_1} b_2^{f_2} b_3^{f_3} b_4^{f_4} T_2(f)| \le \delta_1^{f_1} \delta^{f_2+f_3+f_4} |T_2(f)| = o(1)$.

Similarly the part involving λ can be written as

$$\sum_{\substack{f=(f_1,f_2,f_3,f_4)\in\{0,1,2,3\}\times\{0,1\}^3,\\ f_1+f_2+f_3+f_4=3}} U(f)\alpha^{f_1} b_2^{f_2} b_3^{f_3} b_4^{f_4}$$

where each $U(f)$ is a sum of terms in the following format

$$\frac{1}{\hat{Q}_3(1+\eta-c_3)}\left(-\frac{(c3-e_1)(2c_3-e_2-e_3)}{\hat{n}_3}\right.$$
$$\left.+(c_3-e_2)(c_3-e_3)\left(\frac{e_1}{\hat{Q}_3}+\frac{(c_3-e_1)}{(1+\eta-c_3)^2}\left(\frac{\eta(1+\lambda e^{\lambda}/f_1(\lambda)-\eta)}{\hat{Q}_3(1+\eta-c_3)}-\frac{1}{\hat{n}_3}\right)\right)\right)$$

where $e_1, e_2, e_3 \in \{1, 2\}$. Since $\delta_1 = o(r)$ and $\delta = o(r^2)$, by Lemma 4.8.9, we have that $\lambda(x^* + x) \sim \lambda(x^*)$. Using this fact and computing the series of $U(f)$ with $r \to 0$, we have that $U(f) = O(1/r^{4-f_1})$, and so $\delta = o(r^4/n)$ and $\delta_1^3 = o(r/n)$ ensure $|\alpha^{f_1} b_2^{f_2} b_3^{f_3} b_4^{f_4} U(f)| \leq \delta_1^{f_1} \delta^{f_2+f_3+f_4} |U(f)| = o(1)$. □

4.8.8 Establishing the maximum

In this section, we prove Lemma 4.8.11 which establishes the maximum of f_{pre} in $\hat{S}$. Recall that the region $\hat{S}$ where we want to optimise $f_{\text{pre}}(\hat{x})$ over is defined by conditions (C1)–(C4). We rewrite these conditions as follows:

(D1) $\hat{Q}_3 \geq 3\hat{n}_3 \geq 0$ and, if $\hat{n}_3 = 0$, then $\hat{Q}_3 = 0$.

(D2) $\hat{P}_2 \geq 0$;

(D3) $\hat{P}_3 \geq 0$;

(D4) $\hat{k}_0, \hat{k}_1, \hat{k}_2 \geq 0$ and $\hat{T}_2 \geq 0$ and $\hat{T}_3 \geq 0$;

These conditions are obviously a subset of the conditions (C1)–(C5), with the $\hat{n}_1 \geq 0$ being the only constraint missing, which is implied by $\hat{T}_2 \geq 0$. First we will show that $\hat{x}^*$ is the only local maximum in the interior of $\hat{S}$:

Lemma 4.8.16. The point $\hat{x}^* = (\hat{n}_1^*, \hat{k}_0^*, \hat{k}_1^*, \hat{k}_2^*)$ is the unique local maximum for f_{pre} in the interior of $\hat{S}$ and its value is

$$2r\ln n - 4r\ln r + \left(-\frac{2}{3}\ln(2) - \frac{1}{3}\ln(3) + \frac{1}{3}\right)\lambda^* + \left(-\frac{2}{9}\ln(2) - \frac{1}{9}\ln(3) + \frac{7}{36}\right)(\lambda^*)^2 + O((\lambda^*)^3).$$

We will then analyse local maximums when some condition in (D1)–(D4) is tight. The following lemma will be useful to reduce the number of cases to be analysed by giving sufficient conditions for a point not being a local maximum.

Lemma 4.8.17. Let k be a fixed positive integer and $S \subseteq \mathbb{R}$ be a bounded set. Let $f : S \to \mathbb{R}$ be a continuous function such that $f(x) = -\sum_{i=1}^{q} \ell_i(x) \ln \ell_i(x) + g(x)$, where $\ell_i(x) = \sum_{j=1}^{k} \alpha_{i,j} x_j \geq 0$ for all $x \in S$. Suppose $x^{(0)} \in S$ is such that $\ell_i(x^{(0)}) = 0$ for some i. Suppose there is $v \in \mathbb{R}^k$ such that $x^{(0)} + tv$ is in the interior of S for small enough t and

$$\frac{\mathrm{d}\, g(x^{(0)} + tv)}{\mathrm{d}\, t}|_{t=0} > C,$$

for some (possibly negative) constant C. Then $x^{(0)}$ is not a local maximum for f in S.

The following lemma gives a bound for the value of $f_{\mathrm{pre}}(\hat{x})$ for any local maximum other than $\hat{x}^*$:

Lemma 4.8.18. Let $\hat{S}_1$ be the points in $\hat{S}_m$ such that any of the constraints in (D1)–(D4) is tight. There exists a constant $\beta < -(2/9)\ln(2) - (1/9)\ln(3) + (7/36)$ such that any local maximum of $\hat{S}_m$ in $\hat{S}_1$ for f_{pre} has value at most

$$2r \ln n - 4r \ln r + \left(-\frac{2}{3}\ln(2) - \frac{1}{3}\ln(3) + \frac{1}{3}\right)\lambda^* + \beta(\lambda^*)^2.$$

Note that the constraint $\hat{Q}_3 = 0$ whenever $\hat{n}_3 = 0$ makes $\hat{S}_m$ not closed. We analyse the value of any sequence of points converging to a point with $\hat{Q}_3 > 0$ and $n_3 = 0$:

Lemma 4.8.19. Let $(\hat{x}(i))_{i\in\mathbb{N}}$ be a sequence of points in $\hat{S}_m$ converging to a point z with $\hat{Q}_3(z) > 0$ and $\hat{n}_3(z) = 0$. Then $\lim_{i\to\infty} f_{\mathrm{pre}}(x_i) = -\infty$.

Lemma 4.8.11 is trivially implied by Lemmas 4.8.16, 4.8.18, and 4.8.19. In the rest of this section, we prove these lemmas.

Proof of Lemma 4.8.16. The computations in this proof are elementary (such as computing resultants) but very lengthy. See Section A.5 for a Maple spreadsheet.

Since any local maximum must have value $\exp(\frac{\mathrm{d}\, f_{\mathrm{pre}}}{\mathrm{d}\, t}) = 1$ for any $t \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}$, by (4.61)

$$4\hat{T}_3^3 \hat{n}_3 \hat{n}_1 \lambda - 9\hat{m}_3^2 \hat{Q}_3 \hat{T}_2^2 f_3(\lambda) = 0 \tag{4.67}$$

$$\hat{n}_3 \hat{T}_2^2 \lambda^2 - 2\hat{Q}_3^2 \hat{k}_0 f_3(\lambda) = 0 \tag{4.68}$$

$$\hat{T}_3 \hat{n}_3 \hat{T}_2 \lambda^2 - \hat{k}_1 \hat{Q}_3^2 f_3(\lambda) = 0 \tag{4.69}$$

$$\hat{T}_3^2 \hat{n}_3 \lambda^2 - 2\hat{k}_2 \hat{Q}_3^2 f_3(\lambda) = 0. \tag{4.70}$$

Next we proceed to take resultants between the LHS of these equations to show that there is only one solution in the interior of $\hat{S}_m$ satisfying all of them. In these computations, we consider $f_3(\lambda)$

and λ as independent variables. The resultant of the RHS of (4.68) and (4.69) by eliminating $f_3(\lambda)$ is

$$\lambda^2\hat{T}_2\hat{n}_3\hat{Q}_3^2(6\hat{n}_1\hat{k}_0 + 4\hat{k}_2\hat{k}_0 + 2\hat{n}_1\hat{k}_1 - \hat{k}_1^2 - 6\hat{m}\hat{k}_0) = 0$$

and, since only the last term may possibly be zero in the interior of $\hat{S}_m$, this implies that any local maximum in the interior of $\hat{S}_m$ must satisfy

$$6\hat{n}_1\hat{k}_0 + 4\hat{k}_2\hat{k}_0 + 2\hat{n}_1\hat{k}_1 - \hat{k}_1^2 - 6\hat{m}\hat{k}_0 = 0, \tag{4.71}$$

and note that this determines $\hat{k}_2$ in terms of $\hat{n}_1$, $\hat{k}_1$ and $\hat{k}_0$ for any local maximum in the interior of $\hat{S}_m$. Similarly, the resultant of the RHS of (4.69) and (4.70) by eliminating λ is

$$f_3(\lambda)^2\hat{T}_3^2\hat{n}_3^2\hat{Q}_3^4(4\hat{k}_2\hat{k}_0 + 3\hat{m}\hat{k}_1 - 3\hat{n}_1\hat{k}_1 - \hat{k}_1^2 - 4\hat{n}_1\hat{k}_2)^2 = 0$$

and it implies that any local maximum in the interior of $\hat{S}_m$ must satisfy

$$4\hat{k}_2\hat{k}_0 + 3\hat{m}\hat{k}_1 - 3\hat{n}_1\hat{k}_1 - \hat{k}_1^2 - 4\hat{n}_1\hat{k}_2 = 0. \tag{4.72}$$

The resultant of the RHS of (4.71) and (4.72) by eliminating $\hat{k}_2$ is

$$4\hat{T}_2(3\hat{m}\hat{k}_0 - 3\hat{n}_1\hat{k}_0 - \hat{n}_1\hat{k}_1) = 0,$$

and it implies that any local maximum in the interior of $\hat{S}_m$ must satisfy

$$3\hat{m}\hat{k}_0 - 3\hat{n}_1\hat{k}_0 - \hat{n}_1\hat{k}_1 = 0, \tag{4.73}$$

which gives determines $\hat{k}_1$ in terms of $\hat{k}_0$ and $\hat{n}_1$.

Taking the resultant of the RHS of (4.67) and (4.71) by eliminating $\hat{k}_2$ and ignoring the factors that cannot be zero in $\hat{S}_m$ gives us

$$\begin{aligned}&-4\lambda\hat{k}_1^3\hat{n}_1\hat{k}_0 - 2\lambda\hat{k}_1^4\hat{n}_1^2 + \lambda\hat{k}_1^5\hat{n}_1 - 2\lambda\hat{k}_1^3\hat{n}_1^2\hat{k}_0 + 6\lambda\hat{k}_1^3\hat{m}\hat{n}_1\hat{k}_0 + 4\lambda\hat{k}_1^4\hat{n}_1\hat{k}_0 + 4\lambda\hat{k}_1^3\hat{n}_1\hat{k}_0^2 + 36\hat{k}_0^3 f_3(\lambda)\hat{k}_1\hat{n}_1^2\\ &- 72\hat{k}_0^3 f_3(\lambda)\hat{k}_1\hat{m}\hat{n}_1 + 36\hat{k}_0^3 f_3(\lambda)\hat{k}_1\hat{m}^2 + 72 f_3(\lambda)\hat{k}_0^4\hat{n}_1^2 - 144 f_3(\lambda)\hat{n}_1\hat{k}_0^4\hat{m} + 72 f_3(\lambda)\hat{m}^2\hat{k}_0^4 = 0\end{aligned} \tag{4.74}$$

and then we take the resultant of the RHS of (4.73) and (4.74) by eliminating $\hat{k}_1$ and ignoring the factors that cannot be zero in $\hat{S}_m$ gives us

$$\begin{aligned}&27\hat{k}_0\lambda\hat{m}^3 - 45\hat{k}_0\lambda\hat{m}^2\hat{n}_1 + 21\hat{k}_0\lambda\hat{m}\hat{n}_1^2 - 3\lambda\hat{n}_1^3\hat{k}_0 + 12\hat{m}\hat{n}_1^3 f_3(\lambda) + 12\hat{n}_1^3\lambda\hat{m}\\ &- 12\lambda\hat{m}\hat{n}_1^2 - 12\hat{n}_1^4\lambda - 4 f_3(\lambda)\hat{n}_1^4 + 12\lambda\hat{n}_1^3 = 0.\end{aligned} \tag{4.75}$$

Taking the resultant of the RHS of (4.68) and (4.71) by eliminating $\hat{k}_2$ and ignoring the factors that cannot be zero in $\hat{S}_m$ gives us

$$8\hat{k}_0^2 f_3(\lambda) + 4\lambda^2\hat{k}_0^2 + 6\lambda^2\hat{m}\hat{k}_0 + 4\lambda^2\hat{k}_1\hat{k}_0 - 4\hat{k}_0\lambda^2 - 2\lambda^2\hat{n}_1\hat{k}_0 + 8\hat{k}_1 f_3(\lambda)\hat{k}_0 + \lambda^2\hat{k}_1^2 + 2 f_3(\lambda)\hat{k}_1^2 - 2\lambda^2\hat{n}_1\hat{k}_1 = 0 \tag{4.76}$$

and then we take the resultant of the RHS of (4.73) and (4.76) by eliminating $\hat{k}_1$ and ignoring the factors that cannot be zero in $\hat{S}_m$ gives us

$$2\hat{k}_0 f_3(\lambda)\hat{n}_1^2 + \lambda^2\hat{n}_1^2\hat{k}_0 - 6\lambda^2\hat{m}\hat{n}_1\hat{k}_0 - 12\hat{k}_0 f_3(\lambda)\hat{m}\hat{n}_1 + 9\lambda^2\hat{m}^2\hat{k}_0 + 18\hat{k}_0 f_3(\lambda)\hat{m}^2 - 4\lambda^2\hat{n}_1^2 + 4\lambda^2\hat{n}_1^3 = 0, \tag{4.77}$$

and note that this determines $\hat{k}_0$ in terms of $\hat{n}_1$ and λ.

Finally we take the resultant of the RHS of (4.75) and (4.77) by eliminating $\hat{k}_0$ and ignoring the factors that cannot be zero in $\hat{S}_m$, we get

$$6\lambda\hat{m}\hat{n}_1 + 6f_3(\lambda)\hat{m}\hat{n}_1 + 3\lambda^2\hat{m}\hat{n}_1 - 6\hat{m}\lambda - 6\lambda\hat{n}_1^2 - 2f_3(\lambda)\hat{n}_1^2 - \lambda^2\hat{n}_1^2 + 6\hat{n}_1\lambda = 0. \tag{4.78}$$

We can then use the equation determining λ (that is, $\lambda f_2(\lambda)/f_3(\lambda) = \hat{Q}_3/\hat{n}_3$) by replacing $\hat{k}_0, \hat{k}_1$ and $\hat{k}_2$ by the values determined by $\hat{n}_1, \lambda$ and m and taking the resultant with (4.78) by eliminating $\hat{k}_0$ and ignoring the factors that cannot be zero in $\hat{S}_m$:

$$3\hat{m}e^{2\lambda} - 9\hat{m}^2e^{2\lambda} + 3\hat{m}e^{2\lambda}\lambda - \lambda e^{2\lambda} + 3\hat{m}e^{\lambda}\lambda - e^{\lambda}\lambda + 2\lambda - 12\hat{m}\lambda + 9\hat{m}^2 - 3\hat{m} + 18\lambda\hat{m}^2 = 0$$

which has two solutions for $\hat{m}$: $\hat{m} = 1/3$ (which is false) or

$$\hat{m} = \frac{1}{3}\frac{\lambda f_1(\lambda)g_2(\lambda)}{f_2(2\lambda)},$$

which has a unique positive solution λ^* by Lemma 4.7.4, which defines $\hat{x}^*$. Thus, $\hat{x}^*$ is the only point in the interior of $\hat{S}_m$ such that all partial derivatives at it are zero. We now show that $\hat{x}^*$ is a local maximum. Using the second partial derivatives computed in (4.66) and the series of the determinants of each leading principal submatrix with $\lambda \to 0$, we have that the Hessian at $\hat{x}^*$ is negative definite, which implies that $\hat{x}^*$ is a local maximum.

By writing $f_{\text{pre}}(x^*)$ in terms of λ^* and computing its series with $\lambda \to 0$, we obtain

$$2r\ln n - 4r\ln r + \left(-\frac{2}{3}\ln(2) - \frac{1}{3}\ln(3) + \frac{1}{3}\right)\lambda^* + \left(-\frac{2}{9}\ln(2) - \frac{1}{9}\ln(3) + \frac{7}{36}\right)(\lambda^*)^2 + O((\lambda^*)^3).$$

□

Proof of Lemma 4.8.17. Let $I \in [q]$ be the set of indices such that $\ell_i(x^{(0)}) = 0$. We compute the derivative of $f(x^{(0)} + tv)$ at $t = 0$, using the fact that ℓ_i is a linear function,

$$\begin{aligned}
\frac{\mathrm{d}\, f(x^{(0)}+tv)}{\mathrm{d}\, t}\Big|_{t=0} &\geq C + \sum_{i=1}^{q} \lim_{t\to 0^+} \frac{\big(-\ell_i(x^{(0)}+tv)\ln \ell_i(x^{(0)}+tv) + \ell_i(x^{(0)})\ln \ell_i(x^{(0)})\big)}{t} \\
&= C + \sum_{i=1}^{q} \lim_{t\to 0^+} \frac{\big(-\ell_i(tv)\ln \ell_i(x^{(0)}+tv) + \ell_i(x^{(0)})(\ln \ell_i(x^{(0)}) - \ln(\ell_i(x^{(0)}+tv)\big)}{t} \\
&= C + \sum_{i=1}^{q} \lim_{t\to 0^+} \Big(-\ell_i(v)\ln \ell_i(x^{(0)}+tv)\Big) - \sum_{i\in[q]\setminus I} \lim_{t\to 0^+} \frac{\ell_i(x^{(0)})}{t} \ln\left(1 + t\frac{\ell_i(v)}{\ell_i(x^{(0)})}\right) \\
&= C + \sum_{i=1}^{q} \lim_{t\to 0^+} \Big(-\ell_i(v)\ln \ell_i(x^{(0)}+tv)\Big) - \sum_{i\in[q]\setminus I} \ell_i(v).
\end{aligned}$$

Since $x_0 + tv$ is in the interior of S for small enough but positive t, we have that $\ell_i(v) > 0$ for all $i \in I$. For $i \in [q] \setminus I$, we have that $\ell_i(v)\ln \ell_i(x^{(0)} + tv) + \ell_i(v)$ is bounded. For $i \in I$, using the fact that $\ell_i(v) > 0$, we have that $\ell_i(v)\lim_{t\to 0^+} \ln \ell_i(x^{(0)} + tv) = -\infty$. Thus, we conclude that

$$\frac{\mathrm{d}\, f(x^{(0)}+tv)}{\mathrm{d}\, t}\Big|_{t=0} > 0,$$

which shows that $x^{(0)}$ is not a local maximum. □

Proof of Lemma 4.8.18. We want to find the local maximums in $\hat{S}_1$, which is the set of points in $\hat{S}_m$ such that any of the constraints in (D1)–(D4) is tight. Recall that the constraints (D1)–(D4) are the following:

(D1) $\hat{Q}_3 \geq 3\hat{n}_3 \geq 0$ and, if $\hat{n}_3 = 0$, then $\hat{Q}_3 = 0$.

(D2) $\hat{P}_2 \geq 0$;

(D3) $\hat{P}_3 \geq 0$;

(D4) $\hat{k}_0, \hat{k}_1, \hat{k}_2 \geq 0$ and $\hat{T}_2 \geq 0$ and $\hat{T}_3 \geq 0$;

We split the analysis in the following cases:

Case 1: $\hat{Q}_3 = \hat{n}_3 = 0$;

Case 2: $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_3 = 0$;

Case 3: $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_2 = 0$;

Case 4: $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_3 \neq 0$ and $\hat{P}_2 \neq 0$;

Case 5: $\hat{Q}_3 > 3\hat{n}_3 > 0$ and $\hat{P}_3 = 0$;

Case 6: $\hat{Q}_3 > 3\hat{n}_3 > 0$ and $\hat{P}_2 = 0$.

We will use the definitions in (4.42) many times in the analysis. For Maple spreadsheets with the computations below see Section A.6 for Case 1, Section A.7 for Case 2, Section A.8 for Case 3, Section A.9 for Case 4, Section A.10 for Case 5, and Section A.11 for Case 6.

Case 1: Assume that $\hat{Q}_3 = \hat{n}_3 = 0$. Recall that, by definition, we have that $\hat{Q}_3 = 3\hat{m} - \hat{n}_1 - 2\hat{k}_0 - 2\hat{k}_1 - 2\hat{k}_2$, $\hat{T}_2 = 2\hat{n}_1 - 2\hat{k}_0 - \hat{k}_1$, and $\hat{T}_3 = 3\hat{m} - 3\hat{n}_1 - \hat{k}_1 - 2\hat{k}_2$. Thus,

$$\hat{Q}_3 = \hat{T}_2 + \hat{T}_3. \tag{4.79}$$

Moreover, $\hat{T}_2 \geq 0$ and $\hat{T}_3 \geq 0$ are constraints in the definition of $\hat{S}_m$. Thus, since $\hat{Q}_3 = 0$, we have that $\hat{T}_2 = \hat{T}_3 = 0$. Recall that $\hat{n}_3 = 1 - \hat{n}_1 - \hat{k}_0 - \hat{k}_1 - \hat{k}_2$. Hence, we obtain the following equations:

$$\begin{aligned}
&1 - \hat{n}_1 - \hat{k}_0 - \hat{k}_1 - \hat{k}_2 = 0,\\
&2\hat{n}_1 - 2\hat{k}_0 - \hat{k}_1 = 0,\\
&3\hat{m} - 3\hat{n}_1 - \hat{k}_1 - 2\hat{k}_2.
\end{aligned}$$

By solving this system of equation, we obtain the following values for $\hat{n}_1$, $\hat{k}_1$, and $\hat{k}_2$ in terms of $\hat{k}_0$ and $\hat{m}$:

$$\begin{aligned}
\hat{n}_1 &= 2 - 3\hat{m};\\
\hat{k}_1 &= 4 - 6\hat{m} - 2\hat{k}_0;\\
\hat{k}_2 &= -5 + 9\hat{m} + \hat{k}_0.
\end{aligned}$$

Moreover, $\hat{P}_3 = 3(m - \hat{n}_1) = -6 + 12\hat{m}$, $\hat{P}_2 = 2(\hat{n}_1 - \hat{k}_0) = 4 - 6\hat{m} - 2\hat{k}_0$ Thus, $f_{\text{pre}}(x)$ depends only on $\hat{k}_0$ and we get

$$\begin{aligned}
f_{\text{pre}}(x) = f(\hat{k}_0) := {}& h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{m}_2) - h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2)\\
&- h_n(\hat{m}_3) - 2h_n(\hat{m}_2^-) - \hat{k}_2 \ln 2 - \hat{m}_2^- \ln 2 - \hat{m}_3 \ln 6,
\end{aligned}$$

where $\hat{k}_0 \in [5 - 9\hat{m}, 2 - 3\hat{m}]$. We have that

$$\exp\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, \hat{k}_0}\right) = \frac{(3\hat{m} - 2 + \hat{k}_0)^2}{(-5 + 9\hat{m} + \hat{k}_0)\hat{k}_0} \quad \text{and} \quad \frac{\mathrm{d}^2 f}{\mathrm{d}^2\, \hat{k}_0} = \frac{3\hat{m}\hat{k}_0 - \hat{k}_0 + 33\hat{m} - 10 - 27\hat{m}^2}{\hat{k}_0(-5 + 9\hat{m} + \hat{k}_0)(3\hat{m} - 2 + \hat{k}_0)}.$$

For $\hat{k}_0 \in [5-9\hat{m}, 2-3\hat{m}]$, the denominator of the second derivative is always nonnegative and its numerator is always negative for sufficiently small r (that is, sufficiently large n). Hence, f is strictly concave. Thus, there is a unique maximum and it satisfies:

$$\frac{(3\hat{m}-2+\hat{k}_0)^2}{(-5+9\hat{m}+\hat{k}_0)\hat{k}_0} = 1,$$

that is,

$$\hat{k}_0 = \frac{(3\hat{m}-2)^2}{3\hat{m}-1}.$$

We then compute the series for $f(\hat{k}_0)$ at this point with λ^* going to zero (by using (4.48)):

$$f(\hat{k}_0) = 2r\ln n - 4r\ln r + \left(-\ln(2) - \frac{1}{3}\ln(3) + \frac{1}{3}\right)\lambda^* + (\lambda^*)^2\ln(\lambda^*) + O((\lambda^*)^2).$$

Case 2: Assume that $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_3 = 0$. Since $\hat{P}_3 = 0$ and $\hat{P}_3 = 3(\hat{m}-\hat{n}_1)$ by definition (see (4.42)), we have that $\hat{n}_1 = \hat{m}$. Moreover, since $\hat{T}_3 = \hat{P}_3 - \hat{k}_1 - 2\hat{k}_2$ and $\hat{T}_3, \hat{k}_1, \hat{k}_1 \geq 0$ are constraints in the definition of $\hat{S}_1$, we have that $\hat{k}_1 = 0$ and $\hat{k}_2 = 0$. Using $\hat{Q}_3 = 3\hat{n}_3$ and their definitions in (4.42), we have that $3\hat{m} - \hat{n}_1 - 2\hat{k}_0 - 2\hat{k}_1 - 2\hat{k}_2 = 3(1-\hat{n}_1-\hat{k}_0-\hat{k}_1-\hat{k}_2)$ and so $\hat{k}_0 = 3 - 3\hat{m} - 2\hat{n}_1 = 3 - 5\hat{m}$. Thus, we only have to compute the value of f_{pre} in the point $(\hat{m}, 3-5\hat{m}, 0, 0)$. By computing the series of f_{pre} in this point with λ^* going to zero (by using (4.48)), we get

$$2r\ln(n) - 4r\ln r + \left(\frac{1}{3} - \ln(2) - \frac{1}{3}\ln(3)\right)\lambda^* + O((\lambda^*)^2).$$

Case 3: Assume that $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_2 = 0$. Since $\hat{P}_2 = 0$ and $\hat{P}_2 = 2(\hat{n}_1 - \hat{k}_0)$ by definition (see (4.42)), we have that $\hat{k}_0 = \hat{n}_1$. Moreover, since $\hat{T}_2 = \hat{P}_2 - \hat{k}_1$ and $\hat{T}_2, \hat{k}_1 \geq 0$ are constraints in the definition of $\hat{S}_m$, we have that $\hat{k}_1 = 0$. Using $\hat{Q}_3 = 3\hat{n}_3$ and their definition in (4.42), we have that $3\hat{m} - \hat{n}_1 - 2\hat{k}_0 - 2\hat{k}_1 - 2\hat{k}_2 = 3(1-\hat{n}_1-\hat{k}_0-\hat{k}_1-\hat{k}_2)$ and so $\hat{k}_2 = 3 - 3\hat{m} - 3\hat{n}_1$. So let $f(\hat{n}_1) := f_{\text{pre}}(\hat{n}_1, \hat{n}_1, 0, 3-3\hat{m}-3\hat{n}_1)$ and $\hat{n}_1 \in [2-3\hat{m}, 1-\hat{m}]$. We have that

$$\exp\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, \hat{n}_1}\right) = \frac{8(1-\hat{n}_1-\hat{m})^3}{(\hat{m}-\hat{n}_1)^2(-2\hat{n}_1+3\hat{m})} \quad \text{and} \quad \frac{\mathrm{d}^2 f}{\mathrm{d}^2\, \hat{n}_1} = \frac{(2\hat{m}-1)(4-7\hat{m}-\hat{n}_1)}{(-2+\hat{n}_1+3\hat{m})(1-\hat{n}_1-m)(\hat{m}-\hat{n}_1)}$$

For $\hat{n}_1 \in [2-3\hat{m}, 1-\hat{m}]$, the denominator of the second derivative is always nonnegative and its numerator is always negative for sufficiently small r. Hence, f is strictly concave. Thus, there is unique maximum satisfying

$$8(1-\hat{n}_1-\hat{m})^3 - (\hat{m}-\hat{n}_1)^2(-2\hat{n}_1+3\hat{m})) = 0,$$

which has a unique real solution at $1/2 + \alpha r$, where $\alpha \approx -2.03566$, which is the real solution for

$$9\alpha^3 + 25\alpha^2 + 19\alpha + 11 = 0.$$

We then compute the value of the function f at $1/2 + \alpha r$:

$$2r \ln(n) + 2r \ln r + \beta,$$

with $\beta \approx 1.9389$.

Case 4: Now suppose that $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_3 > 0$ and $\hat{P}_2 > 0$. By Lemma 4.8.17, we do not need to consider the cases $\hat{k}_0 = 0$, $\hat{k}_1 = 0$, $\hat{k}_2 = 0$, $\hat{T}_3 = 0$, $\hat{T}_2 = 0$ and $\hat{m}_3 = 0$.

Since $\hat{Q}_3 = 3\hat{n}_3$, we have that $\hat{k}_0 = 3 - 3\hat{m} - 2\hat{n}_1 - \hat{k}_1 - \hat{k}_2$. Thus we analyse the function

$$f(\hat{n}_1, \hat{k}_1, \hat{k}_2) := f_{\text{pre}}(\hat{n}_1, 3 - 3\hat{m} - 2\hat{n}_1 - \hat{k}_1 - \hat{k}_2, \hat{k}_1, \hat{k}_2).$$

We have that, for any local maximum in this case,

$$\exp\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, \hat{n}_1}\right) = \frac{8\hat{P}_3^3\hat{k}_0^2\hat{n}_3^2\hat{n}_1}{\hat{m}_3^2\hat{T}_2^6} = 1;$$
$$\exp\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, \hat{k}_1}\right) = \frac{2\hat{P}_3\hat{k}_0}{\hat{T}_2\hat{k}_1} = 1;$$
$$\exp\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, \hat{k}_2}\right) = \frac{\hat{P}_3^2\hat{k}_0}{\hat{T}_2^2\hat{k}_2} = 1;$$

and so

$$8\hat{P}_3^3\hat{k}_0^2\hat{n}_3^2\hat{n}_1 - \hat{m}_3^2\hat{T}_2^6 = 0; \tag{4.80}$$
$$2\hat{P}_3\hat{k}_0 - \hat{T}_2\hat{k}_1 = 0; \tag{4.81}$$
$$\hat{P}_3^2\hat{k}_0 - \hat{T}_2^2\hat{k}_2 = 0. \tag{4.82}$$

By taking the resultant of the RHS of (4.81) and (4.82), by eliminating $\hat{k}_1$, we get

$$9\hat{n}_3^2(\hat{k}_2 - 3 + 3\hat{m} + 4\hat{n}_1)$$
$$(9\hat{m}^2\hat{k}_2 - 6\hat{m}\hat{n}_1\hat{k}_2 + n1^2\hat{k}_2 + 27\hat{m}^3 - 27\hat{m}^2 - 36\hat{m}^2\hat{n}_1 - 9\hat{m}\hat{n}_1^2 + 54\hat{m}\hat{n}_1 - 27\hat{n}_1^2 + 18\hat{n}_1^3) = 0.$$

Using $\hat{Q}_3 = 3\hat{n}_3$ and their definition in (4.42), we have that $3\hat{m} - \hat{n}_1 - 2\hat{k}_0 - 2\hat{k}_1 - 2\hat{k}_2 = 3(1 - \hat{n}_1 - \hat{k}_0 - \hat{k}_1 - \hat{k}_2)$ and so $3\hat{m} - 3 = \hat{k}_0 + \hat{k}_1 + \hat{k}_2 - 2\hat{n}_1$. Thus, $\hat{k}_2 - 3 + 3\hat{m} + 4\hat{n}_1 = \hat{k}_0 + \hat{k}_1 + 2\hat{k}_2 + 2\hat{n}_1 > 0$ since we already excluded the case $\hat{k}_0 = 0$. Recall that in this case we have $\hat{n}_3 > 0$. Thus, for any local maximum in this case,

$$9\hat{m}^2\hat{k}_2 - 6\hat{m}\hat{n}_1\hat{k}_2 + n1^2\hat{k}_2 + 27\hat{m}^3 - 27\hat{m}^2 - 36\hat{m}^2\hat{n}_1 - 9\hat{m}\hat{n}_1^2 + 54\hat{m}\hat{n}_1 - 27\hat{n}_1^2 + 18\hat{n}_1^3 = 0. \tag{4.83}$$

This implies that $\hat{k}_2$ can be determined in terms of $\hat{n}_1$:

$$\hat{k}_2 = \frac{9(-3\hat{m} + 3 - 2\hat{n}_1)(\hat{m} - \hat{n}_1)^2}{(3\hat{m} - \hat{n}_1)^2}.$$

By taking the resultant of the RHS of (4.81) and (4.82), by eliminating $\hat{k}_2$, we get

$$44\hat{n}_3^2(\hat{k}_1 - 2\hat{n}_1)(9\hat{m}^2\hat{k}_1 - 6\hat{m}\hat{n}_1\hat{k}_1 + n1^2\hat{k}_1 + 36\hat{m}^2\hat{n}_1 - 36\hat{m}\hat{n}_1 + 36\hat{n}_1^2 - 24\hat{n}_1^3 - 12\hat{m}\hat{n}_1^2) = 0.$$

In this case $\hat{n}_3 > 0$. Moreover, $2\hat{n}_1 - \hat{k}_1 = 0$ implies, by the definitions in (4.42), that $\hat{T}_2 = \hat{P}_2 - \hat{k}_1 = 2\hat{n}_1 - 2\hat{k}_0 - \hat{k}_1 \leq 0$ since $\hat{k}_0 \geq 0$ in $\hat{S}_m$. But we have already excluded the case $\hat{T}_2 = 0$. Thus,

$$9\hat{m}^2\hat{k}_1 - 6\hat{m}\hat{n}_1\hat{k}_1 + n1^2\hat{k}_1 + 36\hat{m}^2\hat{n}_1 - 36\hat{m}\hat{n}_1 + 36\hat{n}_1^2 - 24\hat{n}_1^3 - 12\hat{m}\hat{n}_1^2 = 0. \tag{4.84}$$

This implies that $\hat{k}_1$ can be determined in terms of $\hat{n}_1$:

$$\hat{k}_1 = \frac{12\hat{n}_1(-3\hat{m} + 3 - 2\hat{n}_1)(\hat{m} - \hat{n}_1)}{(3\hat{m} - \hat{n}_1)^2}.$$

We take the resultant of the RHS of (4.80) and (4.84) by eliminating $\hat{k}_1$ and then the resultant of the polynomial obtained with the RHS of (4.83) by eliminating $\hat{k}_2$ and ignoring the factors that cannot be zero in $\hat{S}_m$ and we obtain:

$$18\hat{m} - 36\hat{m}^2 + 18\hat{m}^3 - 18\hat{n}_1 + 18\hat{m}\hat{n}_1 - 3\hat{m}^2\hat{n}_1 + 22\hat{n}_1^2 - 16\hat{m}\hat{n}_1^2 - 7\hat{n}_1^3 = 0.$$

This cubic equation has one real solution for $\hat{n}_1$ and two complex solutions because the discriminant Δ of the polynomial above is $-63/4 + O(r)$, which is negative for sufficiently large n. For we have that the real solution is $1/2 - r - 6r^2 - O(r^3)$ and so the value of the function f_{pre} at this point is, by using (4.48),

$$2r\ln n - 4r\ln r + \left(\frac{1}{3} - \frac{1}{3}\ln(3) - \frac{2}{3}\ln(2)\right)\lambda^* + \left(\frac{11}{72} - \frac{2}{9}\ln(2) - \frac{1}{9}\ln(3)\right)(\lambda^*)^2 + O((\lambda^*)^3).$$

Case 5: Now suppose that $\hat{P}_3 = 0$ and $\hat{Q}_3 > \hat{n}_3 > 0$. Since $\hat{P}_3 = 0$ and $\hat{P}_3 = 3(\hat{m} - \hat{n}_1)$ by definition (see (4.42)), we have that $\hat{n}_1 = \hat{m}$. Moreover, since $\hat{T}_3, \hat{k}_1, \hat{k}_2 \geq 0$ in $\hat{S}_m$ and $\hat{T}_3 = \hat{P}_3 - \hat{k}_1 - 2\hat{k}_2$ by definition, we have that $\hat{k}_1 = 0$ and $\hat{k}_2 = 0$. Thus, for any local maximum with $\hat{P}_3 = 0$, it suffices to analyse

$$f(\hat{n}_3) := f_{\text{pre}}(\hat{m}, 1 - \hat{m} - \hat{n}_3, 0, 0),$$

where $\hat{n}_3 \in (0, 1 - \hat{m})$, since by definition $\hat{k}_0 = 1 - \hat{n}_1 - \hat{k}_1 - \hat{k}_2 - \hat{n}_3 = 1 - \hat{m} - \hat{n}_3 \geq 0$ and $\hat{Q}_3 = 3\hat{m} - \hat{n}_1 - 2\hat{k}_0 - 2\hat{k}_1 - 2\hat{k}_2 = 4\hat{m} - 2 + 2\hat{n}_3 \geq 0$. We do not have to analyse the value at the endpoints of the interval for $\hat{n}_3$ as they were already considered in cases before. Also, in this case $\hat{Q}_3 = \hat{P}_2$, thus we do not have to check the case $\hat{P}_2 = 0$. Thus, it suffices to consider points satisfying

$$\exp\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, \hat{n}_3}\right) = \frac{2(1 - \hat{m} - \hat{n}_3) f_3(\lambda)}{\hat{n}_3 \lambda^2} = 1,$$

where $\lambda f_2(\lambda)/f_3(\lambda) = \hat{Q}_3/\hat{n}_3$. The equation below is equivalent to

$$\hat{n}_3 = \frac{(1 - m) f_3(\lambda)}{f_2(\lambda)}.$$

Combining this with the equation defining λ implies:

$$r = \frac{1}{2} \frac{-2e^\lambda + 2 + \lambda + \lambda e^\lambda}{2e^\lambda - 2 - 3\lambda + \lambda e^\lambda},$$

and since r goes to zero so does λ. We have that

$$r = \frac{1}{24}\lambda + O(\lambda^2),$$

which implies

$$\lambda = 2\lambda^* + O(\lambda^*)^2.$$

We then compute the series of $f(\hat{n}_3)$ with λ going to zero:

$$\begin{aligned} &2r \ln n - 4r \ln r + \left(-\frac{1}{2}\ln(2) - \frac{1}{6}\ln(3) + \frac{1}{6}\right)\lambda + O(\lambda^2) \\ =&2r \ln n - 4r \ln r + \left(-\ln(2) - \frac{1}{3}\ln(3) + \frac{1}{3}\right)\lambda^* + O((\lambda^*)^2). \end{aligned}$$

Case 6: Now suppose that $\hat{P}_2 = 0$ and $\hat{Q}_3 > \hat{n}_3 > 0$. We have that $\hat{P}_2 = 2(\hat{n}_1 - \hat{k}_0)$. Thus, we have $\hat{n}_1 = \hat{k}_0$ since $\hat{P}_2 = 0$. Moreover, since $\hat{T}_2, \hat{k}_1 \geq 0$ in $\hat{S}_m$ and $\hat{T}_2 = \hat{P}_2 - \hat{k}_1$ by definition, we have that $\hat{k}_1 = 0$. Thus, we only need to analyse

$$f(\hat{n}_1, \hat{k}_2) := f_{\mathrm{pre}}(\hat{n}_1, \hat{n}_1, 0, \hat{k}_2),$$

where $\hat{Q}_3 > 3\hat{n}_3 > 0$ and $\hat{P}_3 > 0$. Thus, it suffices to consider points satisfying

$$\exp\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, \hat{n}_1}\right) = \frac{2}{9} \frac{\hat{n}_3^2 \lambda^3}{\hat{m}_3^2 f_3(\lambda)^2} = 1 \quad \text{and} \quad \exp\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, \hat{k}_2}\right) = \frac{1}{2} \frac{\hat{n}_3 \lambda^2}{\hat{k}_2 f_3(\lambda)} = 1,$$

where $\lambda f_2(\lambda)/f_3(\lambda) = \hat{Q}_3/\hat{n}_3$. The second equation implies that for any local maximum

$$\hat{k}_2 = \frac{1}{2} \frac{(1 - 2\hat{n}_1)\lambda^2}{f_2(\lambda)}.$$

By using this with the derivative w.r.t. $\hat{n}_1$, we get

$$\hat{n}_1 = \frac{\lambda^{3/2}\sqrt{2} - 3f_2(\lambda)\hat{m}}{2\lambda^{3/2}\sqrt{2} - 3f_2(\lambda)}.$$

By putting this together with the equation defining λ, we have that

$$\frac{(-e^{\lambda} + 1 + \sqrt{2\lambda})\lambda}{f_3(\lambda)} = 0,$$

which has a unique solution $\ell^* \approx 0.8267$. For $\lambda = \ell^*$, we have $\hat{n}_1 = \frac{1}{2} + \alpha r$, with $\alpha \approx 1.4887$ and $\hat{k}_2 = \beta(1/2 - \hat{n}_1)$ with $\beta \approx 0.1173$. By using this values of $\hat{n}_1$ and $\hat{k}_2$, we evaluate the function $f(\hat{n}_1, \hat{k}_2)$ as

$$2r \ln n - 4r \ln r + 6 \ln r + O(r),$$

since $\alpha < 0$, $0 < \beta < 2$ and $\lambda > 0$. □

Proof of Lemma 4.8.19. Let $\hat{x}(i) = (\hat{n}_1(i), \hat{k}_0(i), \hat{k}_1(i), \hat{k}_2(i))$ and similarly for $\hat{Q}_3(i), \hat{n}_3(i)$, etc. Let $\lambda(i)$ be such that $\lambda(i) f_2(\lambda(i))/f_1(\lambda(i)) = \hat{Q}_3(i)/\hat{n}_3(i)$. Recall that

$$\begin{aligned} f_{\text{pre}}(\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2) =& h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(\hat{m}_2) \\ & - h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{n}_3) - h_n(\hat{m}_3) \\ & - h_n(\hat{T}_3) - h_n(\hat{T}_2) - 2h_n(\hat{m}_2^-) \\ & - \hat{k}_2 \ln 2 - \hat{m}_2^- \ln 2 - \hat{m}_3 \ln 6 \\ & + \hat{n}_3 \ln f_3(\lambda) - \hat{Q}_3 \ln \lambda. \end{aligned}$$

Since $S \subseteq [0,1]^4$ and the fact that $|y \ln y| \leq 1/e$ for $y \in [0,1]$, we have that $f_{\text{pre}}(x) \leq C + \hat{n}_3 \ln f_3(\lambda) - \hat{Q}_3 \ln \lambda$ for some constant C. Thus, it suffices to show that $\hat{n}_3(i) \ln f_3(\lambda(i)) - \hat{Q}_3(i) \ln \lambda(i) \to -\infty$ as $i \to \infty$.

Since $\hat{Q}_3(i)$ converges to a positive number and $\hat{n}_3(i)$ converges to 0, we have that $\hat{Q}_3(i)/\hat{n}_3(i) \to \infty$. This implies that $\lambda(i) \to \infty$. Thus,

$$\begin{aligned} & \hat{n}_3(i) \ln f_3(\lambda(i)) - \hat{Q}_3(i) \ln \lambda(i) \\ & \leq \hat{n}_3(i)\lambda(i) - \hat{Q}_3(i) \ln \lambda(i), \quad \text{since } f_3(\lambda) \leq \exp(\lambda) \\ & \leq \hat{n}_3(i) \frac{\hat{Q}_3(i)}{\hat{n}_3(i)} - \hat{Q}_3(i) \ln \lambda(i), \quad \text{since } \lambda(i) \leq \hat{Q}_3(i)/\hat{n}_3(i) \\ & = \hat{Q}_3(i)(1 - \ln(\lambda(i))) \to -\infty, \quad \text{since } \lambda(i) \to \infty \text{ and } \liminf_{i\to\infty} \hat{Q}_3(i) > 0. \end{aligned}$$

□

4.8.9 Approximation around the maximum and bounding the tail

In this section, we approximate the sum of $\exp(nf_{\text{pre}}(x))$ over a set of points 'close' to x^* and bound the sum for the points 'far' from x^*. More specifically, we prove Lemmas 4.8.12 and 4.8.13.

Proof of Lemma 4.8.12. We use Lemma 4.8.15 and Lemma 4.8.16, which were proved in Section 4.8.7 and Section 4.8.8, resp. Let $x \in B$. By Lemma 4.8.15, since $\delta_1^3 = o(r/n)$ and $\delta^3 = o(r^4/n)$, we have that

$$n\frac{\partial f_{\text{pre}}(\hat{x}^* + \hat{x})}{\partial t_1 \partial t_2 \partial t_3} t_1(\hat{x}) t_2(\hat{x}) t_3(\hat{x}) = o(1),$$

for any $t_1, t_2, t_3 \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}$. By Lemma 4.8.11, we have that

$$\frac{\partial f_{\text{pre}}(\hat{x}^*)}{\partial t} = 0,$$

for any $t \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}$. Thus, by Taylor's approximation (Theorem 2.4.1),

$$\exp\left(nf_{\text{pre}}(\hat{x}^* + \hat{x})\right) = \exp\left(nf_{\text{pre}}(\hat{x}^*) + \frac{n\hat{x}^T H \hat{x}}{2} + o(1)\right), \tag{4.85}$$

where H is the Hessian of f_{pre} at x^*. Using the fact that $\hat{B} \cap ((\mathbb{Z}^4 - x^*)/n)$ is a finite set for each n and Lemma 2.7.1, this implies that

$$\sum_{\substack{\hat{x} \in \hat{B} \\ \hat{x} \in (\mathbb{Z}^4 - x^*)/n}} \exp\left(nf_{\text{pre}}(\hat{x}^* + \hat{x})\right) \sim \sum_{\substack{\hat{x} \in \hat{B} \\ \hat{x} \in (\mathbb{Z}^4 - x^*)/n}} \exp\left(nf_{\text{pre}}(\hat{x}^*) + \frac{n\hat{x}^T H \hat{x}}{2}\right). \tag{4.86}$$

So we need to show that

$$\sum_{\substack{\hat{x} \in \hat{B} \\ \hat{x} \in (\mathbb{Z}^4 - x^*)/n}} \exp\left(\frac{n\hat{x}^T H \hat{x}}{2}\right) \sim 144\sqrt{3}\pi^2 r^{7/2} n^2. \tag{4.87}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We rewrite the summation in the LHS of (4.87) over $\hat{C} := \{y : Ay \in \hat{B}\}$ as

$$\sum_{\substack{y\in\hat{C} \\ y\in(\mathbb{Z}^4-A^{-1}x^*)/n}} \exp\left(\frac{ny^T(A^THA)y}{2}\right)$$

$$= \sum_{\substack{y\in\hat{C} \\ y\in(\mathbb{Z}^4-A^{-1}x^*)/n}} \exp\left(\frac{-ny^T(A^TH_0A)y}{2r^2} - \frac{ny^T(A^TTA)y}{2r} + \frac{O(ny^TJy)}{2}\right), \quad (4.88)$$

by Lemma 4.8.14 (for the definitions of H_0 and T, see (4.58)). Note that the condition "$\hat{x} \in (\mathbb{Z}^4 - x^*)/n$" became "$y \in (\mathbb{Z}^4 - A^{-1}x^*)/n$" because A is an integer invertible matrix and

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is also an integer matrix. Using the definition of H_0 and T in (4.58), we have that

$$\sum_{\substack{y\in\hat{C} \\ y\in(\mathbb{Z}^4-A^{-1}x^*)/n}} \exp\left(\frac{-ny(A^TH_0A)y}{2r^2} - \frac{ny(A^TTA)y}{2r} + \frac{nO(y^TJy)}{2}\right) =$$

$$= \sum_{\substack{y\in\hat{C} \\ y\in(\mathbb{Z}^4-A^{-1}x^*)/n}} \exp\Bigg(- \frac{n}{12r^2}y_2^2 - \frac{n}{6r^2}y_2y_3 - \frac{n}{6r^2}y_2y_4 - \frac{7n}{72r^2}y_3^2 - \frac{4n}{9r^2}y_3y_4 - \frac{n}{6r^2}y_4^2$$
$$- \frac{n}{2r}y_1^2 + \frac{n}{r}y_1y_2 + \frac{n}{r}y_1y_3$$
$$- \frac{11n}{30r}y_2^2 - \frac{2n}{5r}y_2y_3 - \frac{n}{15r}y_2y_4 - \frac{31n}{180r}y_3^2 + \frac{2n}{45r}y_3y_4 + \frac{n}{45r}y_4^2 + \frac{nO(y^TJy)}{2}\Bigg), \quad (4.89)$$

The set $\hat{C} = \{y : Ay \in \hat{B}\}$ can be described as

$$\hat{C} = \big\{y \in \mathbb{R}^4 : |y_1| \leq \delta_1,\ |y_i| \leq \delta \text{ for } i = 2, 3, 4\big\},$$

since $\hat{B}$ was defined as

$$\hat{B} = \Big\{\hat{x} \in \mathbb{R}^4 : \hat{x} = \gamma_1 z_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4,\ |\gamma_1| \leq \delta_1 \text{ and } |\gamma_i| \leq \delta \text{ for } i = 2, 3, 4\Big\}.$$

Thus, the ranges of the summation for different variables y_i's are independent. We have that

$$\sum_{\substack{|y_1|\le\delta_1\\ y_1\in(\mathbb{Z}-(A^{-1}x^*)_1)/n}} \exp\left(-\frac{n}{2r}y_1^2+\frac{n}{r}y_1y_2+\frac{n}{r}y_1y_3+\sum_{j=1}^4 O(ny_1y_j)\right)$$
$$=\sum_{\substack{|\tilde{y}_1|\le\delta_1\sqrt{n/r}\\ \tilde{y}_1\in(\mathbb{Z}-(A^{-1}x^*)_1)/\sqrt{rn}}} \exp\left(-y_1^2/2+\tilde{y}_1\tilde{y}_2+\tilde{y}_1\tilde{y}_3+\sum_{j=1}^4 O(r\tilde{y}_1\tilde{y}_j)\right),$$

where $\tilde{y}_i=\sqrt{n}y_i/\sqrt{r}$ for $i=2,3$. We apply Lemma 4.6.1 with $\alpha=1/2$, $\beta=\tilde{y}_2+\tilde{y}_3$, $\phi=O(r)=o(1)$, $\psi=O(r\tilde{y}_2+r\tilde{y}_3+r\tilde{y}_4)=O(r)=o(1)$, $s_n=\sqrt{rn}\to\infty$ and $T_n=\delta_1\sqrt{n/r}\to\infty$:

$$\sum_{\substack{|\tilde{y}_1|\le\delta_1\sqrt{n/r}\\ y_1\in(\mathbb{Z}-(A^{-1}x^*)_1)/(r\sqrt{n})}} \exp\left(-y_1^2/2+\tilde{y}_1\tilde{y}_2+\tilde{y}_1\tilde{y}_3+\sum_{j=1}^4 O(r\tilde{y}_1\tilde{y}_j)\right)\sim\sqrt{2rn\pi}\exp((\tilde{y}_2+\tilde{y}_3)^2/2).$$

We then proceed similarly for y_2, y_3 and y_4. Fix y_3 and y_4. Set $\check{y}_i=\sqrt{n}y_i/r$ for $i=3,4$. We apply Lemma 4.6.1 with $\alpha=1/12$, $\beta=-(1/6)\check{y}_3-(1/6)\check{y}_4$, $\phi=-(2r/15)+O(r^2)=o(1)$, $\psi=(3r/5)\check{y}_3-r/15+\sum_{j=3}^4 O(r^2\check{y}_j)=o(1)$, $s_n=r\sqrt{n}\to\infty$ and $T_n=\delta\sqrt{n}/r\to\infty$:

$$\sum_{\substack{|y_2|\le\delta\\ y_2\in(\mathbb{Z}-(A^{-1}x^*)_2)/n}} \exp\left(-\frac{n}{12r^2}y_2^2-\frac{n}{6r^2}y_2y_3-\frac{n}{6r^2}y_2y_4-\frac{2n}{15r}y_2^2+\frac{3n}{5r}y_2y_3-\frac{n}{15r}y_2y_4+\sum_{j=2}^4 O(ny_2y_j)\right)$$
$$=\sum_{\substack{|\check{y}_2|\le\delta\sqrt{n}/r\\ \check{y}_2\in(\mathbb{Z}-(A^{-1}x^*)_2)/(r\sqrt{n})}} \exp\left(-\frac{1}{12}\check{y}_2^2-\frac{1}{6}\check{y}_2\check{y}_3-\frac{1}{6}\check{y}_2\check{y}_4-\frac{2r}{15}\check{y}_2^2+\frac{3r}{5}\check{y}_2\check{y}_3-\frac{r}{15}\check{y}_2+\sum_{j=2}^4 O(r^2\check{y}_2\check{y}_j)\right)$$
$$\sim 2\sqrt{3\pi}r\sqrt{n}\exp((\check{y}_3+\check{y}_4)^2/12).$$

Fix y_4 and set $\check{y}_4=\sqrt{n}y_4/r$. We apply Lemma 4.6.1 with $\alpha=1/72$, $\beta=-(1/8)\check{y}_4$, $\phi=O(r)=o(1)$, $\psi=O(r\check{y}_4)=O(r)=o(1)$, $s_n=r\sqrt{n}\to\infty$ and $T_n=\delta\sqrt{n}/r\to\infty$:

$$\sum_{\substack{|y_3|\le\delta\\ y_3\in(\mathbb{Z}-(A^{-1}x^*)_3)/n}} \exp\left(-\frac{n}{72r^2}y_3^2-\frac{n}{18r^2}y_3y_4+\sum_{j=3}^4 O(ny_3y_j/r)\right)$$
$$=\sum_{\substack{|\check{y}_3|\le\delta\sqrt{n}/r\\ \check{y}_3\in(\mathbb{Z}-(A^{-1}x^*)_3)/(r\sqrt{n})}} \exp\left(-\frac{1}{72}\check{y}_3^2-\frac{1}{18}\check{y}_3\check{y}_4+\sum_{j=3}^4 O(r\check{y}_3\check{y}_j)\right)$$
$$\sim 6\sqrt{2\pi}r\sqrt{n}\exp(\check{y}_4^2/18)).$$

Finally, for y_4, we apply Lemma 4.6.1 with $\alpha = 1/36$, $\beta = 0$, $\phi = O(r) = o(1)$, $\psi = 0$, $s_n = r\sqrt{n} \to \infty$ and $T_n = \delta\sqrt{n}/r \to \infty$:

$$\sum_{\substack{|y_4|\le\delta \\ y_4\in(\mathbb{Z}-(A^{-1}x^*)_4)/n}} \exp\left(-\frac{n}{36r^2}y_4^2 + O(ny_4y_4/r)\right) = \sum_{\substack{|\check{y}_4|\le\delta\sqrt{n}/r \\ \check{y}_4\in(\mathbb{Z}-(A^{-1}x^*)_4)/(r\sqrt{n})}} \exp\left(-\frac{1}{36}\check{y}_4^2 + O(r\check{y}_4\check{y}_4)\right)$$
$$\sim 6r\sqrt{\pi n}.$$

Hence,

$$\sum_{\substack{\hat{x}\in\hat{B} \\ \hat{x}\in(\mathbb{Z}^4-x^*)/n}} \exp\left(\frac{n\hat{x}^T H\hat{x}}{2}\right) \sim \sqrt{2rn\pi}\cdot 2\sqrt{3\pi}r\sqrt{n}\cdot 6\sqrt{2\pi}r\sqrt{n}\cdot 6r\sqrt{\pi n} = 144\sqrt{3}\pi^2 n^2 r^{7/2},$$

completing the proof. □

Proof of Lemma 4.8.13. Recall that $h_n(y) = y\ln(yn) - y$,

$$w_{\text{pre}}(x) = \begin{cases} \dfrac{P_3!P_2!Q_3!(m_2-1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2^- - 1)!m_2^-!2^{k_2}2^{m_2^-}6^{m_3}}\dfrac{f_3(\lambda)^{n_3}}{\lambda^{Q_3}}, & \text{if } Q_3 > 3n_3; \\ \dfrac{P_3!P_2!Q_3!(m_2-1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2^- - 1)!m_2^-!2^{k_2}2^{m_2^-}6^{m_3}}\dfrac{1}{6^{n_3}}, & \text{otherwise.} \end{cases}$$

and, for $x \in S$ such that $Q_3 > 3n_3$

$$\begin{aligned} f_{\text{pre}}(\hat{x}) =& h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(m_2) \\ & - h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{n}_3) - h_n(\hat{m}_3) \\ & - h_n(\hat{P}_3 - \hat{k}_1 - 2\hat{k}_2) - h_n(\hat{P}_2 - \hat{k}_1) - 2h_n(\hat{m}_2^-) \\ & - \hat{k}_2\ln 2 - \hat{m}_2^-\ln 2 - \hat{m}_3\ln 6 \\ & + \hat{n}_3\ln f_3(\lambda) - \hat{Q}_3\ln\lambda, \end{aligned}$$

and, if $Q_3 = 3n_3$,

$$\begin{aligned} f_{\text{pre}}(\hat{x}) =& h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(m_2) \\ & - h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{n}_3) - h_n(\hat{m}_3) \\ & - h_n(\hat{P}_3 - \hat{k}_1 - 2\hat{k}_2) - h_n(\hat{P}_2 - \hat{k}_1) - 2h_n(\hat{m}_2^-) \\ & - \hat{k}_2\ln 2 - \hat{m}_2^-\ln 2 - \hat{m}_3\ln 6 \\ & - \hat{n}_3\ln 6. \end{aligned}$$

Thus, by Lemma 2.5.1 (which states that Stirling's approximation is correct up to a constant factor), there is a polynomial $Q(n)$ such that for $\hat{x} \in \hat{S}_m$

$$w_{\text{pre}}(x) \le Q(n) n! \exp(n f_{\text{pre}}(\hat{x})).$$

Hence, if we obtain an upper bound for the tail $\sum_{x \in (S \setminus (x^*+B)) \cap \mathbb{Z}^4} n! \exp(n f_{\text{pre}}(\hat{x}))$, we also get an upper bound for the tail $\sum_{x \in (S \setminus (x^*+B)) \cap \mathbb{Z}^4} w_{\text{pre}}(x)$ although it is a weaker bound because of the polynomial factor $Q(n)$.

Let $x \in (S \setminus (x^* + B)) \cap \mathbb{Z}^4$. Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be such that $x = x^* + \gamma_1 z_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4$. Let $\delta_1' = \omega(\delta_1)$ be such that δ_1'/δ_1 goes to infinity arbitrarily slowly and let δ' be such that δ'/δ goes to infinity arbitrarily slowly. If $\delta_1 \le |\gamma_1| \le \delta_1'$ and $\delta \le |\gamma_i| \le \delta'$ for $i = 2, 3, 4$, by (4.85),

$$\frac{\exp\left(n f_{\text{pre}}(\hat{x})\right)}{\exp\left(n f_{\text{pre}}(\hat{x}^*)\right)} \sim \exp\left(\frac{n(\hat{x} - \hat{x}^*)^T H (\hat{x} - \hat{x}^*)}{2}\right),$$

where H is the Hessian of f_{pre} at x^*. Recall that, by Lemma 4.8.14, $H = (-1/r^2)H_0 - (1/r)T + J$, where H_0 and T are defined in (4.58) and $J = J(n)$ is a matrix with bounded entries. Thus, there exists a positive constant α such that

$$\begin{aligned}
\exp\left(\frac{n(\hat{x} - \hat{x}^*)^T H (\hat{x} - \hat{x}^*)}{2}\right) &= \exp\left(\frac{n(\hat{x} - \hat{x}^*)^T \left((-1/r^2)H_0 - (1/r)T + J\right)(\hat{x} - \hat{x}^*)}{2}\right) \\
&\le \exp\left(-\frac{\alpha \gamma_1^2 n}{r} - \sum_{i=2}^{4} \frac{\alpha \gamma_i^2 n}{r^2}\right) \\
&\le \max\left(\exp\left(-\frac{\alpha \delta_1^2 n}{r}\right), \exp\left(-\frac{\alpha \delta^2 n}{r^2}\right)\right) \\
&= \frac{1}{n^{\omega(1)}},
\end{aligned}$$

where the last relation follows from $\delta_1^2 n/r = \omega(\ln n)$ and $\delta^2 n/r^2 = \omega(\ln n)$.

By Lemma 4.8.11, for any local maximum x in S other than x^*,

$$\frac{\exp(n f_{\text{pre}}(x))}{\exp(n f_{\text{pre}}(x^*))} = \frac{1}{\exp(\Omega(r^2 n))} = \frac{1}{\exp(\Omega(R^2/n))} = \frac{1}{\exp(\Omega(\ln^{3/2} n))},$$

since $R = \omega(n^{1/2} \ln^{3/2}(n))$. Hence, for any $x \in S \setminus (x^* + B)$,

$$\frac{\exp(n f_{\text{pre}}(x))}{\exp(n f_{\text{pre}}(x^*))} = \frac{1}{\exp(\Omega(\ln^{3/2} n))}.$$

Thus,

$$\sum_{x\in(S\setminus(x^*+B))\cap\mathbb{Z}^4} w_{\text{pre}}(x) \le Q(n)n! \sum_{x\in(S\setminus(x^*+B))\cap\mathbb{Z}^4} \exp(nf_{\text{pre}}(\hat{x})) \le Q(n)n!n^4 \frac{\exp(nf_{\text{pre}}(\hat{x}^*))}{\exp(\Omega(\ln^{3/2} n))}$$
$$= o\left(n!\exp(nf_{\text{pre}}(\hat{x}^*))\right).$$

□

4.9 Combining pre-kernels and forests

In this section, we will obtain a formula for the number of connected graphs with vertex set $[n]$ and m edges, proving Theorem 4.1.1. We defer the proof of some lemmas to Section 4.9.1. We will perform Step 4 as described in the overview of the proof in Section 4.4: we will analyse the summation $\sum_n g_{\text{forest}}(N,n)g_{\text{core}}(n, M-(N-n)/2)$ by combining the formula obtained for forests (Section 4.5) and for cores (Section 4.7). We relate the formulae for cores and pre-kernels (Section 4.8) so that we can deduce the asymptotic value of $\sum_n g_{\text{forest}}(N,n)g_{\text{pre}}(n, M-(N-n)/2)$, which is the number of connected graphs.

For $\check{n} \in [0,1]$, let

$$t(\check{n}) = -\frac{(1-\check{n})}{2}\ln(1-\check{n}) + \frac{1-\check{n}}{2} + \check{n}f_{\text{core}}(\hat{n}_1^*), \tag{4.90}$$

where $\hat{n}_1^* = \hat{n}_1^*(n) = 3\hat{m}/g_2(\lambda^*)$ and $\lambda^* = \lambda^*(n)$ is the unique positive solution of the equation $\lambda f_1(\lambda)g_2(\lambda)/f_2(2\lambda) = 3m/n$. We have already discussed the existence and uniqueness of λ^* in Section 4.7.

Elementary but lengthy computations show that

$$\begin{aligned} t(\check{n}) = &- \frac{(1-\check{n})}{2}\ln(1-\check{n}) + \frac{1-\check{n}}{2} \\ &2\check{R}\ln(N) + (2\ln(3)-\ln(2)-2)\check{R} + 2\check{R}\ln(\check{n}) + \left(\ln 3 - \frac{1}{2}\ln(2)\right)\check{n} \\ &+\ln\left(\frac{f_2(2\lambda^*)}{g_1(\lambda^*)}\right)\check{n} + \left(\frac{1}{2}\check{n}+\check{R}\right)\ln\left(\frac{\hat{m}^2 g_1(\lambda^*)^3}{g_2^2(\lambda^*)f_1(\lambda^*)(\lambda^*)^3}\right), \end{aligned} \tag{4.91}$$

where $\check{R} = R/N$. See Section A.13 for a Maple spreadsheet. In this section, we use $\check{y}$ to denote y/N. We obtain the following asymptotic formulae.

Theorem 4.9.1. We have that

$$\sum_{\substack{n\in[N]\\ N-n \text{ even}}} \binom{N}{n} g_{\text{forest}}(N,n)g_{\text{core}}(n,m) \sim \frac{\sqrt{3}}{\sqrt{\pi\lambda^{**}N}}\exp(Nt(\check{n}^*) + N\ln N - N) \tag{4.92}$$

and

$$\sum_{\substack{n\in[N]\\ N-n \text{ even}}} \binom{N}{n} g_{\text{forest}}(N,n) g_{\text{pre}}(n,m) \sim \frac{\sqrt{3}}{\sqrt{\pi N}} \exp(Nt(\check{n}^*) + N\ln N - N) \tag{4.93}$$

where λ^{**} is the unique positive solution to

$$\frac{2\lambda f_1(\lambda)g_2(\lambda) - 3f_2(2\lambda)}{f_1(\lambda)g_1(\lambda)} = \frac{6R}{N} \tag{4.94}$$

and

$$\check{n}^* = \frac{f_2(2\lambda^{**})}{f_1(\lambda^{**})g_1(\lambda^{**})}. \tag{4.95}$$

Theorem 4.1.1 follows immediately from Theorem 4.9.1 by simplifying $t(\check{n}^*)$ by using (4.91) with (4.95) and (4.19). The rest of this section is dedicated to prove Theorem 4.9.1.

The following lemma shows that λ^{**} is well-defined.

Lemma 4.9.2. The equation

$$\frac{2\lambda f_1(\lambda)g_2(\lambda) - 3f_2(2\lambda)}{f_1(\lambda)g_1(\lambda)} = \alpha_n$$

has a unique solution for $\alpha_n > 0$ and it goes to 0 if $\alpha_n \to 0$.

Proof. For the first part, it suffices to show that the function

$$f(\lambda) = \frac{2\lambda f_1(\lambda)g_2(\lambda) - 3f_2(2\lambda)}{f_1(\lambda)g_1(\lambda)}$$

is strictly increasing and it goes to zero as $\lambda \to 0$. By computing the series of $f(\lambda)$ with $\lambda \to 0$, we obtain $f(\lambda) = \lambda^2/2 + O(\lambda^3) \to 0$ as $\lambda \to 0$. To show $f(\lambda)$ is strictly increasing, we compute its derivative:

$$\frac{\mathrm{d}\, f(\lambda)}{\mathrm{d}\,\lambda} = \frac{2(e^{4\lambda} + e^{3\lambda} - e^{\lambda} - 1 - \lambda e^{3\lambda} - 4\lambda e^{2\lambda} - \lambda e^{\lambda})}{f_1(\lambda)^2 g_1(\lambda)^2}$$

while it is obvious that the denominator is positive for $\lambda > 0$, it is not immediate that so is the numerator.

Let $g(\lambda) = e^{4\lambda} + e^{3\lambda} - e^{\lambda} - 1 - \lambda e^{3\lambda} - 4\lambda e^{2\lambda} - \lambda e^{\lambda}$. We will use the following strategy: starting with $i = 1$, we check that $\frac{\mathrm{d}^{i-1}\, g(\lambda)}{\mathrm{d}^{i-1}\,\lambda}|_{\lambda=0} = 0$ and compute $\frac{\mathrm{d}^{i}\, g(\lambda)}{\mathrm{d}^{i}\,\lambda}$. If for some i we can show that $\frac{\mathrm{d}^{i}\, g(\lambda)}{\mathrm{d}^{i}\,\lambda} > 0$ for any λ, then we obtain $g(\lambda) > 0$ for $\lambda > 0$. We omit the computations here. See Section A.14 in the Appendix for a maple spreadsheet. We have that

$$\frac{\mathrm{d}^5\, g(x)}{\mathrm{d}^5\, x} = 2048e^{4\lambda} - 12e^{\lambda} - 324e^{3\lambda} - 486\lambda e^{3\lambda} - 640e^{2\lambda} - 256\lambda e^{2\lambda} - 2\lambda e^{x},$$

which is trivially positive since $\exp(x) > 1 + x$ for all $x \in \mathbb{R}$ and the sum of the coefficients of the negative terms is less than 2048. □

It will be useful to know how λ^{**} compares to R and $\check{n}^*$. By Lemma 4.9.2, $\lambda^{**} = o(1)$ since $R = o(N)$. We can write $\check{R}$ and $\check{n}^*$ in terms of λ^{**} by using (4.94) and (4.95). By expanding the LHS of (4.94) and the RHS (4.95) as functions of λ^{**} about 0, we have that

$$\begin{aligned} \check{R} &= \frac{(\lambda^{**})^2}{12} + O((\lambda^{**})^4), \\ \check{n}^* &= \lambda^{**} - \frac{(\lambda^{**})^2}{3} + O((\lambda^{**})^4). \end{aligned} \tag{4.96}$$

Next, we state the main lemmas for the proof of Theorem 4.9.1. We defer their proofs to Section 4.9.1. First we show the relation between $g_{\text{pre}}(n, m)$ and $g_{\text{core}}(n, m)$ for a certain range of n. The next lemma follows from Theorem 4.7.1 and Theorem 4.8.1 and a series of simplifications that show that $f_{\text{core}} n_1^* = f_{\text{pre}}(x^*)$. For the simplifications see Section A.12.

Lemma 4.9.3. Let $\alpha_1 < \alpha_2$ be positive constants. If $\alpha_1\sqrt{RN} \leq n \leq \alpha_2\sqrt{RN}$, then

$$\frac{g_{\text{pre}}(n, m)}{g_{\text{core}}(n, m)} \sim 2\sqrt{3r}. \tag{4.97}$$

Moreover, for all $n \geq 0$,

$$g_{\text{pre}}(n, m) \leq g_{\text{core}}(n, m) \tag{4.98}$$

This will allow us to obtain the formula for connected hypergraphs from the formula for simple hypergraphs. We compute the point of maximum for $t(\check{n})$:

Lemma 4.9.4. The point $\check{n}^*$ is the unique maximum of the function $t(\check{n})$ in the interval $[0, 1]$. Moreover, $\check{n}^*$ is the unique point such that the derivative of $t(\check{n})$ is 0 in $(0, 1)$, and $t'(\check{n}) > 0$ for $\check{n} < \check{n}^*$ and $t'(\check{n}) < 0$ for $\check{n} < \check{n}^*$.

We then expand the summation around this maximum and approximate it by an integral that can be easily computed.

Lemma 4.9.5. Suppose $\delta^3 = o(\lambda^{**}/N)$ and $\delta = \omega(1/N^{1/2})$. Then

$$\sum_{\substack{n \in [n^* - \delta N, n^* + \delta N] \\ N - n \text{ even}}} \exp(Nt(\check{n})) \sim \sqrt{\frac{\pi N}{2}} \exp(Nt(\check{n}^*)).$$

Finally, we show that the terms far from the maximum do not contribute significantly to the summation:

Lemma 4.9.6. Suppose that $\delta^3 = o(\lambda^{**}/N)$ and $\delta^2 = \omega((\ln N)/N)$. Then

$$\sum_{n\in[0,N]\setminus[n^*-\delta N, n^*+\delta N]} \binom{N}{n} g_{\text{forest}}(N,n) g_{\text{core}}(n,m) = \frac{N!\exp(Nt(\check{n}^*))}{N^{\omega(1)}}.$$

We are now ready to prove Theorem 4.9.1.

Proof of Theorem 4.9.1. In order to use Lemma 4.9.5 and Lemma 4.9.6, we need to check if there exists δ such that $\delta^3 = o(\lambda^{**}/N)$ and $\delta^2 = \omega(\ln N/N)$. This is true if and only if

$$(\lambda^{**})^2 = \omega\left(\frac{\log^3 N}{N}\right),$$

which, by (4.96), is true if and only if

$$R = \omega(\log^3 N),$$

which is true by assumption. Thus, assume that δ satisfies $\delta^3 = o(\lambda^{**}/N)$ and $\delta^2 = \omega(\ln N/N)$.

Let $J(\delta) = [n^* - \delta N, n^* + \delta N] \cap (2\mathbb{Z} - N)$. By (4.96), we have that

$$n^* = \Theta(\lambda^{**}N) = \Theta(\sqrt{RN}).$$

Moreover, since $\delta^3 = o(\lambda^{**}/N)$ and $R \to \infty$,

$$\delta N = o(\sqrt[6]{RN^3}) = o\left(\frac{\sqrt{RN}}{R^{1/3}}\right) = o(n^*).$$

Thus, there are constants $\alpha_1 > 0$ and $\alpha_2 > 0$ such that any $n \in J(\delta)$ satisfies $\alpha_1\sqrt{RN} < n < \alpha_2\sqrt{RN}$. By Lemma 4.9.3

$$\frac{g_{\text{pre}}(n,m)}{g_{\text{core}}(n,m)} \sim 2\sqrt{3r},$$

for any $n \in J(\delta)$ and $m = n/2 + R$. Since $R = o(n)$ and $R = \omega(\log n)$, by Theorem 4.7.1 and by Theorem 4.5.1, for $n \in J(\delta)$,

$$g_{\text{core}}(n,m) \sim \frac{1}{2\pi n\sqrt{r}} \cdot n!\exp\left(nf_{\text{core}}(\hat{n}_1^*)\right) \quad \text{and} \quad g_{\text{forest}}(n,N) = \frac{n}{N} \cdot \frac{(N-n)!N^{(N-n)/2}}{\left(\frac{N-n}{2}\right)!2^{(N-n)/2}}.$$

Thus, for $n \in J(\delta)$, with $m = n/2 + R$, by Stirling's approximation and using the fact that $n = o(N)$ by (4.96),

$$
\begin{aligned}
\binom{N}{n} g_{\text{forest}}(N,n) g_{\text{core}}(n,m) &\sim \binom{N}{n} \cdot \frac{n}{N} \cdot \frac{(N-n)! N^{(N-n)/2}}{\left(\frac{N-n}{2}\right)! 2^{(N-n)/2}} \cdot \frac{1}{2\pi n\sqrt{r}} \cdot n! \exp\Big(n f_{\text{core}}(\hat{n}_1^*)\Big) \\
&= \frac{\sqrt{n}}{2\pi N\sqrt{R}} \cdot \frac{N! N^{(N-n)/2}}{\left(\frac{N-n}{2}\right)! 2^{(N-n)/2}} \exp\Big(n f_{\text{core}}(\hat{n}_1^*)\Big) \\
&\sim \frac{\sqrt{2nN}}{2\pi N\sqrt{R(N-n)}} \cdot \exp\Big(N t(\check{n}) + N\ln N - N\Big), \text{ by (4.90)} \\
&\sim \frac{\sqrt{n}}{\pi N\sqrt{2R}} \cdot \exp\Big(N t(\check{n}) + N\ln N - N\Big).
\end{aligned}
$$

By (4.96),

$$
\frac{\sqrt{n}}{\pi N\sqrt{2R}} \sim \frac{\sqrt{6}}{\pi N\sqrt{\lambda^{**}}}
$$

and so

$$
\binom{N}{n} g_{\text{forest}}(N,n) g_{\text{core}}(n,m) \sim \frac{\sqrt{6}}{\pi N\sqrt{\lambda^{**}}} \tag{4.99}
$$

for $n \in J(\delta)$. Since $J(\delta)$ is a finite set for each n, by Lemma 2.7.1, we have that there exists a function $q(n) = o(1)$ such that the $o(1)$ in (4.99) is bounded by $q(n)$ for any $n \in J(\delta)$. Thus,

$$
\begin{aligned}
\sum_{n\in J(\delta)} \binom{N}{n} g_{\text{forest}}(N,n) g_{\text{core}}(n,m) &\sim \sum_{n\in J(\delta)} \frac{\sqrt{6}}{\pi N\sqrt{\lambda^{**}}} \cdot \exp\Big(N t(\check{n}) + N\ln N - N\Big) \\
&\sim \frac{\sqrt{6}}{\pi N\sqrt{\lambda^{**}}} \sqrt{\frac{\pi N}{2}} \exp(N t(\check{n}^*) + N\ln N - N)
\end{aligned} \tag{4.100}
$$

by Lemma 4.9.5. Together with Lemma 4.9.6, this proves Equation (4.92) of Theorem 4.9.1.

Equations (4.100) and (4.97) implies that

$$
\begin{aligned}
\sum_{n\in J(\delta)} \binom{N}{n} g_{\text{forest}}(N,n) g_{\text{pre}}(n,m) &\sim 2\sqrt{\frac{3R}{n}} \cdot \frac{\sqrt{6}}{\pi N\sqrt{\lambda^{**}}} \sqrt{\frac{\pi N}{2}} \exp(N t(\check{n}^*) + N\ln N - N) \\
&\sim 2\sqrt{\frac{3}{12}} \cdot \frac{\sqrt{3}}{\sqrt{\pi N}} \exp(N t(\check{n}^*) + N\ln N - N) \\
&\sim \frac{\sqrt{3}}{\sqrt{\pi N}} \exp(N t(\check{n}^*) + N\ln N - N),
\end{aligned} \tag{4.101}
$$

which, together with Lemma 4.9.6 and the fact that $g_{\text{pre}}(n,m) \le g_{\text{core}}(n,m)$, proves Equation (4.93) of Theorem 4.9.1. □

4.9.1 Proof of the lemmas in Section 4.9

In this section, we prove Lemmas 4.9.4, 4.9.5 and 4.9.6. See Section A.15 for a Maple spreadsheet. We start by computing the derivatives of t. For that, we need to compute $\frac{\mathrm{d}\,\lambda^*(\check{n})}{\mathrm{d}\,\check{n}}$. This can be done by implicit differentiation using Equation (4.19) that defines λ^* and recalling $m = n/2 + R$. We obtain

$$\frac{\mathrm{d}\,\lambda^*}{\mathrm{d}\,\check{n}} = -\frac{\check{R}}{\check{n}^2\hat{m}a(\lambda^*)}, \tag{4.102}$$

where

$$a(\lambda) = \frac{1}{\lambda} + \frac{\exp(\lambda)}{f_1(\lambda)} + \frac{\exp(\lambda)}{g_2(\lambda)} - \frac{2\exp(2\lambda)}{f_2(2\lambda)} + \frac{2}{f_2(2\lambda)}. \tag{4.103}$$

Thus, the first derivative of $t(\check{n})$, which is defined in (4.91), is

$$\frac{\ln(1-\check{n})}{2} + \ln(3) - \frac{\ln 2}{2} + \ln\left(\frac{f_2(2\lambda^*)}{g_1(\lambda^*)}\right) + \frac{1}{2}\ln\left(\frac{\hat{m}^2 g_1(\lambda^*)^3}{g_2^2(\lambda^*)f_1(\lambda^*)(\lambda^*)^3}\right). \tag{4.104}$$

The second derivative is

$$-\frac{1}{2(1-\check{n})} - \frac{2\check{R}}{(\check{n}+2\check{R})\check{n}} - \frac{4\check{R}^2}{\check{n}(\check{n}+2\check{R})^2}\frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)} \tag{4.105}$$

where

$$b(\lambda) = 2F_1(\lambda) - \frac{f_2(2\lambda)\exp(\lambda)}{g_1(\lambda)}.$$

The third derivative is

$$\begin{aligned} &-\frac{1}{2(1-\check{n})^2} + \frac{4\check{R}(\check{n}+\check{R})}{\check{n}^2(\check{n}+2\check{R})^2} \\ &+ \frac{\mathrm{d}}{\mathrm{d}\,\check{n}}\left(-\frac{4\check{R}^2}{\check{n}(\check{n}+2\check{R})^2}\right)\frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)} - \frac{4\check{R}^2}{\check{n}(\check{n}+2\check{R})^2}\frac{\mathrm{d}}{\mathrm{d}\,\lambda^*}\left(\frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)}\right)\frac{\mathrm{d}\,\lambda^*}{\mathrm{d}\,\check{n}}. \end{aligned} \tag{4.106}$$

Lemma 4.9.7. For $\delta = o(\check{n}^*)$ and $n \in [n^* - \delta N, n^* + \delta N]$, we have that $|\lambda^*(n) - \lambda^{**}| = o(\check{n}^*)$.

Proof. Given a connected (N, M)-graph such that its core has n vertices and m edges, we have that $m = M - (N-n)/2$. Recall that $\check{n}^* = f_2(2\lambda^{**})/f_1(\lambda^{**})g_1(\lambda^{**})$ by (4.95) and

$$\frac{6R}{N} = \frac{2\lambda^{**}f_1(\lambda^{**})g_2(\lambda^{**}) - 3f_2(2\lambda^{**})}{f_1(\lambda^{**})g_1(\lambda^{**})},$$

by (4.94). Thus,

$$3M = \frac{\lambda^{**}(1+\exp(2\lambda^{**}) + \exp(\lambda^{**}))}{\exp(2\lambda^{**}) - 1}.$$

Hence,

$$\frac{\lambda^*(n^*)f_1(\lambda^*(n^*))g_2(\lambda^*(n^*))}{f_2(2\lambda^*(n^*))} = \frac{3m}{n^*} = \frac{3M}{N}\frac{1}{\check{n}^*} - \frac{3}{2\check{n}^*} + \frac{3}{2} = \frac{\lambda^{**})f_1(\lambda^{**})g_2(\lambda^{**})}{f_2(2\lambda^{**})}$$

and so $\lambda^* = \lambda^*(n^*)$. The lemma then follows directly from the fact that $\lambda^{**} = \lambda^*(\check{n}^*)$ and Lemma 4.7.4. ☐

Now we bound the third derivative for points close to $\check{n}^*$:

Lemma 4.9.8. The third derivative of $t(\check{n})$ is $O(1/\lambda^{**})$ for $|\check{n} - \check{n}^*| = o(\check{n}^*)$.

Proof. We analyse the terms in (4.106). By Lemma 4.7.4, since $n = \check{n}(1 + o(1))$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,\check{n}}\left(-\frac{1}{2(1-\check{n})^2} + \frac{4\check{R}(\check{n}+\check{R})}{\check{n}^2(\check{n}+2\check{R})^2}\right) &= -\frac{1}{2(1-\check{n})^2} + \frac{4\check{R}(\check{n}+\check{R}))}{\check{n}^2(\check{n}+2\check{R})^2} \\ &= \left(-\frac{1}{2(1-\check{n}^*)^2} + \frac{4\check{R}(\check{n}^*+\check{R})}{(\check{n}^*)^2(\check{n}^*+2\check{R})^2}\right)(1+o(\check{n}^*)) \\ &= \frac{1}{3\lambda^{**}} + O(1),\end{aligned}$$

where the last equality is obtained by computing the series of the expression in the previous equation using (4.96). For $\lambda \to 0$,

$$a(\lambda) = \frac{1}{6} + \frac{\lambda}{12} + O(\lambda^2) \tag{4.107}$$

$$b(\lambda) = 4\lambda + O(\lambda^2); \tag{4.108}$$

Thus, by Lemma 4.7.4 and (4.96),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,\check{n}}\left(-\frac{4\check{R}^2}{\check{n}(\check{n}+2\check{R})^2}\right)\frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)} &= \frac{4\check{R}^2(3\check{n}-2\check{R})}{\check{n}^2(\check{n}+2\check{R})^3}\frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)} \\ &\sim \frac{4\check{R}^2(3\check{n}^*-2\check{R})}{(\check{n}^*)^2(\check{n}^*+2\check{R})^3}\frac{b(\lambda^{**})}{a(\lambda^{**})f_2(2\lambda^{**})} = \frac{1}{\lambda^{**}} + O(1).\end{aligned}$$

We have that

$$\frac{\mathrm{d}\,b(\lambda)}{\mathrm{d}\,\lambda} = 4\exp(2\lambda) - \frac{\exp(\lambda)(3\exp(2\lambda) - 3 - 2\lambda)}{g_1(\lambda)} + \frac{f_2(2\lambda)\exp(2\lambda)}{g_1(\lambda)^2}$$

and

$$\frac{\mathrm{d}\,a(\lambda)}{\mathrm{d}\,\lambda} = -\frac{1}{\lambda^2} + \frac{\exp(\lambda)}{f_1(\lambda)} - \frac{\exp(2\lambda)}{f_1(\lambda)^2} + \frac{\exp(\lambda)}{g_2(\lambda)} - \frac{\exp(2\lambda)}{g_2(\lambda)^2} - \frac{4\exp(2\lambda)}{f_2(2\lambda)} - \frac{4F_1(\lambda)^2}{f_2(2\lambda)^2}.$$

Thus, by (4.102)

$$
\begin{aligned}
&-\frac{4\check{R}^2}{\check{n}(\check{n}+2\check{R})^2}\frac{\mathrm{d}}{\mathrm{d}\,\lambda^*}\left(\frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)}\right)\frac{\mathrm{d}\,\lambda^*}{\mathrm{d}\,\check{n}}\\
&=-\frac{4\check{R}^2}{\check{n}(\check{n}+2\check{R})^2}\left(\frac{\mathrm{d}\,b(\lambda)}{\mathrm{d}\,\lambda}\Big|_{\lambda=\lambda^*}\frac{1}{a(\lambda^*)f_2(2\lambda^*)}\right.\\
&\qquad\qquad\left.-\frac{b(\lambda^*)}{a(\lambda^*)^2f_2(2\lambda^*)^2}\left(\frac{\mathrm{d}\,a(\lambda)}{\mathrm{d}\,\lambda}\Big|_{\lambda=\lambda^*}f_2(2\lambda^*)+2F_1(\lambda^*)a(\lambda^*)\right)\right)\left(-\frac{\check{R}}{\check{n}^2\hat{m}a(\lambda^*)}\right).
\end{aligned}
$$

By Lemma 4.7.4, the above is the value applied at λ^{**} with an error of $o(\lambda^{**})$ and the series for it with $\lambda^{**}\to 0$ is

$$\frac{2}{3\lambda^{**}}+O(1).$$

□

We now present the proofs of Lemmas 4.9.4, 4.9.5 and 4.9.6.

Proof of Lemma 4.9.4. By setting (4.104) to zero and using $\hat{m}=\lambda^* f_1(\lambda^*)g_2(\lambda^*)/f_2(2\lambda^*)$, we get following value for $\check{n}$

$$\check{n}^*=\frac{f_2(2\lambda^*)}{f_1(\lambda^*)g_1(\lambda^*)}. \tag{4.109}$$

We also know that, by (4.19),

$$\frac{\lambda^* f_1(\lambda^*)g_2(\lambda^*)}{f_2(2\lambda^*)}=3\hat{m}=\frac{3}{2}+\frac{3\check{R}}{\check{n}}. \tag{4.110}$$

Thus, by combining (4.109) and (4.110), we get the following equation:

$$\check{R}=\frac{1}{6}\frac{-3f_2(2\lambda^*)+2\lambda^* f_1(\lambda^*)g_2(\lambda^*)}{f_1(\lambda^*)g_1(\lambda^*)}, \tag{4.111}$$

which has a unique solution λ^{**} for $\check{R}>0$ by Lemma 4.9.2. By computing the series of the second derivative as $\lambda^*\to 0$, we get that the second derivative at $\check{n}^*$ is

$$-1+O(\lambda^*),$$

which is negative for big enough n and so $\check{n}^*$ is a local maximum. □

Proof of Lemma 4.9.5. Let $J(\delta) = [n^* - \delta N, n^* + \delta N] \cap (2\mathbb{Z} - N)$. Using Taylor's approximation, Lemma 4.9.4 and Lemma 4.9.8, for $n \in J(\delta)$,

$$\begin{aligned}\exp(Nt(\check{n})) &= \exp\left(Nt(\check{n}^*) + \frac{Nt''(\check{n}^*)|\check{n} - \check{n}^*|^2}{2} + O\left(\frac{\delta^3 N}{\lambda^{**}}\right)\right)\\ &\sim \exp\left(Nt(\check{n}^*) + \frac{Nt''(\check{n}^*)|\check{n} - \check{n}^*|^2}{2}\right)\end{aligned}$$

since $\delta^3 = o(\lambda^*/N)$, and so

$$\begin{aligned}\sum_{\check{n}\in J(\delta)} \exp(Nt(\check{n})) &\sim \sum_{\check{n}\in J(\delta)} \exp\left(Nt(\check{n}^*) + \frac{Nt''(\check{n}^*)|\check{n} - \check{n}^*|^2}{2}\right)\\ &= \exp\left(Nt(\check{n}^*)\right) \sum_{\substack{x\in[-\delta N, \delta N]\\ (n^*+x)\in(-N+2\mathbb{Z})}} \exp\left(\frac{t''(\check{n}^*)\hat{x}}{2N}\right).\end{aligned} \tag{4.112}$$

We change variables from x to $y = \ell x/2$ with $\ell = \sqrt{|t''(\check{n}^*)|/2} \sim \frac{1}{2}$. Using $\delta = \omega(1/\sqrt{N})$ and Lemma 4.6.1,

$$\sum_{\substack{x\in[-\delta N, \delta N]\\ N-(\check{n}^* N+x)\in 2\mathbb{Z}}} \exp\left(\frac{t''(\check{n}^*)x^2}{2N}\right) = \sum_{\substack{y\in[-\delta\ell\sqrt{N}/2, \delta\ell\sqrt{N}/2]\\ y\in\mathbb{Z}\cdot(\ell/\sqrt{N})}} \exp\left(-4y^2\right) \sim \sqrt{\frac{\pi N}{2}}.$$

□

Proof of Lemma 4.9.6. From Theorems 4.7.1 and 4.5.1, and the definition of t, we have that there is a polynomial $Q(N)$ such that

$$\sum_{n\notin[(\check{n}^*-\delta)N, (\check{n}^*+\delta)N]} \binom{N}{n} g_{\text{forest}}(N, n) g_{\text{core}}(n, m) \le Q(N)N! \sum_{\substack{n\in[0,N]\\ n\notin[n^*-\delta N, n^*+\delta N]}} \exp(Nt(\check{n}))$$

Using Lemma 4.9.4 and (4.112), we have that

$$\sum_{\substack{n\in[0,N]\\ n\notin[n^*-\delta N, n^*+\delta N]}} \exp(Nt(\check{n})) \le N \exp(Nt(\check{n}^*) - \Omega(N\delta^2)),$$

and $N\delta^2 = \omega(\ln N)$ for $\delta^2 = \omega(\ln N)/N$. □

Glossary for Chapter 4

$C(N, M)$	number of connected 3-uniform hypergraphs on $[N]$ with M edges
N	used for the number of vertices in the graph
M	used for the number of edges in the graph
R	$M - N/2$ used as an excess function in the graph
n	used for the number of edges in the core
r	R/n, scaled R
$f_k(\lambda)$	$e^\lambda - \sum_{i=0}^{k-1} \lambda^i/i!$
$g_k(\lambda)$	$e^\lambda + k$
$\lambda(k, c)$	the unique positive solution to $\lambda f_{k-1}(\lambda)/f_k(\lambda) = c$
$g_{\text{core}}(n, m)$	number of (simple) cores with vertex set $[n]$ and m edges
$g_{\text{forest}}(N, n)$	number of forest with vertex set $[N]$ and $[n]$ as its roots
$g_{\text{pre}}(n, m)$	number of (simple) pre-kernels with vertex set $[n]$ and m edges that are connected
λ^{**}	unique positive solution to $\lambda e^{2\lambda} + e^\lambda + 1/(f_1(\lambda)g_1(\lambda)) = 3M/N$. This is used to define a point achieving the maximum when combining cores and pre-kernels, p. 58.
$\check{n}^*$	$f_2(2\lambda^{**})/(f_1(\lambda^{**})g_1(\lambda^{**}))$. This is the point achieving the maximum when combining cores and pre-kernels, p. 58.
2-edge	an edge that contains exactly one vertex of degree 1
3-edge	an edge that contains no vertices of degree 1

For the core:

	For any symbol y, $\hat{y} = y/n$ denotes the scaled version of y
$h_n(y)$	$y \ln(yn) - y$.
f_{core}	a function used to approximate the exponential part of w_{core}, p. 72
w_{core}	a function used to count cores, p. 78
n_1	used as the number of vertices of degree 1
$\mathcal{D}_{n_1}$	set of all $\mathbf{d} \in (\mathbb{N} \setminus \{0, 1\})^{n-n_1}$ with $\sum_i d_i = 3m - n_1$
λ_{n_1}	unique positive solution to $\lambda f_1(\lambda)/f_2(\lambda) = c_2(n_1)$
$n_2(n_1)$	$n - n_1$, the number of vertices of degree at least 2.
$m_3(n_1)$	$m - n_1$, the number of 3-edges

$Q_2(n_1)$	$3m - n_1$, the sum of degrees of vertices of degree at least 2
$c_2(n_1)$	$Q_2(n_1)/n_2(n_1)$, the average degree of the vertices of degree at least 2.
$\eta_2(n_1)$	$\lambda_{n_1} \exp(\lambda_{n_1}/f_1(\lambda_{n_1})$
$\mathcal{G}(n_1, \mathbf{d})$	random core with n_1 vertices of degree 1 and degree sequence $\mathbf{d}$ for the vertices of degree at least 2, p. 73
λ^*	unique positive solution to $\lambda f_1(\lambda) g_2(\lambda)/f_2(2\lambda) = 3m/n$. This is used to define a point achieving the maximum for f_{core}, p. 77
n_1^*	$3m/ng_2(\lambda^*)$. This the point achieving the maximum for f_{core}, p. 72
$\mathbf{Y}$	$(Y_1, \ldots, Y_{n_2}$, where the Y_i's are independent random variables with truncated Poisson distribution $\text{Po}(2, \lambda_{\hat{n}_1})$
Σ_{n_1}	event that a random variable $\mathbf{Y}$ satisfies $\sum_i Y_i = 3m - n_1$

For the pre-kernel:

	For any symbol y, $\hat{y} = y/n$ denotes the scaled version of y
$h_n(y)$	$y \ln(yn) - y$.
f_{pre}	a function used to approximate the exponential part of w_{pre}, p. 86
w_{pre}	a function used to count pre-kernels, p. 110
n_1	used as the number of vertices of degree 1
k_0	used as the number of vertices of degree 2 that are in two 2-edges
k_1	used as the number of vertices of degree 2 that are in one 2-edge and in one 3-edge
k_2	used as the number of vertices of degree 2 that are in two 3-edges
x	used as (n_1, k_0, k_1, k_2)
$\mathcal{D}(x)$	subset of $\mathbb{N}^{n_3(x)}$ such that $\mathbf{d} \in \mathcal{D}(x)$ if $d_i \geq 3$ for all i and $\sum_{i=1}^{n_3(x)} d_i = Q_3(x)$, p. 89
$n_2(x)$	$k_0 + k_1 + k_2$, the number of vertices of degree 2
$n_3(x)$	$n - n_1 - n_2(x)$, the number of vertices of degree at least 3
$m_2(x)$	n_1, the number of 2-edges in the pre-kernel
$m_2^-(x)$	$n_1 - k_0$, the number of 2-edges in the kernel
$P_2(x)$	$2m_2^-(x)$, the number of points in 2-edges in the kernel
$m_3(x)$	$m - n_1$, the number of 3-edges in the pre-kernel
$P_3(x)$	$3m_3(x)$, the number of points in 3-edges in the pre-kernel
$Q_3(x)$	$3m - n_1 - 2n_2(x)$, the sum of the degrees of the vertices of degree at least 3

$c_3(x)$	$Q_3(x)/n_3(x)$, the average degree of the vertices of degree at least 3
$T_3(x)$	$P_3(x) - k_1(x) - 2k_2(x)$, the number of points in 3-edges that will be matched to points in vertices of degree at least 3
$T_2(x)$	$P_2(x) - k_1(x)$, the number of points in 2-edges that will be matched to points in vertices of degree at least 3
$\lambda(x)$	unique positive solution to $\lambda f_2(\lambda)/f_3(\lambda) = c_3(x)$
$\eta_3(x)$	$Q_3(x)/n_3(x)$
λ^*	unique positive solution to $\lambda f_1(\lambda) g_2(\lambda)/f_2(2\lambda) = 3m/n$. This is used to define a point achieving the maximum for f_{pre}, p. 77
x^*	This the point achieving the maximum for f_{pre}, p. 87
$\mathcal{K}$	random kernel (it receives parameters $(V, M_3, k_1, k_2, \mathbf{d})$), p. 90
$\mathcal{P}(x, \mathbf{d})$	random pre-kernel with parameters $x = (n_1, k_0, k_1, k_2)$ and degree sequence $\mathbf{d}$ for the vertices of degree at least 3, p. 90
$\mathbf{Y}$	$(Y_1, \ldots, Y_{n_3})$, where the Y_i's are independent random variables with truncated Poisson distribution $\text{Po}(3, \lambda(x))$.
$\Sigma(x)$	event that a random variable $\mathbf{Y}$ satisfies $\sum_i Y_i = 3m - n_1 - 2n_2$
S^*_ψ	a set of points 'close' to x^*, p. 94

Chapter 5

Robustness of random k-cores

We consider the following procedure: for a fixed integer $k \geq 3$, given a k-core G with n vertices, delete an edge e chosen uniformly at random from the edges of G and obtain the k-core of the new graph $G - e$. Recall that the k-core of a graph can be obtained by a deletion procedure that iteratively deletes vertices of degree less than k and stops when the remaining graph has minimum degree at least k. Note that even a vertex with very high degree can be eventually deleted since a cascading sequence of deletions may occur. From now on, assume we are obtaining the k-core by applying this deletion procedure (sometimes we will use variants of this procedure but they are all essentially the same). The deletion of a single edge can have a catastrophic effect on the k-core. For example, by deleting any edge of a connected k-regular graph, the resulting graph has an empty k-core. In fact, for any graph G such that there is Hamilton path P where each vertex but the first is adjacent to at most $k-1$ vertices that appear after it in P and the degree of the first vertex is k, the deletion of any edge incident with the first vertex in P triggers a cascading effect that causes the whole k-core to collapse when applying the deletion procedure. If the deletion of a single random edge has a nonnegligible probability of making the k-core empty or at least much smaller than the original k-core, then one could say that the graph was weak as a k-core. If, on the other hand, the number of vertices of the new k-core is close to the original one with high probability, then one could say that the graph was a robust k-core. So the number of vertices that are excluded from the original k-core can be seen as a measure of robustness of the graph as a k-core. In this chapter, we will show how this procedure behaves for random k-cores. In particular, we are interested in the robustness of the k-core of $G(n, m)$.

It is then relevant to know an estimate for the number of vertices in the k-core of $G(n, m)$. Let us recapitulate some results that were already mentioned in the Introduction. Łuczak[46, 45] has proved that the appearance of a nonempty k-core is a remarkable phenomenon: in the random graph process for $G(n, m)$, the first nonempty k-core is of order at least $0.0002n$ vertices. Informally, this means that the k-core is born 'giant'. Pittel, Spencer and Wormald [54] proved a very precise result that determines the threshold c_k for the emergence of a giant k-core in $G(n,m)$: when the average degree $2m/n$ is below $c_k - n^{-1/2-\varepsilon}$ for some positive $\varepsilon < 1/2$, the k-core of $G(n,m)$ is empty a.a.s. and when the average degree is above $c_k + n^{-1/2-\varepsilon}$ it has order $\Theta(n)$ a.a.s. Not only that, they also give quite precise estimates in the number of vertices in the k-core throughout the whole graph process. After this threshold result, many proofs were given for the emergence of a giant k-core in graphs and hypergraphs, using a variety of techniques; see [20, 28, 51, 17, 40, 35, 36, 59]. We show that, if the average degree is above $c_k + \varepsilon$ for a positive constant ε, the k-core is quite robust: the number of vertex deletions triggered by the deletion of a single random edge is bounded in probability (that is, for any $h(n) \to \infty$, however slowly, the probability that more than $h(n)$ vertices are deleted goes to zero). We actually obtain more general results, which we describe next.

We consider k-cores chosen uniformly at random from the k-cores with vertex set $[n]$ and m edges. As we have already mentioned, we use a deletion procedure to find the k-core of a graph. We analyse the number the vertices deleted when this procedure is applied to $G - e$, where G is a random k-core with given number of vertices and edges and e is an edge chosen u.a.r. in $E(G)$. We show that, if the average degree $c = 2m/n$ of G tends to k, the k-core of $G - e$ is empty a.a.s. An intuitive reason for why this happens is the following. As mentioned before, for any k-regular connected graph the graph obtained by deleting any single edge has an empty k-core. Thus, if c is exactly k, that is G is k-regular, the deletion of any edge makes the deletion procedure delete the whole connected component that contained that edge. Moreover, the k-core G is k-connected a.a.s., and so $G - e$ has an empty k-core a.a.s. It is then reasonable to expect that, when c is very close to k, the k-core of $G - e$ is very small or even empty. We can conclude that the k-core of $G - e$ is empty for $c \to k$, because we can show that a.a.s. it should have less than γn vertices for a constant γ (which we can choose to be as small as we want) and one can deduce from a result by Janson and Luczak [36] that, there exists a constant $\gamma_0 > 0$ such that, a.a.s., either the k-core of $G - e$ has at least $\gamma_0 n$ vertices or it is empty. This is similar to the aforementioned result by Łuczak that says that, a.a.s., either the k-core of $G(n,m)$ has at least $0.0002n$ vertices or it is empty.

Next we consider the case when the average degree c of G is greater than $k + \varepsilon$. We define a constant $c'_k > k$ and analyse the behaviour of the deletion procedure when the average degree c is below and when it is above c'_k. For bounded $c > c'_k + \psi(n)$ with $\psi = \omega(n^{-1/4})$, we have that, for any $h(n) = \omega(\psi(n)^{-1})$, the probability that more than $h(n)$ vertices are deleted goes to zero. Roughly speaking, this means that G is quite robust as a k-core. In the intermediate case $k + \varepsilon < c < c'_k - \varepsilon$, all vertices are deleted with probability bounded away from zero. Since $c > k + \varepsilon$, the deletion procedure has a nonneglible probability of stopping without deleting any vertices at all and so it is not possible to determine the outcome of the deletion procedure a.a.s. Nonetheless, we also prove that, for any $h(n) \to \infty$, a.a.s. either less than $h(n)$ vertices are deleted or the whole k-core collapses.

For the k-core of $G(n,m)$, the relation between the threshold c_k for the appearance of the

k-core and the constant c'_k we defined plays an important role. For any constant $\varepsilon > 0$, if $c > c_k + n^{-1/2+\varepsilon}$, the average degree of the k-core of G is asymptotic to a certain increasing function of c with probability tending to 1. The constant c'_k is actually defined as the value of this function at the threshold c_k. This implies that, if the average degree of $G(n, m)$ is above $c_k + \omega(n^{-1/4})$, the average degree of its k-core is above c'_k a.a.s. Therefore, using our result for random k-cores with average degree $c'_k + \omega(n^{-1/4})$, we can deduce that, if the average degree of $G(n, m)$ is $c_k + \phi(n)$ with $\phi(n) = \omega(n^{-1/4})$, then, for any function $h(n) = \omega(\phi(n)^{-1})$, less than $h(n)$ vertices are deleted a.a.s.

5.1 Main results

Let $G_k = G_k(n, m)$ be a graph sampled uniformly at random from the (simple) k-cores with vertex set $[n]$ and $m = m(n)$ edges. For any graph H, let $K(H)$ denote the k-core of H and let $W(H)$ be the random variable $|V(H)| - |V(K(H - e))|$, where e is an edge chosen uniformly at random from the edges of H. Note that $|V(H)| - |V(K(H - e))|$ is the number of vertices we delete from $H - e$ to obtain its k-core.

Recall that, for every integer $k \geq 0$, $f_k(\lambda) := e^\lambda - \sum_{i=0}^{k-1} \lambda^i/i!$. For every integer $k \geq 1$, let

$$h_k(\mu) = \frac{e^\mu \mu}{f_{k-1}(\mu)}.$$

For every integer $k \geq 3$, let

$$c_k = \inf\{h_k(\mu) : \mu > 0\} \tag{5.1}$$

and let

$$\mu_{k,c_k} \text{ be the unique positive solution of } c_k = h_k(\mu). \tag{5.2}$$

We discuss the existence of μ_{k,c_k} later. Let

$$c'_k = \frac{\mu_{k,c_k} f_{k-1}(\mu_{k,c_k})}{f_k(\mu_{k,c_k})}. \tag{5.3}$$

Our main result describes the behaviour of $W(G_k(n, m))$ according to the range of m. We remark that $W(G_k(n, m))$ is a random variable in the uniform probability space with ground set $\{(G, e) : G \in \mathcal{G},\ e \in E(G)\}$, where $\mathcal{G}$ is the set of all k-cores with vertex set $[n]$ and m edges. One consequence of working with this probability space is that proving that an event holds with probability bounded away from zero in it still leaves the possibility that, in a non-negligible proportion of graphs in $\mathcal{G}$, the event does not hold.

Theorem 5.1.1. Let $k \geq 3$ be a fixed integer. Let $m = m(n)$ and $c = 2m/n$. Then the following hold.

(i) If $c \geq k$ and $c \to k$, then $W(G_k(n,m)) = n$ a.a.s.

(ii) Let $\varepsilon > 0$ be a fixed real. Suppose that $k + \varepsilon \leq c \leq c'_k - \varepsilon$. For any function $h(n) \to \infty$, we have that a.a.s. $W(G_k(n,m)) \leq h(n)$ or $W(G_k(n,m)) = n$. Moreover, $W(G_k(n,m)) = n$ with probability bounded away from zero.

(iii) Let $\psi(n) = \omega(n^{-1/4})$ be a positive function and let C_0 be a constant. Suppose that $c'_k + \psi(n) \leq c \leq C_0$. For every $h(n) = \omega(\psi(n)^{-1})$, we have that $\mathbb{P}(W(G_k(n,m)) \geq h(n)) \to 0$.

We remark that there are some known results about the k-core of random graphs with given degree sequence under some constraints on the degree sequences (see [35, 20, 28]). Since the degree sequence of a graph G and the degree sequence of $G - e$ for any edge $e \in E(G)$ are very similar, it is intuitive that one can draw some conclusions about $W(G_k(n,m))$ by using these results. Indeed, in the case $c > c_k + \varepsilon$ is bounded, one can use [35] to conclude that $W(G_k(n,m)) = o(n)$ a.a.s. We were not able to derive results for the cases (i) and (ii) directly from known results about the k-core of random graphs with given degree sequence. These results classify the degree sequences and provide results according to such classification. The k-core of $G_k(n,m)$ is the whole graph while, according to our results, the k-core of $G_k(n,m) - e$ can be empty in cases (i) and (ii) with non-negligible probability. Since the deletion of a single edge has very little effect in the degree sequence of the graph, the existing results we mentioned cannot apply to these cases since they do not draw different conclusions for two almost identical degree sequences.

We apply Theorem 5.1.1 to study the robustness of the k-core of $G(n,m)$, the random graph with uniform distribution on all graphs with vertex set $[n]$ and m edges.

Corollary 5.1.2. Let $k \geq 3$ be a fixed integer. Let $m = m(n)$ and suppose that $c = 2m/n = c_k + \psi(n) \geq c_k + n^{-\delta}$ and $c \leq C_0$, where δ is a constant in $(0, 1/4)$ and C_0 is a constant. Then, for every $h(n) = \omega(\psi(n)^{-1})$, we have that $\mathbb{P}(W\big(K\big(G(n,m)\big)\big) \geq h(n)) \to 0$.

The proof of Theorem 5.1.1(iii) has a simple strategy. We define a deletion procedure to obtain the k-core of a random multigraph with minimum degree at least k after deleting one random edge. We then define a branching process and couple it with the deletion procedure in a way that the number of particles alive in the branching process is an upper bound for the number of edges marked to be deleted in the deletion procedure. We show that the branching process faces extinction in at most $h(n)$ steps a.a.s., which implies that the deletion procedure stops in at most $h(n)$ steps a.a.s. This implies that the number of deleted vertices is at most $h(n)$ a.a.s. We then use the well-known fact (Theorem 2.2.2) that the probability that the random multigraph is simple is $\Omega(1)$ and so the number of vertices that have to be deleted from the random simple k-core is also at most $h(n)$ a.a.s.

We now describe the strategy for the proofs of Theorems 5.1.1(i) and (ii). First we define a random walk and couple it with the deletion procedure for a random multigraph in a way that the position of the walk is always bounded from above by the number of endpoints of edges marked to be deleted. The coupling holds for at least $\Theta(n)$ steps. We then show that a.a.s. the random walk either reaches zero in at most $h(n)$ steps or it remains in the positive axis for at least $\Theta(n)$ steps. The latter holds with probability bounded away from 0 for $c \in (k, c'_k)$ bounded away from the ends of this interval, and with probability going to 1 for $c \to k$ when $h(n)$ goes to infinity sufficiently slowly. We can then conclude that a.a.s. the deletion procedure for the random multigraph either deletes at most $h(n)$ vertices or it deletes at least $\Theta(n)$ vertices, with the latter holding with probability bounded away from 0 for $k \leq c \leq c'_k - \varepsilon$ (and with probability going to one in the case $c \to k$). Since the probability that the random multigraph is simple is $\Omega(1)$, the same result can be deduced for the random simple k-core except for the part that states that the number of vertices deleted is $\Theta(n)$ with probability bounded away from zero. To deal with this, we show how to couple the deletion process of the random multigraph and the random simple k-core for $h(n)$ steps, where $h(n)$ goes to infinity sufficiently slowly.

We then use the differential equation method as described in Section 2.9 to show that, conditioning upon deleting at least $\Theta(n)$ vertices, the deletion procedure continues until we have only γn vertices a.a.s., where we can choose the constant $\gamma > 0$ to be arbitrarily small. We finish the proof by applying a result by Janson and Luczak [35] from which we can deduce that for a small enough $\gamma > 0$ the k-core should either be empty or have more than γn vertices a.a.s.

The differential equations related to deletion procedures for finding the k-cores of hypergraphs are strongly related to those for graphs (see, e.g. [17]). For this reason, we expect that the techniques we used can be extended to analyse robustness aspects of k-cores in hypergraphs.

This chapter is organized as follows. In Section 5.2, we define a deletion procedure for finding the k-core of a random multigraph $G - e$ and random walks that approximate the behaviour of the deletion procedure. In Section 5.3, we prove Theorem 5.1.1(iii) and Corollary 5.1.2. In Section 5.4, we prove an intermediate result for the cases in (i) and (ii) in Theorem 5.1.1: we show that, for $c \leq c'_k - \varepsilon$ and any $h(n) \to \infty$, the k-core of a random multigraph $G - e$ either has $n - \Omega(n)$ vertices or it has at least $n - h(n)$ vertices a.a.s., and, for $c \to k$, it has $n - \Omega(n)$ vertices a.a.s. In Section 5.5, we prove Theorem 5.1.1(i) and (ii), except for the claim in part (ii) that says that $W(G_k(n, m)) = n$ with probability bounded away from zero, which is handled in Section 5.6.

5.2 Random walks and a deletion procedure

In this section, we will describe a deletion procedure for finding the k-core of a random multigraph after the deletion of a random edge and we will define random walks that will help us to analyse this procedure.

We will use the allocation model $G_k^{\mathrm{multi}}(n, m)$ restricted to k-cores as described in Section 2.2. As we mentioned before in Section 2.2, by conditioning $G_k^{\mathrm{multi}}(n, m)$ to be simple, we obtain a uniform probability space on k-cores with vertex set $[n]$ and m edges. That is, $G_k(n,m)$ has the same distribution as $G_k^{\text{multi}}(n,m)$ conditioned upon simple graphs.

Let $\mathcal{D}_k(n,m)$ be the set of $\mathbf{d} \in \mathbb{N}^n$ with $\sum_{i=1}^n d_i = 2m$ and $d_i \geq k$ for all $i \in [n]$. Recall that $G^{\text{multi}}(\mathbf{d})$ denotes the graph generated using the pairing model and degree sequence $\mathbf{d}$, which we have introduced in Section 2.2. We recall a few properties of $G^{\text{multi}}(\mathbf{d})$ that we have mentioned in Section 2.2. The random graph $G_k^{\text{multi}}(n,m)$ conditioned upon having degree sequence $\mathbf{d}$ has the same distribution as $G^{\text{multi}}(\mathbf{d})$. By Lemma 2.10.2, we have that the degree sequence of $G_k^{\text{multi}}(n,m)$ has the same distribution as $\mathbf{Y} = (Y_1, \ldots, Y_n)$ where the Y_i's are independent truncated Poisson random variables with parameters (k, λ) conditioned upon the event Σ that $\sum_{i=1}^n Y_i = 2m$. For more information on truncated Poisson random variables, see Section 2.10.

For every multigraph H, let $\mathbf{d}(H)$ denote the degree sequence of H. For any $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, let $D_j(\mathbf{d})$ be the number of occurrences of j in $\mathbf{d}$ and let $\eta(\mathbf{d}) = \sum_{i=1}^n d_i(d_i - 1)/(2m)$.

5.2.1 A deletion procedure

In this section, we describe a deletion procedure for finding the k-core of the random multigraph $G_k^{\text{multi}}(\mathbf{d}) - e$, where e is an edge chosen u.a.r. from the edges of $G_k^{\text{multi}}(\mathbf{d})$. We will sample $G_k^{\text{multi}}(\mathbf{d})$ using the pairing model by discovering one edge at a time. We start by choosing e by picking two points uniformly at random from the set of all points.

Deletion procedure ($\mathbf{d}$)

- Partition $[2m]$ into n bins $S_1, \ldots, S_n$ such that $|S_i| = d_i$ for every $1 \leq i \leq n$.
- Iteration 0: choose e by picking distinct points u and v uniformly at random from $[2m]$. Delete u and v and **mark** all points in bins of size less than k.
- Loop: while there is a marked undeleted point, choose one such point u u.a.r. and find the other end v of the edge incident to u. Delete u and v. If v was in a bin of size exactly k (now of size $k-1$ because we deleted v), mark all the other points in the bin.

After the deletion procedure is over, the k-core can be obtained by adding a random matching uniformly at random on the undeleted points. Let $Z_0(\mathbf{d})$ denote the number of marked points after the deletion of the edge e chosen in Iteration 0. Note that $Z_0(\mathbf{d}) \in \{0, k-2, k-1, 2(k-1)\}$. The case $Z_0(\mathbf{d}) = k-2$ is very uncommon since it means the edge e has both ends in the same bin and such bin has size k.

Let $Y_j(\mathbf{d})$ be the number of undeleted marked points after the j-th iteration of the loop (and $Y_0(\mathbf{d}) = Z_0(\mathbf{d})$). The procedure stops when $Y_j(\mathbf{d}) = 0$. Let $Z_j(\mathbf{d})$ be the number of points that are marked in the j-th iteration of the loop.

We mark new points in an iteration of the loop if v was in a bin of size k. The probability that this happens (denoted by $p_j(\mathbf{d})$) is the ratio of the number of unmarked points in bins of (current) size k and the number of undeleted points other than u. If v is also a marked point, then no new points will be marked and v is deleted. In this case, $Z_j(\mathbf{d}) = -1$ and the probability that this happens (denoted by $p'_j(\mathbf{d})$) is the ratio of the marked undeleted points other than u and the number of undeleted points other than u. Thus, in the j-th iteration of the loop,

$$Z_j(\mathbf{d}) = \begin{cases} k-1, & \text{with probability } p_j(\mathbf{d}); \\ -1, & \text{with probability } p'_j(\mathbf{d}); \\ 0, & \text{otherwise.} \end{cases}$$

The probabilities of $p_j(\mathbf{d})$ and $p'_j(\mathbf{d})$ are analysed later. We have that

$$W(G_k^{\text{multi}}(\mathbf{d})) \geq \sum_{j\geq 0} \frac{Z_j(\mathbf{d})}{k-1}$$

and the number of edges deleted is the number of steps performed by the deletion procedure.

We have that, in the j-th iteration of the loop,

$$\frac{kD_k(\mathbf{d}) - (j+1)(k-1)}{2m-2j-1} \leq p_j(\mathbf{d}) \leq \frac{kD_k(\mathbf{d}) + jk}{2m-2j-1}, \tag{5.4}$$

which implies

$$p_j(\mathbf{d}) = \frac{kD_k(\mathbf{d})}{2m-2j-1} + O\left(\frac{j}{2m-2j-1}\right). \tag{5.5}$$

5.2.2 Random walks

Given c and k, we will define random walks in $\mathbb{Z}$ that will help us to study the behaviour of the deletion procedure as we explained in Section 5.1. Recall that $\lambda(k,c)$ is the unique positive root of $\lambda f_{k-1}(\lambda)/f_k(\lambda) = c$. For more properties of $\lambda(k,c)$, see Section 2.10.

Let

$$q(k,c) = \frac{\lambda(k,c)^{k-1}}{(k-1)!f_{k-1}(\lambda(k,c))}. \tag{5.6}$$

Let $Z(k,c)$ be a random variable such that

$$Z(k,c) = \begin{cases} k-1, & \text{with probability } q(k,c); \\ 0, & \text{otherwise.} \end{cases}$$

The variable Z_0 is set to be equal to $Z_0(\mathbf{d})$. Let $Y_0 = Z_0$. For $j > 0$, let $Y_j = Y_{j-1} + Z_j - 1$ where Z_j has same distribution as $Z(k,c)$ and the variable Z_j is independent from $Z_0, Z_1, \ldots, Z_{j-1}$. Thus, we defined a random walk such that the position in iteration j is Y_j and the drift is given by $Z_j - 1$. Similarly, for $\xi = \xi(n) \geq 0$ and $\xi \leq 1 - q(k,c)$, define the random variable $Z^+(k,c,\xi)$ by

$$Z^+(k,c,\xi) = \begin{cases} k-1, & \text{with probability } q(k,c)+\xi; \\ 0, & \text{otherwise.} \end{cases}$$

The variable Z_0^+ is set to be equal to $Z_0(\mathbf{d})$. Let $Y_0^+ = Z_0^+$. For $j > 0$, let $Y_j^+ = Y_{j-1}^+ + Z_j^+ - 1$ where Z_j^+ has same distribution as $Z^+(k,c,\xi)$ and the variable Z_j^+ is independent from $Z_0^+, Z_1^+, \ldots, Z_{j-1}^+$. Note that $(Y_j)_{j\in\mathbb{N}}$ and $(Y_j^+)_{j\in\mathbb{N}}$ are actually branching processes.

For $\xi = \xi(n) \geq 0$ and $\xi \leq q(k,c)$, define the random variable $Z^-(k,c,\xi)$ by

$$Z^-(k,c,\xi) = \begin{cases} k-1, & \text{with probability } q(k,c)-\xi; \\ -1, & \text{with probability } \xi; \\ 0, & \text{otherwise.} \end{cases}$$

The variable Z_0^- is set to be equal to $Z_0(\mathbf{d})$. Let $Y_0^- = Z_0^-$. For $j > 0$, let $Y_j^- = Y_{j-1}^- + Z_j^- - 1$ where Z_j^- has same distribution as $Z^-(k,c,\xi)$ and the variable Z_j^- is independent from $Z_0^-, Z_1^-, \ldots, Z_{j-1}^-$.

We say that Y_j is the number of particles alive in iteration j and that Z_j is the number of particles born in iteration j (and similarly for Y_j^+, Z_j^+, and Y_j^-, Z_j^-).

The random walk given by $Z^+(k,c,\xi)$ is going to be used to bound the number of marked points in the deletion process by above, while the random walk given by $Z^-(k,c,\xi)$ will bound it from below. Here we will prove some properties of these random walks.

Recall that $h_k(\mu) = \mu e^\mu / f_{k-1}(\mu)$ and $c_k = \inf\{h_k(\mu) : \mu > 0\} = h_k(\mu_{k,c_k})$. Here we justify why the infimum is reached by a unique μ.

Lemma 5.2.1. The infimum of $\{h_k(\mu) : \mu > 0\}$ is reached by a unique positive μ that satisfies

$$\frac{\mu^{k-1}}{(k-2)!} = f_{k-1}(\mu). \tag{5.7}$$

Moreover, $h_k(\mu) = c$ has exactly two positive solutions when $c > c_k$.

Proof. It is easy to see that h_k is differentiable and its first derivative is

$$\frac{e^\mu}{f_{k-1}(\mu)}\Big(1 + \mu - \mu\frac{f_{k-2}(\mu)}{f_{k-1}(\mu)}\Big). \tag{5.8}$$

When $\mu \to 0$ and $k \geq 3$, we have that

$$h_k(\mu) = \frac{\mu e^\mu}{f_{k-1}(\mu)} = \frac{\mu}{\mu^{k-1}/(k-1)!}(1 + O(\mu)) = \frac{(k-1)!}{\mu^{k-2}}(1 + O(\mu)),$$

which goes to ∞ as μ goes to 0. When $\mu \to \infty$, using $e^\mu \geq f_{k-1}(\mu)$,

$$h_k(\mu) = \frac{\mu e^\mu}{f_{k-1}(\mu)} \geq \mu,$$

which goes to ∞ as μ goes to ∞. Thus, it suffices to show that there is a unique μ such that (5.8) is zero. Using the fact that $f_{k-2}(\mu) = f_{k-1}(\mu) + \mu^{k-2}/(k-2)!$, we have that

$$\frac{e^\mu}{f_{k-1}(\mu)}\Big(1 + \mu - \mu\frac{f_{k-2}(\mu)}{f_{k-1}(\mu)}\Big) = \frac{e^\mu}{f_{k-1}(\mu)}\Big(1 - \frac{\mu^{k-1}}{(k-2)!f_{k-1}(\mu)}\Big)$$

and so the first derivative is 0 if and only if

$$\frac{\mu^{k-1}}{(k-2)!} = f_{k-1}(\mu). \tag{5.9}$$

Moreover, $f'_\ell(\mu) = f_{\ell-1}(\mu) > 0$ for $\mu > 0$ and any integer ℓ. Thus, the functions on both sides of (5.9) are convex and increasing for $\mu > 0$ and so (5.9) has a unique solution. This implies that the equation $h_k(\mu) = c$ has exactly one positive solution for $c = c_k$ and exactly two solutions when $c > c_k$. □

In view of Lemma 5.2.1, define $\mu_{k,c}$ as the largest solution of $h_k(\mu) = c$.

Lemma 5.2.2. The following hold:

(i) $\mathbb{E}\left(Z(k,c)\right)$ is a strictly decreasing function of c for $c > k$, and $\mathbb{E}\left(Z(k,c'_k)\right) = 1$.

(ii) For any $\varepsilon > 0$ with $c'_k - \varepsilon > k$, there exists a positive constant α such that $\mathbb{E}\left(Z(k,c'_k - \varepsilon)\right) > 1 + \alpha$.

(iii) Let C_0 be a nonnegative integer. There exists a positive constant β such that, for any nonnegative function $\psi(n) \leq C_0$, we have that $\mathbb{E}\left(Z(k,c'_k + \psi(n))\right) \leq 1 - \beta\psi(n)$.

Proof. Let $g(c) = \mathbb{E}\left(Z(k,c)\right)$. Note that $g(c) = (k-1)q(k,c)$ by the definition of $Z(k,c)$. By the definition of c'_k in (5.3), we have that $\lambda(k,c'_k) = \mu_{k,c_k}$ and so

$$g(c'_k) = (k-1)q(k,c'_k) = \frac{\lambda(k,c'_k)}{(k-2)!f_{k-1}(\lambda(k,c'_k))} = \frac{\mu_{k,c_k}}{(k-2)!f_{k-1}(\mu_{k,c_k})} = 1,$$

since (5.7) holds for $\mu = \mu_{k,c_k}$.

We have that $\lambda(k,c)$ is a strictly increasing function of c and vice-versa by Lemma 2.10.4. If $c > k$, then $\lambda(k,c) > 0$. Thus, by considering $c = c(\lambda) = \lambda f_{k-1}(\lambda)/f_k(\lambda)$ and differentiating with respect to λ, we get

$$\begin{aligned}\frac{d}{d\lambda}q(k,c) &= \frac{1}{(k-1)!}\left(\frac{(k-1)\lambda^{k-2}}{f_{k-1}(\lambda)} - \frac{\lambda^{k-2}f_{k-2}(\lambda)}{f_{k-1}(\lambda)^2}\right)\\ &= \frac{\lambda^{k-2}}{(k-1)!f_{k-1}(\lambda)}\left(k-1-\frac{\lambda f_{k-2}(\lambda)}{f_{k-1}(\lambda)}\right) < 0\end{aligned}$$

since $\lambda f_{k-2}(\lambda)/f_{k-1}(\lambda) > k-1$ for $\lambda > 0$ (see Lemma 2.10.4). Thus, $g(c)$ is strictly decreasing for $c > k$. This finishes the proof of (i).

We now prove (ii) and (iii). By Lemma 2.10.4, for $\varepsilon' > 0$ sufficiently small we have that $k < c(\varepsilon') < k + \varepsilon$. It is easy to see that $c(\lambda)$ is a continuously differentiable function on $\lambda \in [\varepsilon', \infty)$ with $\varepsilon' > 0$. Moreover, as already mentioned, $c(\lambda)$ is a strictly increasing function of λ. By the Inverse Function Theorem, this implies that $\lambda(k,c)$ is a continuously differentiable function on $c \in [c(\varepsilon'), C_0]$ and so $g(c)$ is a continuously differentiable function on c. Thus, the infimum $\inf\{g'(c) : c(\varepsilon') \le c \le c'_k\}$ and the supremum $\sup\{g'(c) : c'_k \le c \le C_0\}$ are both achieved and are both negative constants since $g(c)$ is strictly decreasing. By the Mean Value Theorem, there are positive constants α (depending on ε) and β (depending on C_0) such that $g(c) \ge 1 + \alpha|c - c'_k|$ for $c(\varepsilon') < c < c'_k$ and $g(c) \le 1 - \beta|c - c'_k|$ for $c'_k < c < C_0$. □

Lemma 5.2.3. Let $k, c(n), \xi(n)$ be such that $\mathbb{E}(Z^-(k,c,\xi)) > 1 + \varepsilon$, for some constant $\varepsilon > 0$. Then $\mathbb{P}(Y_j^- > 0,\ \forall j \ge 0 \,|\, Z_0^- > 0)$ is bounded away from 0, and, for any function $h(n) \to \infty$,

$$\mathbb{P}\left(Y_j^- > 0\ \forall j \ge h(n)\right) = 1 + o(1).$$

Proof. The first part follows from the fact that $(Y_j^-)_{j\ge0}$ is a random walk in $\mathbb{R}$ with positive expected drift (see e.g. [27, p. 366]).

For the rest of the proof the probabilities are always conditioned upon $Z_0^- > 0$. Note that Z_j^- are independent random variables with range $[-1, k-1]$. Thus, by Hoeffding's inequality (Theorem 2.3.1), since $Y_j^- = Z_0^- + \sum_{i=1}^{j}(Z_i^- - 1)$,

$$\mathbb{P}\left(Y_j^- < Z_0^- + (\varepsilon - \varepsilon')j\right) \le \exp\left(-\frac{\varepsilon'^2 j}{(k+1)^2}\right), \tag{5.10}$$

where $\varepsilon' \in (0, \varepsilon)$. For each $i \in \mathbb{N}$, let A_i denote the event that $Y_{h+it}^- > (\varepsilon - \varepsilon')(h + it)$ and let $\overline{A}_i$ denote its complement. For $j = h(n)$ and $i \in \mathbb{N}$ we have that (5.10) implies

$$\mathbb{P}(\overline{A_i}) \le \exp\left(-\frac{\varepsilon'^2(h+it)}{(k+1)^2}\right). \tag{5.11}$$

Let $t(n)$ be a function such that $t(n) \to \infty$ and $t(n) < (\varepsilon - \varepsilon')h(n)/2$. Since $Z_j^- \geq -1$, we have that $Y_{j'}^- > 0$ for all $j' \in [h(n) + it(n), h(n) + (i+1)t(n)]$ conditioned upon the event A_i. Thus,

$$\begin{aligned}
&\mathbb{P}\left(Y_j^- > 0\ \forall j \geq h(n)\right) = 1 - \mathbb{P}\left(Y_j^- \leq 0 \text{ for some } j \geq h(n)\right) \\
&\geq 1 - \sum_{i\geq 0} \mathbb{P}\left(Y_j^- \leq 0 \text{ for some } j \in [h+it, h+(i+1)t] \,\middle|\, A_i\right) \mathbb{P}\left(A_i\right) - \sum_{i\geq 0} \mathbb{P}(\overline{A_i}) \\
&= 1 - \sum_{i\geq 0} \mathbb{P}(\overline{A_i}) \geq 1 - \sum_{i=0} \exp\left(-\frac{\varepsilon'^2(h+it)}{(k+1)^2}\right) \quad \text{by (5.11)} \\
&= 1 - \exp\left(-\frac{\varepsilon'^2 h}{(k+1)^2}\right)\left(1 - \exp\left(\frac{\varepsilon'^2 t}{(k+1)^2}\right)\right)^{-1} = 1 + o(1).
\end{aligned}$$

□

5.3 The case $c > c_k' + \omega(n^{-1/4})$

In this section, we will prove that the random k-core of $G_k(n,m) - e$ with $c = 2m/n > c_k' + \omega(n^{-1/4})$ (and bounded) has size 'close' to n. More precisely, we prove Theorem 5.1.1(iii). We start by proving a version of Theorem 5.1.1(iii) for random multigraphs with given degree sequence.

Theorem 5.3.1. Let $\psi(n) = \omega(n^{-1/4})$ be a positive function and let C_0 be a constant. Suppose that $m = m(n)$ is such that $c = 2m/n$ satisfies $c_k' + \psi(n) \leq c \leq C_0$. Let $\mathbf{d} \in \mathcal{D}_k(n,m)$ be such that $|D_k(\mathbf{d}) - \mathbb{E}\left(D_k(\mathbf{Y})\right)| \leq n\phi(n)$ for $\phi(n) = o(\psi(n))$, where $\mathbf{Y} = (Y_1, \ldots, Y_n)$ and the Y_i's are independent truncated Poisson random variables with parameters $(k, \lambda(k,c))$. For every $h(n) = \omega(\psi(n)^{-1})$, we have that $\mathbb{P}(W(G^{\text{multi}}(\mathbf{d})) \geq h(n)) = o(1)$.

Using Theorem 5.3.1, we can deduce a result about multigraphs with given number of vertices and edges, which is then used to prove Theorem 5.1.1(iii).

Corollary 5.3.2. Let $\psi(n) = \omega(n^{-1/4})$ be a positive function and let C_0 be a constant. Suppose that $c = 2m/n$ is such that $c_k' + \psi(n) \leq c \leq C_0$. For every $h(n) = \omega(\psi(n)^{-1})$, we have that $\mathbb{P}(W(G_k^{\text{multi}}(n,m)) \geq h(n)) = o(1)$.

Proof of Theorem 5.3.1. We will choose ξ big enough so that $Z_j(\mathbf{d})$ is stochastically bounded from above by Z_j^+ for $j \leq t(n)$ steps, where we choose $t(n)$ later. That is, we will couple $(Z_j(\mathbf{d}))_{j=0,\ldots,t}$ and $(Z_j^+)_{j=0,\ldots,t}$ so that $Z_j(\mathbf{d}) \leq Z_j^+$ for $j \leq t(n)$.

For the first step we set $Z_0^+ = Z_0(\mathbf{d})$. To couple $Z_j(\mathbf{d})$ and Z_j^+, we will show that one can choose $\xi > 0$ so that for $j \leq t(n)$,

$$\mathbb{P}(Z_j^+ = k-1) = q(k,c) + \xi > p_j(\mathbf{d}) = \mathbb{P}(Z_j(\mathbf{d}) = k-1). \tag{5.12}$$

Then one way of coupling $Z_j(\mathbf{d})$ and Z_j^+ would be to use a random variable X that has uniform distribution in $[0,1]$ and set

- $Z_j^+ = Z_j(\mathbf{d}) = k-1$ if $X \in [0, p_j(\mathbf{d})]$;
- $Z_j^+ = Z_j(\mathbf{d}) = 0$ if $X \in (p_j(\mathbf{d}), p_j(\mathbf{d}) + 1 - q(k,c) - \xi]$;
- $Z_j^+ = k-1$ if $X \in (p_j(\mathbf{d}) + 1 - q(k,c) - \xi, 1]$;
- $Z_j(\mathbf{d}) = -1$ if $X \in (p_j(\mathbf{d}) + 1 - q(k,c) - \xi, p_j(\mathbf{d}) + 1 - q(k,c) - \xi + p'_j(\mathbf{d})]$;
- $Z_j(\mathbf{d}) = 0$ if $X \in (p_j(\mathbf{d}) + 1 - q(k,c) - \xi + p'_j(\mathbf{d}), 1]$.

Recall we start the deletion process with n bins with d_i points inside each bin i. Let p denote the initial ratio between the number of points in bins of size k and the total number of points. Since $|D_k(\mathbf{d}) - \mathbb{E}(D_k(\mathbf{Y}))| \leq n\phi(n)$ for $\phi(n) = o(\psi(n))$, we have that, for some function $\phi_1(n)$ such that $\phi_1(n) = O(\phi(n))$,

$$\begin{aligned} p &= \frac{kD_k(\mathbf{d})}{2m} = \frac{k\mathbb{E}(D_k(\mathbf{Y}))}{2m} + \phi_1 = k\frac{\lambda(k,c)^k}{k! f_k(\lambda(k,c))}\frac{n}{2m} + \phi_1 \\ &= k\frac{\lambda(k,c)^k}{k! f_k(\lambda(k,c))}\frac{f_k(\lambda(k,c))}{\lambda(k,c) f_{k-1}(\lambda(k,c))} + \phi_1 = \frac{\lambda(k,c)^{k-1}}{(k-1)! f_{k-1}(\lambda(k,c))} + \phi_1 \\ &= q(k,c) + \phi_1. \end{aligned} \tag{5.13}$$

Choose $t(n) = \psi(n)^{-1} n^{\alpha}$, where α is constant in $(0, 1/2)$. By (5.5),

$$p_j(\mathbf{d}) \leq \frac{kD_k(\mathbf{d})}{2m}\left(1 + O\left(\frac{j}{n}\right)\right) + O\left(\frac{j}{2m-2j}\right) = p\left(1 + O\left(\frac{j}{n}\right)\right) + O\left(\frac{j}{2m-2j}\right).$$

Since $t(n) = \psi(n)^{-1} n^{\alpha}$ with $\alpha \in (0, 1/2)$ and $\psi(n)^{-1} = o(n^{1/4})$, we have that $t(n) = o(n^{3/4})$. Together with the fact that $2m \geq kn$, this implies that

$$p_j(\mathbf{d}) = p\left(1 + O\left(\frac{j}{n}\right)\right) + O\left(\frac{j}{2m-2j}\right) = p + O\left(\frac{t}{n}\right). \tag{5.14}$$

Moreover, since $\alpha < 1/2$ and $\psi(n) = \omega(n^{-1/4})$,

$$\psi^2 = \omega\left(n^{-1/2}\right) = \omega\left(n^{\alpha-1}\right) = \omega\left(\frac{t\psi}{n}\right)$$

and so $t/n = o(\psi)$, which implies

$$p_j(\mathbf{d}) = p + o(\psi) \tag{5.15}$$

by (5.14). Thus, by (5.13) and (5.15),

$$p_j(\mathbf{d}) = q(k,c) + o(\psi) + O(\phi) = q(k,c) + o(\psi),$$

So we can choose $\xi = o(\psi)$ satisfying (5.12). Note that we also need to choose ξ so that $1 - q(k,c) - \xi > 0$. Lemma 5.2.2 implies that $q(k,c) \le 1/(k-1)$ and so any $\xi < 1 - 1/(k-1)$ satisfies $1 - q(k,c) - \xi > 0$.

We can assume that $h(n) \le t(n)$. We will show that

$$\mathbb{P}\left(Y_j^+ = 0 \text{ for some } j \le t(n)\right) = 1 + o(1).$$

That is the random walk $(Y_j^+)_{j\in\mathbb{N}}$ reaches 0 in at most $t(n)$ steps a.a.s., which implies by our coupling that so does $(Y_j(\mathbf{d}))_j$ and so $W(G^{\text{multi}}(\mathbf{d})) \le t(n)$ a.a.s., proving Theorem 5.3.1.

We have that $\mathbb{E}\left(Z^+(k,c,\xi)\right) \le 1 - \beta\psi(n) + (k-1)\xi$ according to Lemma 5.2.2 for some positive constant β. Since $\xi = o(\psi)$, we have $\mathbb{E}\left(Z^+(k,c,\xi)\right) \le 1 - \beta'\psi(n)$ for some positive constant β'. Thus, we have that

$$\begin{aligned}\mathbb{E}\left(Y_t^+\right) &\le 2(k-1)(1-\beta'\psi(n))^{t(n)} = 2(k-1)\exp\left(t(n)\log(1-\beta'\psi(n))\right)\\ &\le 2(k-1)\exp\left(-t(n)\beta'\psi(n)\right) = 2(k-1)\exp\left(-\beta' n^{\alpha}\right) = o(1)\end{aligned}$$

because $t(n) = n^{\alpha}/\psi(n)$ with $\alpha > 0$. By Markov's inequality this implies

$$\mathbb{P}\left(Y_t^+ \ge 1\right) \le \mathbb{E}\left(Y_t^+\right) = o(1),$$

and we are done. □

5.3.1 Proof of Corollary 5.3.2 and Theorem 5.1.1(iii)

In this section we use Theorem 5.3.1, which is a result for the random multigraph $G^{\text{multi}}(\mathbf{d})$, to prove a result about the random multigraph $G_k^{\text{multi}}(n,m)$ which is then used to deduce a result for the random simple graph $G_k(n,m)$. More specifically, we prove Corollary 5.3.2 and Theorem 5.1.1(iii).

Let $h(n) = \omega(\psi(n)^{-1})$. Choose $\phi(n)$ such that $\phi(n) = o(\psi(n))$ and $\phi(n) = \omega(n^{-1/4})$. First we will prove Corollary 5.3.2. We will show that $\mathbf{d}(G_k^{\text{multi}}(n,m))$ is in the set of sequences that satisfy the hypotheses in Theorem 5.3.1 a.a.s. Intuitively, this means that such set of sequences contains all the 'typical' degree sequences for $G_k^{\text{multi}}(n,m)$. Let $\tilde{\mathcal{D}}_k(n,m)$ be the set of degree sequences $\mathbf{d}$ satisfying $|D_k(\mathbf{d}) - \mathbb{E}\left(D_k(\mathbf{Y})\right)| \le n\phi(n)$. By Corollary 2.10.2, $\mathbf{d}(G_k^{\text{multi}}(n,m))$ has the same distribution as $\mathbf{Y} = (Y_1,\ldots,Y_n)$ such the Y_i's are independent truncated Poisson random variables with parameters $(k, \lambda(k,c))$ and conditioned upon the event Σ that $\sum_i Y_i = 2m$. Using Chebyshev's inequality,

$$\mathbb{P}\left(\left|D_k(\mathbf{Y}) - \mathbb{E}\left(D_k(\mathbf{Y})\right)\right| \geq n\phi(n)\right) \leq \frac{n}{n^2\phi(n)^2}.$$

By Theorem 2.10.8, we have that

$$\mathbb{P}(\Sigma) \sim \frac{1}{\sqrt{2\pi nc(1+\eta_c - c)}} = \Omega\left(\frac{1}{\sqrt{n}}\right)$$

since c is bounded and $\eta_c \leq c$ by (2.7). Thus,

$$\mathbb{P}\left(\mathbf{d}(G_k^{\text{multi}}(n,m)) \notin \tilde{\mathcal{D}}_k(n,m)\right) \leq \frac{\mathbb{P}\left(\mathbf{Y} \notin \tilde{\mathcal{D}}_k(n,m)\right)}{\mathbb{P}(\Sigma)} = O\left(\frac{n\sqrt{n}}{n^2\phi(n)^2}\right) = o(1) \tag{5.16}$$

since $\phi = \omega(n^{-1/4})$.

For each n, we have that $\tilde{\mathcal{D}}_k(n,m)$ is a finite set and for every $\mathbf{d} \in \tilde{\mathcal{D}}_k(n,m)$, we have that $\mathbb{P}(W(G_k^{\text{multi}}(\mathbf{d})) \geq h(n)) = o(1)$ by Theorem 5.3.1. Thus, by Lemma 2.7.1, there exists $f(n) = o(1)$ such that

$$\mathbb{P}\left(W(G_k^{\text{multi}}(\mathbf{d})) \geq h(n)\right) \leq f(n), \tag{5.17}$$

for all $\mathbf{d} \in \tilde{\mathcal{D}}_k(n,m)$. We have that

$$\begin{aligned}
&\mathbb{P}\left(W(G_k^{\text{multi}}(n,m)) \geq h(n)\right) \\
&\leq \mathbb{P}\left(W(G_k^{\text{multi}}(n,m)) \geq h(n) \,\middle|\, \mathbf{d}(G_k^{\text{multi}}(n,m)) \in \tilde{\mathcal{D}}_k(n,m)\right) \mathbb{P}\left(\mathbf{d}(G_k^{\text{multi}}(n,m)) \in \tilde{\mathcal{D}}_k(n,m)\right) \\
&\quad + \mathbb{P}\left(\mathbf{d}(G_k^{\text{multi}}(n,m)) \notin \tilde{\mathcal{D}}_k(n,m)\right) \\
&= \mathbb{P}\left(W(G_k^{\text{multi}}(n,m)) \geq h(n) \,\middle|\, \mathbf{d}(G_k^{\text{multi}}(n,m)) \in \tilde{\mathcal{D}}_k(n,m)\right) + o(1), \quad \text{by (5.16)} \\
&= \sum_{\mathbf{d} \in \tilde{\mathcal{D}}_k(n,m)} \mathbb{P}\left(W(G_k^{\text{multi}}(\mathbf{d})) \geq h(n)\right) \mathbb{P}\left(\mathbf{d}(G_k^{\text{multi}}(n,m)) = \mathbf{d} \,\middle|\, \mathbf{d}(G_k^{\text{multi}}(n,m)) \in \tilde{\mathcal{D}}_k(n,m)\right) \\
&\leq \sum_{\mathbf{d} \in \tilde{\mathcal{D}}_k(n,m)} f(n)\, \mathbb{P}\left(\mathbf{d}(G_k^{\text{multi}}(n,m)) = \mathbf{d} \,\middle|\, \mathbf{d}(G_k^{\text{multi}}(n,m)) \in \tilde{\mathcal{D}}_k(n,m)\right) \quad \text{by (5.17)} \\
&\leq f(n) = o(1),
\end{aligned}$$

proving Corollary 5.3.2.

We will now prove Theorem 5.1.1(iii). To deduce the result for simple graphs, we impose further conditions on the degree sequences: let $\hat{\mathcal{D}}_k(n,m)$ be the set of degree sequences in $\tilde{\mathcal{D}}_k(n,m)$ that satisfy the conditions that $\max_i d_i \le n^{\varepsilon}$ for some $\varepsilon \in (0, 0.25)$ and that $|\eta(\mathbf{d}) - \mathbb{E}(\eta(\mathbf{Y}))| \le \phi(n)$. We have that, by Lemma 2.10.10, $\mathrm{Var}(Y_i(Y_i - 1)) = O(1)$ since c is bounded and so by Chebyshev's inequality,

$$\mathbb{P}\Big(|\eta(\mathbf{Y}) - \mathbb{E}(\eta(\mathbf{Y}))| \ge \phi(n)\Big) = O\left(\frac{1}{n\phi(n)^2}\right).$$

By (2.18), we have that

$$\mathbb{P}\left(\max_j Y_j \ge n^{\varepsilon}\right) = O\big(n\exp(-n^{\varepsilon}/2)\big).$$

This implies that $\mathbb{P}(\mathbf{d}(G_k^{\mathrm{multi}}(n,m)) \in \hat{\mathcal{D}}_k(n,m))$ is also $1+o(1)$. For $\mathbf{d} \in \hat{\mathcal{D}}_k(n,m)$, the probability of that $G_k^{\mathrm{multi}}(\mathbf{d})$ is simple is already known: by Theorem 2.2.2,

$$\begin{aligned}\mathbb{P}(G^{\mathrm{multi}}(\mathbf{d}) \text{ simple}) &= \exp\left(-\frac{\eta(\mathbf{d})}{2} - \frac{\eta(\mathbf{d})^2}{4} + O\left(\frac{\max_i d_i^4}{n}\right)\right) \\ &\sim \exp\left(-\frac{\eta_c}{2} - \frac{\eta_c^2}{4} + O\left(\frac{\max_i d_i^4}{n}\right)\right),\end{aligned} \tag{5.18}$$

where $\eta_c = \lambda(k,c) f_{k-2}(\lambda(k,c))/f_{k-1}(\lambda(k,c))$. For each n, we have that $\hat{\mathcal{D}}_k(n,m)$ is a finite set and so Lemma 2.7.1 and (5.18) imply that there exists $g(n) = o(1)$ such that

$$\left|\mathbb{P}(G^{\mathrm{multi}}(\mathbf{d}) \text{ simple}) - \exp\left(-\frac{\eta_c}{2} - \frac{\eta_c^2}{4}\right)\right| \le g(n), \tag{5.19}$$

for all $\mathbf{d} \in \hat{\mathcal{D}}_k(n,m)$. Hence,

$$\begin{aligned}\mathbb{P}\left(G_k^{\mathrm{multi}}(n,m) \text{ is simple}\right) &\ge \sum_{\mathbf{d} \in \hat{\mathcal{D}}_k(n,m)} \mathbb{P}\left(G^{\mathrm{multi}}(\mathbf{d}) \text{ simple}\right) \mathbb{P}\left(\mathbf{d}(G_k^{\mathrm{multi}}(n,m)) = \mathbf{d}\right) \\ &\sim \exp\left(-\frac{\eta_c}{2} - \frac{\eta_c^2}{4}\right) = \Omega(1)\end{aligned}$$

and so

$$\begin{aligned}&\mathbb{P}\Big(W(G_k(n,m)) \ge h(n)\Big) \\ &= \mathbb{P}\Big(W(G_k^{\mathrm{multi}}(n,m)) \ge h(n) \,\Big|\, G_k^{\mathrm{multi}}(n,m) \text{ is simple}\Big) \\ &\le \frac{\mathbb{P}\left(W(G_k^{\mathrm{multi}}(n,m)) \ge h(n)\right)}{\mathbb{P}\left(G_k^{\mathrm{multi}}(n,m) \text{ is simple}\right)} \\ &= o(1).\end{aligned}$$

This finishes the proof of Theorem 5.1.1(iii).

5.3.2 The k-core of $G(n,m)$

In this section, we use Theorem 5.1.1(iii) to deduce that the k-core of $G(n,m)$ is 'robust'. More specifically, we prove Corollary 5.1.2. We will use a result by Pittel, Spencer and Wormald [54, Theorem 2]. Although this result does not state the number of edges in the k-core, this can be obtained from its proof with the main steps in [54, Equations (6.18),(6.34)] and [54, Corollary 1] applied to J_1. We state [54, Theorem 2] with the addition of a concentration result for the number of edges here.

Theorem 5.3.3. Suppose $c > c_k + n^{-\delta}$, $\delta \in (0,1/2)$ being fixed. Fix $\sigma \in (3/4, 1-\delta/2)$ and $\bar{\zeta} = \min\{2\sigma - 3/2, 1/6\}$. Then with probability $\geq 1 + O(\exp(-n^{\zeta}))$ $(\forall \zeta < \bar{\zeta})$, the random graph $G(n, m = cn/2)$ contains a giant k-core with $e^{-\mu_{k,c}} f_k(\mu_{k,c})n + O(n^{\sigma})$ vertices and $(1/2)\mu_{k,c}e^{-\mu_{k,c}} f_{k-1}(\mu_{k,c})n + O(n^{\sigma})$ edges.

We will now prove Corollary 5.1.2.

Proof of Corollary 5.1.2. Recall that c is bounded and $c \geq c_k + n^{-\delta}$, where δ is a constant in $(0,1/4)$. So $\delta = 1/4 - \varepsilon$, where ε is a constant in $(0,1/4)$. Let $\varepsilon' < \varepsilon$ be a constant such that $\varepsilon' < 1/4 - \delta/2$. Fix $\sigma = 3/4 + \varepsilon'$. Thus, the average degree of the k-core is

$$\frac{\mu_{k,c} f_{k-1}(\mu_{k,c})}{f_k(\mu_{k,c})}\Big(1 + O(n^{-1/4+\varepsilon'})\Big).$$

The proof of Lemma 5.2.1 shows that $h'(\mu_{k,c_k}) = 0$ and $h'(\mu) > 0$ for $\mu > \mu_{k,c_k}$. Since c is bounded and by the definition of $\mu_{k,c}$, this implies that $\mu_{k,c} = \mu_{k,c_k} + \Omega(c - c_k)$. Moreover, the function $x \mapsto x f_{k-1}(x)/f_k(x)$ is smooth. Thus, the average degree of the k-core of $G(n,m)$ is $(c'_k + \Theta(c - c_k))(1 + O(n^{-1/4+\varepsilon'}))$. Since $c - c'_k > n^{-\delta} = n^{-1/4+\varepsilon}$ with $\varepsilon > \varepsilon'$, the average degree of the k-core is $c'_k + \Omega(c - c_k)$. Since $c - c_k = \omega(n^{-1/4})$, we can now apply Theorem 5.1.1(iii) to obtain the desired result. ☐

5.4 The case $k \leq c \leq c'_k - \varepsilon$: deleting $\Theta(n)$ vertices

In this section, we prove that, for $c \leq c'_k - \varepsilon$ and any $h(n) \to \infty$, the k-core of $G_k^{\text{multi}}(n,m) - e$ either has $n - \Omega(n)$ vertices or it has at least $n - h(n)$ vertices a.a.s., and it has $n - \Omega(n)$ vertices a.a.s. when $c \to k$. This is is an intermediate step for the proof of Theorem 5.1.1(i) and (ii).

First we obtain a result for $G^{\text{multi}}(\mathbf{d})$:

Theorem 5.4.1. Let $\varepsilon > 0$ be a fixed real and let $\phi(n) = o(1)$. Suppose that $k \leq c \leq c'_k - \varepsilon$. Let $\mathbf{d}$ be such that $D_k(\mathbf{d}) \geq \mathbb{E}\left(D_k(\mathbf{Y})\right)(1 - \phi(n))$, where $\mathbf{Y} = (Y_1, \ldots, Y_n)$ and the Y_i's are independent truncated Poisson random variables with parameters $(k, \lambda_{k,c})$. Then there exists a constant $\varepsilon' > 0$ (depending on ε) such that, for any function $h(n) \to \infty$, we have that a.a.s. $W(G^{\text{multi}}(\mathbf{d})) \le h(n)$ or $W(G^{\text{multi}}(\mathbf{d})) \ge \varepsilon' n$. Moreover, $W(G^{\text{multi}}(\mathbf{d})) \ge \varepsilon' n$ with probability bounded away from zero.

We can use Theorem 5.4.1 to prove the same result for $G_k^{\text{multi}}(n,m)$.

Corollary 5.4.2. Let $\varepsilon > 0$ be a fixed real. Suppose that $k \le c \le c_k' - \varepsilon$. Then there exists a constant $\varepsilon' > 0$ (depending on ε) such that, for any function $h(n) \to \infty$, we have that a.a.s. $W(G_k^{\text{multi}}(n,m)) \le h(n)$ or $W(G_k^{\text{multi}}(n,m)) \ge \varepsilon' n$. Moreover, $W(G_k^{\text{multi}}(n,m))) \ge \varepsilon' n$ with probability bounded away from zero.

For the case $c \to k$, Theorem 5.4.1 implies a stronger result because there is a function $h(n) \to \infty$ such that $W(G_k^{\text{multi}}(n,m)) \ge h(n)$ a.a.s. From this one can deduce the following result.

Corollary 5.4.3. There exists a constant $\varepsilon' > 0$ such that, if $c \to k$, then $W(G_k^{\text{multi}}(n,m))) \ge \varepsilon' n$ a.a.s.

Proof of Theorem 5.4.1. We will choose ξ so that $Z_j(\mathbf{d})$ is stochastically bounded from below by Z_j^-, where Z_j^- has the same distribution as $Z^-(k,c,\xi)$, and so that $\mathbb{E}\left(Z^-(k,c,\xi)\right)$ is bounded away from 1 from above.

Let $p = kD_k(\mathbf{d})/2m$. First we compute the probability that $Z_0(\mathbf{d}) > 0$. In Iteration 0, the probability that e has its endpoint u in a of bin of size k (and so $Z_0(\mathbf{d}) \ge k-2$) is p, since p is the ratio of points in bins of size k and the total number of points. Similarly to (5.13),

$$p \ge q(k,c) + O(\phi_1)$$

for some function $\phi_1 = O(\phi)$. Lemma 5.2.2 implies that $q(k,c) > 1/(k-1)$. Thus, $Z_0(\mathbf{d}) > 0$ with probability bounded away from zero.

For $j \ge 1$, by (5.5) we have that

$$p_j(\mathbf{d}) \ge q(k,c) + O(\phi_1) + O\left(\frac{t}{2m-2t}\right),$$

where $\phi_1 = O(\phi)$. Moreover, for $j \ge 1$,

$$p_j'(\mathbf{d}) \le \frac{(k-1)(j+1)}{2m-2j-1} = O\left(\frac{j}{n}\right).$$

Lemma 5.2.2 implies that $\mathbb{E}\left(Z^-(k,c,\xi)\right) \ge 1 + \alpha' - (k-1)\xi$ for some constant $\alpha' > 0$. Choose a fixed ξ in $(0, \alpha'/(k-1))$. Thus, we have $\mathbb{E}\left(Z^-(k,c,\xi)\right) \ge 1 + \alpha$ for some $\alpha > 0$.

We choose $\varepsilon'' > 0$ small enough so that, by setting $t(n) = \varepsilon'' n$, we have $p_j(\mathbf{d}) \geq q(k,c) - \xi$ and $\mathbb{P}(Z_j(\mathbf{d}) = -1) \leq \xi$ for any $j \leq t(n)$. This is possible since

$$\frac{t}{2m - t} = \frac{\varepsilon'' n}{2m - \varepsilon'' n} \tag{5.20}$$

goes to 0 as $\varepsilon'' \to 0$. Thus, we can couple the processes for at least $t(n) = \varepsilon'' n$ steps.

By Lemma 5.2.3, a.a.s. either $Y_j^- \leq 0$ for some $j \leq h(n)$ or $Y_j^- > 0$ for all j. Moreover, the latter occurs with probability bounded away from zero. Note that this means that

$$\mathbb{P}\left(\exists j \in [h(n), \varepsilon'' n] \text{ s.t. } Y_j^- \leq 0\right) = o(1)$$

and so, by the coupling,

$$\mathbb{P}\left(\exists j \in [h(n), \varepsilon'' n] \text{ s.t. } Y_j(\mathbf{d}) \leq 0\right) \leq \mathbb{P}\left(\exists j \in [h(n), \varepsilon'' n] \text{ s.t. } Y_j^- \leq 0\right) = o(1),$$

which implies that a.a.s. either $Y_j(\mathbf{d}) = 0$ for some $j \leq h(n)$ or $Y_j(\mathbf{d}) > 0$ for $1 \leq j \leq \varepsilon'' n$. Moreover, since

$$\mathbb{P}(\exists j \leq h(n) \text{ s.t. } Y_j(\mathbf{d}) = 0) \leq \mathbb{P}(\exists j \leq h(n) \text{ s.t. } Y_j^- = 0)$$

by the coupling, we have that $Y_j(\mathbf{d}) > 0$ for $1 \leq j \leq \varepsilon'' n$ with probability bounded away from zero. Thus, a.a.s. either $W(G^{\text{multi}}(\mathbf{d})) \leq h(n) + 2$ or $W(G_k^{\text{multi}}(n,m)) \geq \varepsilon'' n/(k-1)$, and the latter holds with probability bounded away from zero. This completes the proof of Theorem 5.4.1. □

5.4.1 Proof of Corollary 5.4.2 and Corollary 5.4.3

In this section, we use Theorem 5.4.1 to show that, for $c \leq c_k' - \varepsilon$ and any $h(n) \to \infty$, the k-core of $G_k^{\text{multi}}(n,m) - e$ either has $n - \Omega(n)$ vertices or it has at least $n - h(n)$ vertices a.a.s., and it has $n - \Omega(n)$ vertices a.a.s. when $c \to k$. More specifically, we prove Corollary 5.4.2 and Corollary 5.4.3. The proof of Corollary 5.4.2 is very similar to the proof of Corollary 5.3.2; we include it here for completeness.

First we prove Corollary 5.4.2. Let $\tilde{\mathcal{D}}_k(n,m)$ be the set of degree sequences $\mathbf{d}$ satisfying $D_k(\mathbf{d}) \geq \mathbb{E}\left(D_k(\mathbf{Y})\right)(1 - \phi(n))$, where we choose $\phi(n)$ later. Recall that $\mathbf{d}(G_k^{\text{multi}}(n,m))$ has the same distribution as $\mathbf{Y} = (Y_1, \ldots, Y_n)$ such the Y_i's are independent truncated Poisson random variables with parameters $(k, \lambda(k,c))$ and conditioned upon the event Σ that $\sum_i Y_i = 2m$. Using Chebyshev's inequality,

$$\mathbb{P}\left(\left|D_k(\mathbf{Y}) - \mathbb{E}\left(D_k(\mathbf{Y})\right)\right| \geq \mathbb{E}\left(D_k(\mathbf{Y})\right)\phi(n)\right) \leq \frac{n}{\mathbb{E}\left(D_k(\mathbf{Y})\right)^2 \phi(n)^2} = O\left(\frac{1}{n\phi(n)^2}\right) \tag{5.21}$$

since c is bounded. If $r = 2m - kn \leq n^{1/4}$, by Theorem 2.10.8 (Equation (2.15)), we have that

$$\mathbb{P}(\Sigma) \sim e^{-r}\frac{r^r}{r!} = \Omega\left(\frac{1}{\sqrt{r}}\right)$$

by Stirling's approximation (Lemma 2.5.1). If $r \geq n^{1/4}$, Theorem 2.10.8 (Equation (2.14)) implies

$$\mathbb{P}(\Sigma) \sim \frac{1}{\sqrt{2\pi nc(1+\eta_c - c)}} = \Omega\left(\frac{1}{\sqrt{n}}\right)$$

since c is bounded and $\eta_c \leq c$ by (2.7). Thus, $\mathbb{P}(\Sigma) = \Omega(1/\sqrt{n})$, which, together with (5.21), implies

$$\begin{aligned}\mathbb{P}\left(\mathbf{d}(G_k^{\text{multi}}(n,m)) \notin \tilde{\mathcal{D}}_k(n,m)(\phi)\right) &\leq \frac{\mathbb{P}\left(\left|D_k(\mathbf{Y}) - \mathbb{E}\left(D_k(\mathbf{Y})\right)\right| \geq \mathbb{E}\left(D_k(\mathbf{Y})\right)\phi(n)\right)}{\mathbb{P}(\Sigma)} \\ &= O\left(\frac{1}{\sqrt{n}\phi(n)^2}\right)\end{aligned} \tag{5.22}$$

So we choose $\phi = \omega(n^{-1/4})$. Let $A(\mathbf{d})$ be the event that $W(G^{\text{multi}}(\mathbf{d})) \leq h(n)$ or $W(G^{\text{multi}}(\mathbf{d})) \geq \varepsilon'' n$ and ε'' is given by Theorem 5.4.1. Let $\overline{A}(\mathbf{d})$ denote the complement of $A(\mathbf{d})$. Since $\tilde{\mathcal{D}}_k(n,m)$ is a finite set for each n and $\mathbb{P}(\overline{A}(\mathbf{d})) = o(1)$ for $\mathbf{d} \in \tilde{\mathcal{D}}_k(n,m)$, Lemma 2.7.1 implies that there exists a function $f(n) = o(1)$ such that

$$\mathbb{P}(\overline{A}(\mathbf{d})) \leq f(n) \tag{5.23}$$

for all $\mathbf{d} \in \tilde{\mathcal{D}}_k(n,m)$. Thus, by (5.23) and (5.21),

$$\begin{aligned}&\mathbb{P}\left(W(G^{\text{multi}}(\mathbf{d})) \in (h(n), \varepsilon'' n)\right) \\ &\leq \mathbb{P}\left(\overline{A}(\mathbf{d}) \,\middle|\, \mathbf{d}(G_k^{\text{multi}}(n,m)) \in \tilde{\mathcal{D}}_k(n,m)\right) \mathbb{P}\left(\mathbf{d}(G_k^{\text{multi}}(n,m)) \in \tilde{\mathcal{D}}_k(n,m)\right) + o(1) \\ &\leq f(n)(1+o(1)) + o(1) = o(1),\end{aligned}$$

proving Corollary 5.4.2.

Now we prove Corollary 5.4.3. Suppose $c \to k$. It suffices to prove that $Z_0(\mathbf{d}) > 0$ a.a.s. and that $p_j(\mathbf{d}) \sim 1$ for all $j \leq t(n)$ for some $t(n)$ that goes to infinity. In the proof of Theorem 5.4.1 we have that

$$p = q(k,c) + O(\phi_1)$$

where $\phi_1 = O(\phi)$ and so

$$p_j(\mathbf{d}) = q(k,c) + O(\phi_1) + O\left(\frac{t}{2m-t}\right)$$

by (5.5). Moreover, $\mathbb{P}(Z_0(\mathbf{d})) > p$. Thus, we only need to show that $q(k,c) \to 1$. Recall that

$$q(k,c) = \frac{\lambda(k,c)^{k-1}}{(k-1)!f_{k-1}(\lambda(k,c))}.$$

Since $c \to 0$, so does $\lambda(k,c)$. Hence, by computing the series of $q(k,c)$ with $\lambda(k,c) \to 0$,

$$q(k,c) = \frac{\lambda(k,c)^{k-1}}{(k-1)!\dfrac{\lambda(k,c)^{k-1}}{(k-1)!}}(1 + O(\lambda(k,c))) \sim 1.$$

Thus, we can choose $t(n)$ going to infinity slowly enough (depending only on ϕ and c) so that $Z_j(\mathbf{d}) = k-1$ for all $1 \le j \le t(n)$. Corollary 5.4.3 is now straightforward since we just proved that there exists $h(n) \to \infty$ such that $W(G^{\text{multi}}(\mathbf{d})) > h(n)$ a.a.s.

5.5 The case $k \le c \le c'_k - \varepsilon$

In this section, we will prove that, for $c \le c'_k - \varepsilon$ and any $h(n) \to \infty$, the k-core of $G_k(n,m) - e$ either has at least $n - h(n)$ vertices or it is empty a.a.s. and, for $c \to k$, it is empty a.a.s. More specifically, we will prove Theorem 5.1.1(i) and Theorem 5.1.1(ii) except for the claim that $W(G_k(n,m)) = n$ with probability bounded away from zero, which is handled in Section 5.6. We use the differential equation method as described in Section 2.9. We will also use some results from [17] about the system of differential equations for the case $c \ge k + \varepsilon$.

The strategy of the proof is the following. First we will show that, for a sufficiently small constant $\gamma > 0$, a.a.s. the k-core of $G_k^{\text{multi}}(n,m) - e$ has either at least γn vertices or it is empty. This is an application of a result by Janson and Luczak [35, Lemma 5.1] about the size of k-cores of graphs with given degree sequence. We define a system of differential equations and use Theorem 2.9.2 to show that the solution to this system of equations approximates the behaviour of the deletion procedure a.a.s. We show that, if the deletion procedure does not stop in $\varepsilon' n$ steps, then the solution to the system of differential equations implies that it will not stop until the number of undeleted vertices is less than γn, in which case we will show that it should be empty.

We will use the pairing-allocation model $\mathcal{P}(M, L, V, k)$ as described in [17]: given a set M of points together with a perfect matching E_M on M and two disjoint sets L, V let h be chosen uniformly at random from the functions mapping M to $L \cup V$ such that $|h^{-1}(v)| \ge k$ for all $v \in V$ and $|h^{-1}(v)| = 1$ for all $v \in L$. Let $G_{\mathcal{P}} = G_{\mathcal{P}}(M, L, V, k)$ be the multigraph obtained by adding edges joining $h(a)$ and $h(b)$ for every $ab \in E_M$ and $h(a), h(b) \in V$. Note that $G_k^{\text{multi}}(n,m) = G_{\mathcal{P}}([2m], \varnothing, [n], k)$ with $E_M = \{\{i, m+i\} : i \in [m]\}$.

We say that the vertices in V are *heavy* vertices and the vertices in L are *light* vertices. We will also say that point $i \in M$ is in v if $h(i) = v$.

Cain and Wormald [17] analyse a deletion procedure for obtaining the k-core. Here we will use a similar procedure with the only modifications being in the first step. The procedure receives as input $h : [2m] \to [n]$ such that $|h^{-1}(v)| \ge k$ for all $v \in [n]$. We remark that this deletion procedure is similar to the one described in Section 5.2.1, but it does not receive the degree sequence as the input.

Deletion procedure – pairing–allocation (h)**:**

- Let $M = [2m]$, $L = \varnothing$ and $V = [n]$.
- Iteration 0: Choose $i \in [m]$ uniformly at random. Find $v = h(i)$ and $u = h(m+i)$ (that is, the edge to be deleted is $e = uv$). Delete i and $m+i$ from M. If $u \neq v$ and $|h^{-1}(v)| = k$, then delete v from V, add $k-1$ new elements to L and redefine the action of h on $h^{-1}(v) \setminus \{i\}$ as a bijection to the new elements. Similarly to u, if $u \neq v$ and $|h^{-1}(u)| = k$, then delete u from V, add $k-1$ new elements to L and redefine the action of h on $h^{-1}(u) \setminus \{m+i\}$ as a bijection to the new elements. If $u = v$ and $|h^{-1}(v)| \leq k+1$, then delete v from V, add $|h^{-1}(v)| - 2$ new elements to L and redefine the action of h on $h^{-1}(v) \setminus \{i, m+i\}$ as a bijection to the new elements.
- Loop: While $L \neq \varnothing$, choose $j \in h^{-1}(L)$ uniformly at random. Delete j and $m+j$ from M and delete $h(j)$ from L. Find $v = h(m+j)$. If $v \in L$, delete v from L. If $v \in V$ and $|h^{-1}(v)| = k$, then delete v from V, add $k-1$ new elements to L and redefine the action of h on $h^{-1}(v) \setminus \{i\}$ as a bijection to the new elements.

Let h_0, M_0, L_0, V_0 be the values of h, M, L, V, resp., after Iteration 0. Let h_i, M_i, L_i, V_i be the values of h, M, L, V, resp., after the i-th iteration of the loop. Then the proof of [17, Lemma 6] gives us the same conclusion as [17, Lemma 6]:

Lemma 5.5.1. Starting with $h = \mathcal{P}([2m], \varnothing, [n], k)$ and conditioning upon the values of M_i, L_i and V_i, we have that h_i has same distribution as $\mathcal{P}(M_i, V_i, L_i, k)$.

The following lemma is a simple concentration result for the vertices of degree k in the pairing-allocation model. It is going to be useful in the analysis of the deletion procedure above. This lemma can be deduced from [17, Lemma 1].

Lemma 5.5.2. Let M, L and V be disjoint sets with $|V| \to \infty$. Let $t := |M| - |L| - k|V|$. Suppose that $t \to \infty$ and that there is a constant $\varepsilon > 0$ such that $|V| \geq \varepsilon |M|$. The number of vertices of degree k in $G_{\mathcal{P}}(M, V, L, k)$ is asymptotic to $\mathbb{P}(\text{Po}(k, \lambda) = k)|V|$, where $\lambda = \lambda(k, t/|V|)$.

Proof. Similarly to the allocation model (see Section 2.2), the degree sequence of the heavy vertices of $G_{\mathcal{P}}(M, L, V, k)$ has the distribution $\text{Multi}_{\geq k}(|V|, |M| - |L|)$. Thus, by Lemma 2.10.1, the degree sequence of the heavy vertices of $G_{\mathcal{P}}(M, L, V, k)$ has the same distribution of $\mathbf{Y} = (Y_1, \ldots, Y_{|V|})$, where the Y_i's have are independent random variables with distribution $\text{Po}(k, \lambda)$, conditioned upon the event Σ that $\sum_{i=1}^{V} Y_i = t$. Let D_k be the number of vertices v of degree k in $G_{\mathcal{P}}$. Let $D_k(\mathbf{Y})$

denote the number of occurrences of k in $\mathbf{Y}$. Then $\mathbb{E}\left(D_k(\mathbf{Y})\right) = \mathbb{P}(\mathrm{Po}(k,\lambda) = k)|V| = \Theta(|V|)$, since $|M|/|V|$ is bounded. By Theorem 2.3.1,

$$\mathbb{P}\left(|D_k(\mathbf{Y}) - \mathbb{E}\left(D_k(\mathbf{Y})\right)| \geq \phi|V|\right) \leq \exp(-\Omega(\phi^2|V|)).$$

By Lemma 2.10.8, $\mathbb{P}(\Sigma) = \Omega(1/\sqrt{|V|})$. Thus,

$$\mathbb{P}\left(|D_k(\mathbf{Y}) - \mathbb{E}\left(D_k(\mathbf{Y})\right)| \geq \phi|V|\right) \leq O(\sqrt{|V|})\exp(-\Omega(\phi^2|V|)),$$

and so it suffices to choose $\Phi = \omega(1/|V|^{2-\alpha})$, for some $\alpha > 0$. □

5.5.1 No small k-cores

In this section, we show that there is a positive constant γ such that the k-core of $G_k^{\mathrm{multi}}(n,m) - e$ either has at least γn vertices or it is empty. This is an application of a result by Luczak and Janson:

Theorem 5.5.3 ([35, Lemma 5.1])**.** If a degree sequence $(\mathbf{d}_n)_{n\in\mathbb{N}}$ satisfies $\sum_i e^{\alpha d_i} \leq Rn$ for constants α and R, then there is a constant γ such that a.a.s. no subgraph of $G^{\mathrm{multi}}(\mathbf{d})$ with less than γn vertices has average degree at least k.

Lemma 5.5.4. Let C_0 be a constant. Suppose that $m = m(n)$ satisfies $kn \leq 2m \leq C_0 n$. Then there exists a constant γ such that a.a.s. the graph obtained from $G_k^{\mathrm{multi}}(n,m)$ by deleting an edge chosen uniformly at random either has a k-core of size at least γn or its k-core is empty.

Proof. We will use Theorem 5.5.3. We set $\alpha \in (0, 1/3)$ and we will choose R later. Let $\check{\mathcal{D}}_k(n,m) \subseteq \mathcal{D}_k(n,m)$ be the set of degree sequences $\mathbf{d}$ such that $\sum_i e^{\alpha d_i} \leq Rn$. Since the degree sequence of $G_k^{\mathrm{multi}}(n,m) - e$ is bounded by the degree sequence of $G_k^{\mathrm{multi}}(n,m)$, it suffices to show that the degree sequence $\mathbf{d} = \mathbf{d}(G_k^{\mathrm{multi}}(n,m))$ is in $\check{\mathcal{D}}_k(n,m)$ a.a.s.

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be such that the Y_i's are independent random variables with distribution $\mathrm{Po}(k, \lambda_{k,c})$. As already mentioned before, $\mathbf{d}$ has the same distribution of $\mathbf{Y}$ conditioned upon the event Σ that $\sum_i Y_i = 2m$. By Theorem 2.10.8, we have that, if $r = 2m - kn < \log n$,

$$\mathbb{P}(\Sigma) = \left(1 + O\left(\frac{r^{5/2}}{n}\right)\right)\frac{e^{-r}r^r}{r!} \geq \left(1 + O\left(\frac{(\log n)^{5/2}}{n}\right)\right)\frac{e^{-r}r^r}{\sqrt{2\pi r}\left(\frac{r}{e}\right)^r} \sim \frac{1}{\sqrt{2\pi r}} = \Omega\left(\frac{1}{\sqrt{\log n}}\right),$$

where we used the fact that $r! \leq \sqrt{2\pi r}\left(\frac{r}{e}\right)^r$. For any $r \to \infty$,

$$\mathbb{P}(\Sigma) = \frac{1 + O(r^{-1})}{\sqrt{2\pi nc(1 + \eta_c - c)}} \geq \frac{1 + O(r^{-1})}{\sqrt{2\pi n C_0}},$$

since $c \le C_0$ and $\eta_c \le c$ by (2.7). Thus, we conclude that

$$\mathbb{P}(\Sigma) = \Omega\left(\frac{1}{\sqrt{n}}\right). \tag{5.24}$$

For J_0 big enough (depending only on C_0), we have that $\lambda(k, C_0)/J_0 \le e^{-1}$, which implies $\lambda(k,c)/J_0 \le e^{-1}$ since $c \le C_0$ and $\lambda(k,c)$ is an increasing function of c (Lemma 2.10.4). Clearly,

$$\sum_{j \le J_0} e^{\alpha j} D_j(\mathbf{Y}) \le e^{\alpha J_0} n.$$

Let $J_1 = J_0 + (1+\beta)\log n$ with $\beta \in (\frac{1}{2}, \frac{1}{2\alpha} - 1)$. Let $p = \lambda(k,c)^{J_0 - 1}/((J_0-1)! f_k(\lambda(k,c)))$. Then

$$\mathbb{P}\left(\exists j > J_1 \text{ with } D_j(\mathbf{Y}) > 0\right) \le np \sum_{i \ge 0} \frac{1}{e^{(1+\beta)\log n + i}} \le \frac{pn^{-\beta}}{1 - e^{-1}} = O(n^{-\beta}).$$

Using (5.24) and the fact that and $\beta \in (1/2, (2\alpha)^{-1} - 1)$, we conclude that

$$\begin{aligned} \mathbb{P}\left(\max_i d_i > J_1\right) &= \mathbb{P}\left(\exists j > J_1 \text{ with } D_j(\mathbf{Y}) > 0 \,\middle|\, \Sigma\right) \le \frac{\mathbb{P}\left(\exists j > J_1 \text{ with } D_j(\mathbf{Y}) > 0\right)}{\mathbb{P}(\Sigma)} \\ &= O\left(n^{-\beta}\sqrt{n}\right) = o(1). \end{aligned}$$

And so

$$\sum_{j > J_1} e^{\alpha j} D_j(\mathbf{Y}) = 0 \text{ a.a.s.}$$

Now we consider $j \in (J_0, J_1]$. By Hoeffding's inequality,

$$\mathbb{P}\left(|D_j(\mathbf{Y}) - p^{(j)} n| \ge a\sqrt{n}\right) \le 2e^{-2a^2}$$

where $p^{(j)} = \lambda(k,c)^j/(j! f_k(\lambda(k,c)))$. Together with (5.24) this implies that

$$\mathbb{P}\left(|D_j(\mathbf{Y}) - p^{(j)} n| \ge a\sqrt{n} \,\middle|\, \Sigma\right) = O(\sqrt{n}) e^{-a^2}.$$

Thus,

$$\begin{aligned} \mathbb{P}\left(|D_j(\mathbf{Y}) - p^{(j)} n| \ge a\sqrt{n} \text{ for some } j \in (J_0, J_1]\right) &= (1+\beta)\log n \cdot O(\sqrt{n}) e^{-a^2} \\ &= O(n^{-\beta'}), \end{aligned}$$

for $a = \sqrt{(1+\beta')\log n}$ with $\beta' > 0$. Thus, a.a.s.

$$\begin{aligned}
\sum_{j=J_0+1}^{J_1} e^{\alpha j} D_j(\mathbf{Y}) &\le e^{\alpha J_0} \sum_{j=1}^{(1+\beta)\log n} e^{\alpha j}\left(p^{(j)} n + a\sqrt{n}\right) \\
&\le e^{\alpha J_0} \sum_{j=1}^{(1+\beta)\log n} e^{\alpha j}\left(p\frac{1}{e^j} n + a\sqrt{n}\right), \quad \text{by our choice of } J_0 \\
&\le e^{\alpha J_0}\left(np \sum_{j=1}^{(1+\beta)\log n} e^{-2j/3} + e^{\alpha(J_1-J_0)}(J_1-J_0)a\sqrt{n}\right) \quad \text{since } \alpha < 1/3; \\
&\le e^{\alpha J_0}\left(\frac{p}{e^{2/3}(1-e^{-2/3})} + \frac{n^{(1+\beta)\alpha+1/2}(\log n)^{3/2}}{n}\sqrt{1+\beta'}(1+\beta)\right) n.
\end{aligned}$$

Using $1+\beta < (2\alpha)^{-1}$ we have that $n^{(1+\beta)\alpha+1/2} \le n^{\alpha'}$ for some $\alpha' < 1$ and so we can set

$$R = e^{\alpha J_0}\left(1 + \frac{p}{e^{2/3}(1-e^{-2/3})} + \sqrt{1+\beta'}(1+\beta)\right).$$

□

5.5.2 The case $c \to k$

In this section, we will prove that, for $c \to k$, the k-core of $G_k(n,m) - e$ is empty a.a.s., proving Theorem 5.1.1(i). So suppose that $c = 2m/n = k + \phi(n)$, where $\phi(n) = o(1)$ and $\phi(n) \ge 0$. Let S_i denote the number of points in heavy vertices just after the i-th iteration of the loop. Let S_0 denote the number of points in heavy vertices after Iteration 0. We will use x as i/n and $y(i/n)$ to approximate S_i/n.

We will use Theorem 2.9.2. Define $D_\gamma = \{(x,y) : -\gamma < 2x < k-\gamma,\ \gamma < y < k+\gamma\}$. Note that D_γ is bounded, connected and open. We choose $\gamma < \min\{\gamma_0/3, k\}$ so that the k-core cannot be non-empty and smaller than $\gamma_0 n$ a.a.s.(γ_0 is given by Lemma 5.5.4). Moreover, we work with n big enough so that $\phi(n) < \gamma$. After the first step there are at most $2(k-1)$ points in L_0 and all the other vertices in V_0 and so $S_0 \ge 2m - 2(k-1)$. Then it is clear that $S_0/n \le k + \phi < k + \gamma$ and $S_0/n > \gamma$.

Let $T_D = \min\{i : (i/n, S_i/n) \notin D\}$. Let W_i denote the number of light vertices after iteration i is performed. We also use the stopping time $T = \min\{i : W_i = 0\}$. That is, there are no light vertices to be deleted and the deletion process has actually ended. We need to check the boundedness hypothesis, trend hypothesis and Lipschitz hypothesis (see Theorems 2.9.1 and 2.9.2). The boundedness hypothesis is trivially true: $|S_i - S_{i+1}| \le k$ always.

Now we check the trend hypothesis. Let $f(x,y) = -ky/(k-2x)$. Let H_i denote the history of the process at iteration $i \geq 1$. We need to show that $\xi_1 := |\mathbb{E}(S_{i+1} - S_i | H_i) - f(i/n, S_i/n)| = o(1)$ while $i < T$ and $i < T_D$. We have that $S_{i+1} - S_i$ is zero if j is matched to a light vertex, is -1 if j is matched to a point in a heavy vertex with degree $> k$ and is $-k$ if j is matched to a point in a heavy vertex with degree exactly k. The probability that j is matched to a point in a heavy vertex is $S_i/(2m-2i-2)$. The probability that such a heavy vertex has degree k is at least $1 - \sum\{d_i : d_i > k\}/S_i$ where $\mathbf{d}$ is the degree sequence of $G_k^{\text{multi}}(n,m)$ (we do not sample the degree sequence, we just decide if the vertex had degree k or not). But, for every degree sequence $\mathbf{d}$ such that $\sum_{i=1}^{n} d_i = 2m = kn + n\Phi(n)$, we have that $\sum\{d_i : d_i > k\} \leq n\phi(n)$. Moreover, $S_i \geq \gamma n$ for $(i/n, S_i/n) \in D_\gamma$. Thus, $1 - \sum\{d_i : d_i > k\}/S_i \geq 1 - \phi(n)/\gamma = 1 - o(1)$ and so we have that

$$\mathbb{E}(S_{i+1} - S_i | H_i) = \frac{-k|S_i|}{2m-2i-2}(1+o(1)) = f(i/n, S_i/n) + o(1)$$

and so the trend hypothesis holds. Straightforward computations show that the Lipschitz hypothesis also holds in $D_\gamma \cap \{(x,y) : x \geq 0\}$.

According to [65, Theorem 6.1], $y'(x) = f(x,y)$ has a unique solution in D_γ, say y^*, with $y(0) = k$ and a unique solution in D_γ, say y^{**}, with $y(0) = S_0/n$. Note that y^* is a fixed function while y^{**} is a random variable because S_0 is a random variable. The Lipschitz condition implies that, for any x with both $(z, y^*(z))$ and $(z, y^{**}(z))$ in D_γ for all $0 \leq z \leq x$, we have that $|y^*(x) - y^{**}(x)| \leq x|k - S_0/n|R =: \xi_3$, where R is some big constant and so $\xi_3 = o(1)$. Let $\xi_2 = o(1)$ and $\xi_2 > \xi_1$ and $\xi_2 > \xi_3$. By [65, Theorem 6.1], there is a constant C and a function $\xi \to 0$, such that, a.a.s. at each step $i < \min\{T, n\sigma\}$ we have that

$$|S_i - ny^*(i/n)| \leq \xi n, \tag{5.25}$$

where σ denotes the supremum of x such that $(z, y^*(z))$ and $(z, y^{**}(z))$ are at ℓ^∞-distance at least $C\xi_2$ of the boundary of D_γ for all $0 \leq z \leq x$.

It is straightforward to check that

$$y^*(x) = k\left(\frac{k-2x}{k}\right)^{k/2}.$$

Let ε' be given by Corollary 5.4.3. For $\varepsilon' < x < (k-\gamma)/2$, we have that

$$(k-2x) - y^*(x) = (k-2x)\left(1 - \left(\frac{k-2x}{k}\right)^{k/2-1}\right) \geq \frac{2\gamma\varepsilon'}{k}.$$

This implies that, if (5.25) holds at i where $\varepsilon' n < i < (k-\gamma)n/2$, then $W_i = 2m-2i-2-S_i = \Omega(n)$. Thus, if (5.25) holds for some step $i \in (\varepsilon' n, \sigma n]$ with $T > i$, then $T > i+1$ because there are still

$\Omega(n)$ points to be deleted. This implies that, conditioning upon $T > \varepsilon' n$, we have that $T > \sigma n$ a.a.s.

For any constant $\alpha \in (0, \gamma)$, using the fact that $\xi_3 = o(1)$, there exists x such that $x \leq \sigma n$ and the ℓ^∞-distances of $(x, y^*(x))$ and $(x, y^{**}(x))$ to the boundary of D_γ are in $(C\xi_2, \alpha)$. For such an x we have $T > x$ a.a.s. because $T > \sigma n$ a.a.s. (conditioning upon $T > \varepsilon' n$). Thus, (5.25) holds a.a.s. Since x is at ℓ^∞-distance at most α of the boundary of D_γ, either $2x \geq k - \gamma - \alpha$ or $y^*(x) \leq \gamma + \alpha$. We excluded $y^*(x) \geq k + \gamma - \alpha$ because $y^*(0) = k$ and y^* decreases as x increases. For n sufficiently large so that $|\xi(n)| < \gamma$, the equation (5.25) with $2x \geq k - \gamma - \alpha$ or $y^*(x) \leq \gamma + \alpha$ shows that $S_i \leq n\gamma_0$ a.a.s.

Since $T > \varepsilon' n$ a.a.s. by Corollary 5.4.3, the k-core would have to be smaller than $\gamma_0 n$ a.a.s. and so it must be empty a.a.s. by Lemma 5.5.4. We conclude that $W(G_k^{\text{multi}}(n, m)) = n$ a.a.s. Since the probability that $G_k^{\text{multi}}(n, m)$ is simple is $\Omega(1)$ (as in (5.18)), we have that $W(G_k(n, m)) = n$ a.a.s.

5.5.3 The case $c \in [k + \varepsilon, c + k' - \varepsilon]$

In this section, we will prove that, for $c \leq c'_k - \varepsilon$ and any $h(n) \to \infty$, the k-core of $G_k(n, m) - e$ either has at least $n - h(n)$ vertices or it is empty a.a.s., proving Theorem 5.1.1(ii) except for the claim that $W(G_k(n, m)) = n$ with probability bounded away from zero, which is addressed in Section 5.6. Let ε' be given by Corollary 5.4.2. Assume that $c \to C \in [k + \varepsilon, c + k' - \varepsilon]$, where C is a constant. We will explain later how to drop this assumption.

We use the deletion procedure described in the beginning of the section. For each i, let S_i denote the number of points in heavy vertices right after iteration i, let T_i denote the number of heavy vertices right after iteration i and let W_i denote the number of points in light vertices right after iteration i. We will use the Differential Equation Method (Theorem 2.9.2) to approximate S_i and T_i. Note that $W_i = 2m - 2i - 2 - S_i$. We will use $y(i/n)$ to approximate S_i/n and $z(i/n)$ to approximate T_i/n.

Let γ be a positive constant with $\gamma < \min\{1, C - k\}$ to be chosen later. Define

$$D_\gamma = \Big\{(x, y, z) : \gamma < z < 1 + \gamma,\ -\gamma < x < C - \gamma,\ \gamma < y < C + \gamma, y > (k + \gamma)z\Big\}.$$

Then D_γ is bounded, connected and open. We have $T_0 \in \{n, n-1, n-2\}$ and $S_0 \in [2m - 2 - 2(k-1), 2m - 2]$. Thus, $T_0/n = 1 + o(1/n)$ and $S_0/n = C + o(1)$. Then D_γ contains the closure of the points $(0, y, z)$ such that $\mathbb{P}(S_i = yn \text{ and } T_i = zn) \neq 0$ for some n. Note that $y > (k + \gamma)z$ implies that $\mu := \lambda(k, y/z) > \varepsilon$ for some constant $\varepsilon > 0$. As we will see later, this will be very important in the proof. Note that, for the case $c \to k$ one cannot add the condition $y > (k + \gamma)z$ to D_γ since this would exclude the initial point from D_γ.

We use the stopping time $T = \min\{i : W_i = 0\}$ again. We have to check the boundedness hypothesis, the trend hypothesis and the Lipschitz hypothesis. The boundedness hypothesis is again easy: $|S_{i+1} - S_i| \leq k$ and $|T_{i+1} - T_i| \leq 1$ always.

We now check the trend hypothesis. We remark that the trend hypothesis is exactly like in [17]. Define

$$f_z(x) = -\frac{y}{C-2x}\left(1 - \frac{\mu z}{y}\right) \quad \text{and} \quad f_y(x) = -\frac{y}{C-2x}\left(k - (k-1)\frac{\mu z}{y}\right),$$

where $\mu = \lambda(k, y/z)$. Let H_i denote the history of the process at iteration $i \geq 1$. We will show that $|\mathbb{E}(S_{i+1} - S_i|H_i) - f_y(i/n)| = o(1)$ and $|\mathbb{E}(T_{i+1} - T_i|H_i) - f_z(i/n)| = o(1)$ while $i < T$ and $i < T_D$. We have that $S_{i+1} - S_i$ is zero if j is matched to a light vertex, is -1 if j is matched to a point in a heavy vertex with degree $> k$ and is $-k$ if j is matched to a point in a heavy vertex with degree exactly k. The probability that j is matched to a point in a heavy vertex is $S_i/(2m - 2i - 2)$. Conditional upon this event, the probability that j is matched to a vertex of degree k is k/S_i times the number of vertices of (current) degree k. By Lemma 5.5.2, the number of vertices of degree k at step i is asymptotic to $T_i \cdot \mathbb{P}(\mathrm{Po}(k,\lambda) = k) = T_i \cdot \lambda^k/(k! f_k(\lambda))$, where $\lambda := \lambda(k, S_i/T_i)$. Thus, the probability that j is matched to a point of degree k vertex, conditioned upon the event that it is matched to a point in a heavy vertex, is asymptotic to

$$\begin{aligned}
\frac{k}{S_i}\frac{T_i\lambda^k}{k! f_k(\lambda)} &= 1 + \frac{(k-1)! f_k(\lambda) S_i - T_i \lambda^k}{(k-1)! f_k(\lambda) S_i} \\
&= 1 + \frac{(k-1)! \lambda f_{k-1}(\lambda) T_i - T_i \lambda^k}{(k-1)! f_k(\lambda) S_i} \qquad \text{since } S_i/T_i = \frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} \\
&= 1 + \frac{((k-1)!\lambda T_i)\left(f_{k-1}(\lambda) - \lambda^{k-1}/(k-1)!\right)}{(k-1)! f_k(\lambda) S_i} \\
&= 1 - \frac{\lambda T_i}{S_i} \qquad \text{since } f_k(\lambda) = f_{k-1}(\lambda) - \frac{\lambda^{k-1}}{(k-1)!}.
\end{aligned}$$

Thus, $|\mathbb{E}(S_{i+1} - S_i|H_i) - f_y(i/n)| = \xi_y = o(1)$. Similarly of T_i, we have that $|\mathbb{E}(T_{i+1} - T_i|H_i) - f_z(i/n)| = \xi_z = o(1)$. The Lipschitz hypothesis is straightforward to check.

According to [65, Theorem 6.1], the system of differential equations $y'(x) = f_y(x)$ and $z'(x) = f_z(x)$ has unique solutions (y^*, z^*) and (y^{**}, z^{**}), with initial conditions $y(0) = C$ and $z(0) = 1$, and $y(0) = S_0/n$ and $z(0) = T_0$, resp. Note that, as in the case $c \to k$, we have that (y^*, z^*) is a pair of fixed functions while (y^{**}, z^{**}) is a random variable since the initial position is random. The Lipschitz hypothesis implies that, there exists a constant R such that, for any x with both $(x, y^*(x), z^*(x))$ and $(x, y^{**}(x), z^{**}(x))$ in D_γ, we have $\max\{|y^*(x) - y^{**}(x)|, |z^*(x) - z^{**}(x)|\} \leq x|k - S_0/n|R =: \xi_3$ with $\xi_3 = o(1)$. Let $\xi_2 = o(1)$ be such that $\xi_2 > \max\{\xi_z, \xi_y, \xi_3\}$. Thus, by [65, Theorem 6.1], there is a constant C_0 and a function $\xi = o(1)$, such that, a.a.s. at each step

$i < \min\{T, n\sigma\}$ we have that

$$\max\left\{\left|S_i - ny^*(i/n)\right|, \left|T_i - nz^*(i/n)\right|\right\} \leq \xi n, \tag{5.26}$$

where σ denotes the supremum of x such that, for all $0 \leq x' \leq x$, we have $(x', y^*(x'), z^*(x'))$ and $(x', y^{**}(x'), z^{**}(x'))$ are at ℓ^∞-distance at least $C_0\xi_2$ of the boundary of D_γ.

In [17], properties of the differential equations system $\{y'(x) = f_y(x), z'(x) = f_z(x)\}$ were proved. Among them is that $\mu^2/(C-2x)$ and $(ze^\mu)/f_k(\mu)$ are constants as long as $C - 2x > 0$, $y > 0$ and $\mu > \varepsilon$ for constant $\varepsilon > 0$. The authors of [17] notified us that a small detail was omitted in their proof: one needs $\mu'(0)$ to be not zero. The reason for that is at some point in their proof they cancel a factor that is zero if and only if $\mu'(x) = 0$. If $\mu'(0) \neq 0$, then $\mu^2/(C-2x)$ is a nonzero constant until $\mu'(x)$ becomes zero. But in this case it is impossible to $\mu'(x)$ to become zero since $\mu'(x)$ is continuous and $\mu^2/(C-2x)$ being a constant implies that $\mu'(x) = \alpha/\mu$ is bounded away from zero, where α is a nonzero constant.

With initial conditions $y(0) = C$ and $z(0) = 1$, we get $\mu^2/(C-2x) = \lambda(k,C)^2/C$ and $ze^\mu/f_k(\mu) = e^{\lambda(k,C)}/f_k(\lambda(k,C))$, which can be used to deduce that

$$y^* = (C-2x)\frac{h_k(\lambda(k,C))}{h_k(\mu)}. \tag{5.27}$$

For $x \geq \varepsilon'/2$, since $\mu^2/(C-2x) = \lambda(k,C)^2/C$, we have $\mu(x) \leq \lambda_{k,C}\sqrt{1-\varepsilon'/C}$ and so $h_k(\mu) \geq (1+\varepsilon'')h_k(\lambda_{k,C})$, for some $\varepsilon'' > 0$. Thus, for every x such that $\varepsilon' \leq 2x \leq C - \gamma$ using (5.27),

$$C - 2x - y^* = C - 2x - (C-2x)\frac{h_k(\lambda_{k,C})}{h_k(\mu)} \geq (C-2x)\left(1 - \frac{1}{1+\varepsilon''}\right) \geq \frac{\gamma\varepsilon''}{1+\varepsilon''}.$$

This implies that, if (5.25) holds at i with $\varepsilon' n < 2i < (C-\gamma)n$, then $W_i = 2m - 2i - 2 - S_i = \Omega(n)$. Thus, if (5.25) holds for some step $i \in (\varepsilon' n, \sigma n]$ with $T > i$, then $T > i + 1$ because there are still $\Omega(n)$ points to be deleted. This implies that, conditioning upon $T > \varepsilon' n$, we have that $T > \sigma n$ a.a.s.

For any constant $\alpha \in (0, \gamma)$, using the fact that $\xi_3 = o(1)$, there exists x such that $x \leq \sigma n$ and $(x, y^*(x), z^*(x))$ and $(x, y^{**}(x), z^{**}(x))$ are at ℓ^∞-distance $(C_0\xi_2, \alpha)$ of the boundary of D_γ. For such an x we have $T > x$ a.a.s. because $T > \sigma n$ a.a.s. Thus, (5.26) holds a.a.s. Since x is at ℓ^∞-distance at most α of the boundary of D_γ, we have that at least one of the following hold:

(a) $z^*(x) \leq \gamma + \alpha$;

(b) $2x \geq C - \gamma - \alpha$;

(c) $y^*(x) \leq \gamma + \alpha$;

(d) There exists $(\hat{x}, \hat{y}, \hat{z})$ such that $\frac{\hat{y}}{\hat{z}} = k + \gamma$ and $|y^*(x) - \hat{y}| \leq \alpha$ and $|z^*(x) - \hat{z}| \leq \alpha$.

We excluded $y^*(x) \geq C + \gamma - \alpha$ and $z^*(x) \geq 1 + \gamma - \alpha$ because $y^*(0) = C$ and $z^*(0) = 1$ and f_y and f_z are decreasing.

Since $\mu^2/(C - 2x) = \lambda(k, C)^2/C$, μ decreases as x increases. Since $\mu(0) = 0$, we have that $h_k(\mu) \geq h_k(\lambda_{k,C})$ and so $y^*(x) \leq C - 2x$ by (5.27). Thus, for $2x \geq C - \gamma - \alpha$, we have that $y \leq \gamma + \alpha$, that is (c) holds.

If (a) or (c) holds, then by (5.26), then a.a.s. $S_i \leq (\gamma + \alpha_\xi)n \leq 3\gamma n$. Now suppose that (d) holds. Then

$$\left|\frac{y^*(x)}{z^*(x)} - \frac{\hat{y}}{\hat{z}}\right| \leq \left|\frac{y^*(x) - \hat{y}}{z^*(x)}\right| + \left|\frac{\hat{y}}{\hat{z}}\right|\left|\frac{z^*(x) - \hat{z}}{z^*(x)}\right| \leq \frac{\alpha(k + \gamma + 1)}{\gamma}.$$

Thus, using $\alpha \leq \gamma^2$, we have that $y^*(x)/z^*(x) = k + O(\gamma)$ and so $\mu = O(\gamma)$. Using that $\mu^2/(C - 2x)$ remains as a positive constant during the process, we then have $C - 2x = O(\gamma^2)$, we can then conclude that the $S_i = O(\gamma n)$ a.a.s. Thus, conditioned upon $T > \varepsilon' n$ the k-core has at most $O(\gamma n)$ vertices a.a.s. Let γ_0 be the constant given by Lemma 5.5.4. By choosing γ small enough, we can conclude that, conditioned upon $T > \varepsilon' n$, the k-core has less than $\gamma_0 n$ vertices a.a.s. and which implies, by Lemma 5.5.4, that the k-core must be empty a.a.s. By Corollary 5.4.2, we have that $W(G_k^{\text{multi}}(n, m)) \leq h(n)$ or $W(G_k^{\text{multi}}(n, m)) = n$ with probability $1 + o(1)$ conditioned upon $T > \varepsilon' n$ (where the convergence depends on c).

Recall that we assumed $c \to C$. We show how to drop this assumption here. Let $(c_i)_{i \in \mathbb{N}}$ such that every $c_i \in [k + \varepsilon, c + k' - \varepsilon]$. Let $r(n)$ be the probability that neither $W(G_k^{\text{multi}}(n, m)) \leq h(n)$ nor $W(G_k^{\text{multi}}(n, m)) = n$. Then every subsequence of $(c_i)_{i \in \mathbb{N}}$ has a subsequence that converges to some constant C_0 and in that subsequence $r(n) \to 0$. So by the subsubsequence principle $r(n) \to 0$. Since the probability that $G_k^{\text{multi}}(n, m)$ is simple is $\Omega(1)$, we have that $W(G_k(n, m)) \leq h(n)$ or $W(G_k(n, m)) = n$ a.a.s.

5.6 Deletion procedure for $G_k(n, m)$

In this section, we show that, for $k + \varepsilon \leq c \leq c_k' - \varepsilon$, there exists a function $h(n) \to \infty$ such that the k-core of $G_k(n, m)$ has fewer than $n - h(n)$ vertices with probability bounded away from zero. Since we have already proved that, for any function $h(n) \to \infty$, either this k-core has at least $n - h(n)$ vertices or it is empty, this implies it is empty with probability bounded away from zero.

Lemma 5.6.1. Let $\varepsilon > 0$ be a fixed real. Suppose that $c = 2m/n$ satisfies $k + \varepsilon \leq c \leq c_k' - \varepsilon$. Then there exists a function $h(n) \to \infty$ such that $\mathbb{P}(W(G_k(n, m)) \geq h(n)) = \Omega(1)$.

Together with Section 5.5.3, this lemma implies that $\mathbb{P}(W(G_k(n,m)) = n) = \Omega(1)$, which completes the proof of Theorem 5.1.1(ii). We now prove Lemma 5.6.1. We will work with a set of 'typical' degree sequences. Let $G(\mathbf{d})$ be chosen uniformly at random from all (simple) k-cores with degree sequence $\mathbf{d}$. We will define a deletion procedure for finding the k-core of $G(\mathbf{d})$ after the deletion of a random edge and for finding the k-core of $G^{\text{multi}}(\mathbf{d})$ after the deletion of a random edge. In this deletion procedure, in each step a vertex is deleted. We show that we can couple the deletion procedures for $G(\mathbf{d})$ and $G^{\text{multi}}(\mathbf{d})$ for $t(n) \to \infty$ steps in such way that the exact same vertices are involved in each step. Since we already know that, for $h(n)$ going to infinity sufficiently slowly, we have $\mathbb{P}(G^{\text{multi}}(\mathbf{d}) \geq h(n)) = \Omega(1)$ by Theorem 5.4.1 (under some constraints in the degree sequence), we must have $\mathbb{P}(G^{\text{multi}}(\mathbf{d}) \geq h_1(n)) = \Omega(1)$ for $h_1(n)$ going to infinity sufficiently slowly (depending on $h(n)$ and $t(n)$). Using the fact that the set of degree sequences analysed is a set of 'typical' degree sequences, we can easily carry the result over to $G_k(n,m)$.

We start by defining a set of 'typical' degree sequences. Let $\phi(n) = o(1)$ with $\phi(n) = \omega(n^{-1/4})$. Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be such that the Y_i's are independent truncated Poisson random variables with parameters $(k, \lambda_{k,c})$. Let $\tilde{\mathcal{D}}_k(n,m)$ be the degree sequences $\mathbf{d}$ such that $|D_k(\mathbf{d}) - \mathbb{E}\left(D_k(\mathbf{Y})\right)| \leq n\phi(n)$ and $\max_i d_i \leq n^\beta$ for some $\beta \in (0, 0.25)$ and $|\eta(\mathbf{d}) - \mathbb{E}\left(\eta(\mathbf{Y})\right)| \leq \phi(n)$. Similarly to the proof in Section 5.3.1, one can prove that $\mathbf{d}(G_k(n,m)) \in \tilde{\mathcal{D}}_k(n,m)$ a.a.s. We will show that there exists $h(n) \to \infty$ such that

$$\mathbb{P}\left(W(G(\mathbf{d})) \geq h(n)\right) = \Omega(1), \tag{5.28}$$

for $\mathbf{d} \in \tilde{\mathcal{D}}_k(n,m)$. Since $\tilde{\mathcal{D}}_k(n,m)$ is a finite set for each n, by Lemma 2.7.2, there exists a constant $\alpha > 0$ such that $\mathbb{P}\left(W(G(\mathbf{d})) \geq h(n)\right) > \alpha$ for sufficiently large n. Thus, together with the fact that $\mathbf{d}(G_k(n,m)) \in \tilde{\mathcal{D}}_k(n,m)$ a.a.s., this implies that

$$\mathbb{P}\left(W(G_k(n,m)) \geq h(n)\right) \geq \mathbb{P}(\mathbf{d}(G_k(n,m)) \in \tilde{\mathcal{D}}_k(n,m))\alpha \sim \alpha,$$

which implies Lemma 5.6.1. We will now prove (5.28).

As mentioned before, we will couple deletion algorithms for $G(\mathbf{d})$ and $G^{\text{multi}}(\mathbf{d})$ so that they coincide for $t(n) \to \infty$ steps, for $\mathbf{d} \in \tilde{\mathcal{D}}_k(n,m)$. We use a deletion algorithm that is essentially the same as the one we used in the other sections. The only difference is that we explore a whole vertex at a time (instead of an edge at a time) and mark the vertices that have to be deleted.

Deletion procedure by vertex:

- Iteration 0: Choose an edge uv uniformly at random, delete uv and mark the vertices with degree less than k.
- Loop: While there is an undeleted marked vertex, say w, find its neighbours, delete w and the edges incident to it, and then mark all neighbours of w that now have degree less than k.

If we can do such a coupling for $t(n) \to \infty$ iterations of the loop, then we can choose $h(n) \to \infty$ such $h(n) \leq \min\{t(n), \varepsilon' n\}$ with ε' as in Theorem 5.4.1 so that $\mathbb{P}(W(G^{\text{multi}}(\mathbf{d})) \geq h(n)) = \Omega(1)$. This would imply that the deletion algorithm did not stop for at least $h(n)$ steps and so $\mathbb{P}(W(G(\mathbf{d})) \geq h(n)) \geq \Omega(1)$.

In the rest of this section, we show that there exists $t(n) \to \infty$ such that we can couple the deletion algorithms for $G(\mathbf{d})$ and $G^{\text{multi}}(\mathbf{d})$ so that they coincide for $t(n)$ iterations of the loop. For now assume that $t(n) \to \infty$ with $t(n) \leq \log n$. Later we add more restrictions on the growth of $t(n)$. We show that the probabilities that a certain edge uv is chosen in the first step are asymptotically equivalent for $G(\mathbf{d})$ and $G^{\text{multi}}(\mathbf{d})$ and so the first step can be coupled. For the other steps $i \leq t(n)$, we show that the probabilities that the set of neighbours of the vertex w is some specific set are again asymptotically equivalent for $G(\mathbf{d})$ and $G^{\text{multi}}(\mathbf{d})$ with some error $\xi(n) = o(1)$. So we can couple the deletion algorithms for $t(n)$ steps, where $t(n)$ will depend on $\xi(n)$. In the computations in this section, we will use $\mathbb{P}_{\text{multi}}$ to denote the probabilities in the deletion procedure for $G^{\text{multi}}(\mathbf{d})$ and we will use $\mathbb{P}$ to denote the probabilities in the deletion procedure for $G(\mathbf{d})$. First we analyse the procedure for multigraphs. Let $uv \in \binom{V}{2}$. Then

$$\begin{aligned}
\mathbb{P}_{\text{multi}}(uv \text{ is chosen in the first step}) &= \frac{\mathbb{P}_{\text{multi}}(uv \in E(G^{\text{multi}}(\mathbf{d})))}{m} \\
&= \frac{d_u d_v}{m} \frac{(2m-2)!2^m m!}{(2m)!2^{m-1}(m-1)!} \\
&= \frac{d_u d_v}{m(2m-1)} = \frac{d_u d_v}{2m^2}(1+\xi_1(n)),
\end{aligned} \tag{5.29}$$

where $\xi_1(n) = O(1/n)$.

In i-th iteration of the loop, we delete a vertex w and find its set of neighbours U. Let $\ell \leq k-1$ be the current degree of w and let $\{u_1, \dots, u_\ell\}$ be a subset of ℓ undeleted vertices. Let $x_1, \dots, x_\ell$ be an enumeration of the points inside w. Let $y_1, \dots, y_\ell$ be the points matched to $x_1 \dots, x_\ell$. Let $\check{m}$ be the number of undeleted edges at the beginning of the i-th iteration of the loop and let $\check{\mathbf{d}}$ be the degree sequence of the current graph. Using $[x]_j = (x)(x-1)\dots(x-j+1)$, we have

$$\mathbb{P}\left(U = \{u_1, \dots, u_\ell\}\right) = \ell!\,\mathbb{P}(y_i \in u_i \ \forall i) = \frac{\ell! \prod_{i=1}^{\ell} \check{d}_{u_i}}{2^\ell [\check{m}]_\ell}(1+\xi_2(n)), \tag{5.30}$$

where $\xi_2(n) = o(1)$ because $\check{m} \geq m - kt(n) \geq m - k\log n$ and $\ell \leq k-1$. Now we have to compute estimates for the probabilities in the deletion algorithm for simple graphs. The following lemma is an application of [47, Theorem 10].

Lemma 5.6.2. Let $\mathbf{d} \in \mathcal{D}_k(n,m)$ be such that $\max_i d_i \leq n^{0.25}$. Let H be a graph on $[n]$ with at most $kt(n)$ edges. Let L be a supergraph of H with at most k edges more than H such that there is a simple graph G with degree sequence $\mathbf{d}$ such that $G \cap L = H$. Then

$$\mathbb{P}\left(L \subseteq G(\mathbf{d}) \,\middle|\, H \subseteq G(\mathbf{d})\right) = \frac{\prod_{v=1}^{n}[d_i - h_i]_{j_i}}{2^{|E(J)|}[m]_{|E(J)|}}(1 + \nu(n)),$$

where $\mathbf{h}$ is the degree sequence of H, $J = L - E(H)$, $\mathbf{j}$ is the degree sequence of J, and $\nu(n) = o(1)$.

Notice that to use this lemma one has to check the existence of a simple graph G with certain properties. In our case, Erdős-Gallai Theorem will be enough to ensure such simple graph exists.

Lemma 5.6.3. Let n be sufficiently large so that $n - n^{0.25} - k\log n > \sqrt{n}$. Let $n' \geq n - \log n$. Let $\mathbf{g}$ be a sequence on $[n']$ such that $g_1 \geq g_2 \geq \cdots \geq g_{n'}$, $\sum_i g_i$ is even, $g_1 \leq n^{0.25}$, $|\{j : g_j = 0\}| \leq k \log n$. Then there exists a simple graph with degree sequence $\mathbf{g}$.

The proofs for these lemmas are presented in Section 5.6.1. Now we can analyse the deletion algorithm for simple graphs. Let $uv \in \binom{V}{2}$. Then

$$\mathbb{P}\left(uv \text{ is chosen in the first step}\right) = \mathbb{P}\left(uv \in E(G(\mathbf{d}))\right)\frac{1}{m}.$$

We need to compute $\mathbb{P}(uv \in E(G(\mathbf{d})))$. Note that this is the same as $\mathbb{P}(L \subseteq G(\mathbf{d}) \,|\, H \subseteq G(\mathbf{d}))$ with $L = ([n], \{uv\})$ and $H = ([n], \varnothing)$. In order to use Lemma 5.6.2, we need to check if there is a simple graph G with $G \cap L = H$ with degree sequence $\mathbf{d}$. This is the same as saying that there exists a simple graph G with degree sequence $\mathbf{d}$ such that $uv \notin E(G)$. It suffices to show that, for every set of vertices $S \subseteq [n] \setminus \{u, v\}$ of size d_v, there is a simple graph with degree sequence $\mathbf{d}'$, where $\mathbf{d}'$ is obtained from $\mathbf{d}$ be deleting v and decreasing the degree of every vertex in S by 1 (that is, S can be the set of neighbours of v and it does not include u). Note that $\sum_i d'_i$ is even because $\sum_j d_j$ is even. Moreover, $n - 1 \geq n - \log n$ and $\max_i d'_i \leq n^{0.25}$ and $\mathbf{d}'$ has no zeroes. By Lemma 5.6.3, there is a simple graph with degree sequence $\mathbf{d}'$ and so we can use Lemma 5.6.2 to show that

$$\mathbb{P}\left(uv \in G\right) = \frac{d_u d_v}{2m}(1 + \xi_3(n)), \tag{5.31}$$

where $\xi_3(n) = o(1)$. Thus, (5.29) and (5.31) show that the first step can be coupled so that the same edge is chosen.

Now suppose we are in the i-th iteration of the loop and deleting a vertex w. Let $\check{n}$ be the number of undeleted vertices in the beginning of iteration i and let $\check{m}$ be the number of undeleted edges at the beginning of iteration i. Let $\check{\mathbf{d}}$ be the current degree sequence (that is, $\check{d}_u$ is the number neighbours u has among the undeleted vertices). At each iteration we delete at most $k-1$ edges and only one vertex. So $\check{n} \geq n - t(n)$ and $\check{m} \geq m - (k-1)t(n)$. Let $\ell := \check{d}_w$ and $\{u_1, \ldots, u_\ell\}$ be a set with ℓ (undeleted) vertices. Let U be the neighbours of w discovered in iteration i. We want to compute the probability that $U = \{u_1, \ldots, u_\ell\}$. In order to use Lemma 5.6.2, we have to check if there exists a simple graph G with degree sequence $\check{\mathbf{d}}$ such that $G \cap L = H$, where H is the graph discovered so far (which includes the deleted vertices) and $L = ([n], E(H) \cup \{wu_1, \dots, wu_\ell\})$, which is the same as checking if it is possible to get a simple graph such that w as no neighbours in $\{u_1, \dots, u_\ell\}$. Let U' be a set of ℓ undeleted vertices such that $U' \cap \{u_1, \dots, u_\ell\} = \varnothing$. There are plenty of choices for U' since $t(n) \le \log(n)$. Let $\mathbf{d}'$ be the degree sequence on $\check{n} - 1$ obtained from $\check{\mathbf{d}}$ by deleting w and decreasing the degree of each vertex in U' by 1. Then $\check{n} \ge n - \log n$, $\max_i d'_i \le n^{0.25}$ and $|\{j : d'_j = 0\}| \le kt(n) \le k \log n$. Using Lemma 5.6.3, there is a simple graph with degree sequence $\mathbf{d}'$ and so, by Lemma 5.6.2,

$$\mathbb{P}(U = \{u_1, \dots, u_\ell\}) = \frac{\ell! \prod_{i=1}^{\ell} \check{d}_{u_i}}{2^\ell [\check{m}]_\ell}(1 + \xi_4(n)), \tag{5.32}$$

where $\xi_4(n) = o(1)$. Thus, using (5.32) and (5.30),there exists a function $\xi(n) = o(1)$ such that

$$\mathbb{P}(U = \{u_1, \dots, u_\ell\}) = \mathbb{P}_{\text{multi}}(U = \{u_1, \dots, u_\ell\})(1 + \xi(n)).$$

We conclude that the deletion algorithms can be coupled for $t(n)$ steps as long as $(1+\xi)^t = 1+o(1)$. Thus, it suffices to choose $t = o(1/\xi)$. This finishes the proof of Lemma 5.6.1.

5.6.1 Proofs of Lemma 5.6.2 and Lemma 5.6.3

In this section, we prove some results about the deletion procedure for simple graphs. More specifically, we prove Lemma 5.6.2 and Lemma 5.6.3.

For any graphs $H \subseteq L$ on $[n]$ and $\mathbf{d} \in \mathbb{N}^n$ such that $\sum_{i=1}^n d_i$ is even, let $N(\mathbf{d}, L, H)$ denote the number of graphs G on $[n]$ with degree sequence $\mathbf{d}$ such that $G \cap L = H$. First we state a result by McKay [47, Theorem 2.10] that estimates the probability that $G(\mathbf{d})$ contains a subgraph L conditioned upon containing a subgraph H of L.

Theorem 5.6.4 ([47, Theorem 2.10])**.** Let $\mathbf{d} \in \mathbb{N}^n$ such that $\sum_{i=1}^n d_i = 2m$ is even. Let $H \subseteq L$ be graphs on $[n]$. Let $\Delta = \max_{i=1}^n d_i$ and let Δ_L denote the maximum degree of L. Let $J = ([n], E(L) \setminus E(H))$. Let $\mathbf{j}$ denote the degree sequence of J and let $\mathbf{h}$ denote the degree sequence of H. Let $\gamma := \Delta(\Delta + \Delta_L)$ and $\overline{\gamma} := \Delta(\Delta + \Delta_L + 2)$. Then the following hold:

(a) If $m - E(H) - E(J) \ge \gamma$,

$$\frac{N(\mathbf{d}, H, H)}{N(\mathbf{d}, L, L)} \le \frac{\prod_{i=1}^n [d_i - h_i]_{j_i}}{2^{|E(J)|}[m - |E(H)| - \gamma]_{|E_j|}}$$

(b) If $m - E(H) - E(J) \ge \overline{\gamma} + \Delta(\Delta_L + 1)$,

$$\frac{N(\mathbf{d}, H, H)}{N(\mathbf{d}, L, L)} \ge \frac{\prod_{i=1}^n [d_i - h_i]_{j_i}}{2^{|E(J)|}[m - |E(H)| - 1]_{|E_j|}} \frac{\left(1 - \frac{\Delta(\Delta_L+1)}{2(m - |E(H)| - |E(J)| - \overline{\gamma})}\right)^{|E(J)|}}{\left(1 + \frac{\Delta^2}{2(m - |E(H)| - \gamma - (e-1)|E(J)|/e)}\right)^{|E(J)|}}$$

We are now ready to prove Lemma 5.6.2.

Proof of Lemma 5.6.2. Let Δ_L be the maximum degree in L and let Δ be the maximum degree in D. Note that $\Delta_L \leq |E(H)| + k \leq k \log n + k$ and $\Delta \leq n^{0.25}$. Recall that $J = L \setminus H$. Then

$$\begin{aligned} m - |E(H)| - |E(J)| &\geq m - kt(n) - k \geq n^{0.25}(2n^{0.25}) \\ &\geq \Delta(\Delta + \Delta_L) =: \gamma. \end{aligned}$$

So we can use part (a) of Theorem 5.6.4 to obtain that

$$\begin{aligned} \mathbb{P}\left(L \subseteq G(\mathbf{d}) \,\middle|\, H \subseteq G(\mathbf{d})\right) &\leq \frac{\prod_{v=1}^{n} [d_i - h_i]_{j_i}}{2^{|E(J)|}[m - |E(H)| - \gamma]_{|E(J)|}} \\ &= \frac{\prod_{v=1}^{n} [d_i - h_i]_{j_i}}{2^{|E(J)|}[m]_{|E(J)|}}(1 + \nu_1(n)) \end{aligned}$$

with $\nu_1(n) = o(1)$ because $|E(J)| \leq k$ and $m - |E(H)| - \gamma \geq m - k\log n - 2\sqrt{n}$. Now we will use part (b) of Theorem 5.6.4. We have to check the conditions for (b):

$$\begin{aligned} m - |E(H)| - |E(J)| &\geq m - kt(n) - k \geq 3\sqrt{n} \\ &\geq \Delta(\Delta + \Delta_L + 2) + \Delta(\Delta_L + 1) \end{aligned}$$

so we can apply Theorem 5.6.4(b). We have that

$$\begin{aligned} 0 &\leq \frac{\Delta(\Delta_L + 1)}{m - |E(H)| - |E(J)| - \Delta(\Delta + \Delta_L + 2)} \\ &\leq \frac{n^{0.25}(n^{0.25} + 1)}{n - k\log n - k - n^{0.25}(2n^{0.25} + 2)} =: \nu_2(n), \end{aligned}$$

with $\nu_2(n) = O(1/\sqrt{n})$, and

$$\begin{aligned} 0 &\leq \frac{\Delta^2}{2(|E(G)| - |E(H)| - \gamma - (1 - 1/e)|E(J)|)} \\ &\leq \frac{\sqrt{n}}{2(n - k\log n - n^{0.25}(2n^{0.25}) - (1 - 1/e)k)} =: \nu_3(n), \end{aligned}$$

with $\nu_3(n) = O(1/\sqrt{n})$. Then Theorem 5.6.4(b) implies that

$$\mathbb{P}\left(L \subseteq G(\mathbf{d}) \,\middle|\, H \subseteq G(\mathbf{d})\right) \geq \frac{\prod_{v=1}^{n} [d_i - h_i]_{j_i}}{2^{|E(J)|}[m]_{|E(J)|}}(1 + \nu_4(n)) \left(\frac{1 + \nu_2(n)}{1 + \nu_3(n)}\right)^{E(J)}.$$

with $\nu_4(n) = o(1)$. Since $\nu_i(n) = o(1)$ for $i = 1, 2, 3, 4$, we can conclude that

$$\mathbb{P}\left(L \subseteq G(\mathbf{d}) \,\middle|\, H \subseteq G(\mathbf{d})\right) = \frac{\prod_{v=1}^{n} [d_i - h_i]_{j_i}}{2^{|E(J)|}[m]_{|E(J)|}}(1 + \nu(n)),$$

where $\nu = o(1)$. □

Proof of Lemma 5.6.3. We will use Erdős-Gallai Theorem: $\mathbf{g}$ is the degree sequence of a simple graph if and only if, for every $1 \leq \ell \leq n'$,

$$\sum_{i=1}^{\ell} g_i \leq \ell(\ell-1) + \sum_{j=\ell+1}^{n'} \min\{\ell, g_j\}.$$

If $\ell \geq n^{0.25} + 1$, then $\sum_{i=1}^{\ell} g_i \leq \ell g_1 \leq \ell(\ell-1)$. If $\ell \leq n^{0.25}$,

$$\sum_{i=1}^{\ell} g_i \leq \ell n^{0.25} \leq \sqrt{n} \leq n - n^{0.25} - (k+1)\log n \leq n' - \ell - |\{j : g_j = 0\}|$$
$$= \sum_{j=\ell+1}^{n'} 1 - |\{j : g_j = 0\}| \leq \sum_{j=\ell+1}^{n'} \min\{\ell, g_j\},$$

and we are done. □

Glossary for Chapter 5

c	$2m/n$, the average degree
c_k	$\inf\{h_k(\mu) : \mu > 0\}$, p. 148
c'_k	$\mu_{k,c_k} f_{k-1}(\mu_{k,c_k})/f_k(\mu_{k,c_k})$, p. 148
$\mathbf{d}(H)$	the degree sequence of a graph H
$D_j(\mathbf{d})$	$\lvert\{i : d_i = j\}\rvert$, the number of vertices of degree j
$\mathcal{D}_k(n,m)$	the set of $\mathbf{d} \in \mathbb{N}^n$ with $\sum_{i=1}^n d_i = 2m$ and $\min_i d_i \geq k$
$\eta(\mathbf{d})$	$\sum_i d_i(d_i - 1)/(2m)$
$f_k(\lambda)$	$e^\lambda - \sum_{i=0}^{k-1} \lambda^i/i!$
G_k	$G_k(n,m)$, the graph sampled uniformly at random from the (simple) k-cores on $[n]$ and m
G^{multi}	$G_k^{\text{multi}}(n,m)$, the random multigraph generated using the allocation model restricted to k-cores
$G^{\text{multi}}(\mathbf{d})$	the random multigraph generated using the pairing model and degree sequence $\mathbf{d}$
$G(n,m)$	the graph sampled uniformly at random from the (simple) graphs on $[n]$ and m edges
$h_k(\mu)$	$e^\mu \mu/f_{k-1}(\mu)$, p. 148
$K(H)$	the k-core of a graph H
$\lambda(k,c)$	the unique positive solution to $\lambda f_{k-1}(\lambda)/f_k(\lambda) = c$
$\mu_{k,c}$	be the largest positive solution to $c = h_k(\mu)$ for $c \geq c_k$, p. 154
$\text{Multi}_{\geq k}(n, 2m)$	multinomial distribution conditioned upon each coordinate being at least k
$P(M, L, V, k)$	the pairing-allocation model, where V is the set of heavy vertices and L the set of light vertices, p. 165
Σ	used to denote the event that $\sum_{i=1}^n Y_i = 2m$ for independent truncated Poisson random variables with parameters $(k, \lambda(k,c))$
$W(H)$	$\lvert V(H)\rvert - \lvert V(K(H-e))\rvert$, where e is an edge chosen uniformly at random from of $E(H)$
$\mathbf{Y}$	used to denote a vector $(Y_1, \ldots, Y_n)$ of independent truncated Poisson random variables with parameters $(k, \lambda(k,c))$

For the random walks and deletion procedure:

$p_j(\mathbf{d})$	probability that $Z_j(\mathbf{d}) = k - 1$, p. 152

$p'_j(\mathbf{d})$	probability that $Z_j(\mathbf{d}) = -1$, p. 152
$q(k,c)$	$\lambda(k,c)^{k-1}/((k-1)!f_{k-1}(\lambda(k,c)))$, p. 152
$Y_j(\mathbf{d})$	the number of undeleted marked points after the j-th iteration of the loop in the deletion procedure, p. 151
Y_j	$Y_{j-1} + Z_j - 1$, p. 153
Y_j^+	$Y_{j-1}^+ + Z_j^+ - 1$, the position after Step j of the random walk that bounds the deletion procedure from above, p. 153
Y_j^-	$Y_{j-1}^- + Z_j^- - 1$, the position after Step j of the random walk that bounds the deletion procedure from below, p. 153
Z_j	random variable with same distribution as $Z(k,c)$ giving the drift of Y_j Step j, p. 153
Z_j^+	random variable with same distribution as $Z^+(k,c,\xi)$ giving the drift of Y_j^+ in Step j, p. 153
Z_j^-	random variable with same distribution as $Z^-(k,c,\xi)$ giving the drift of Y_j^- in Step j, p. 153
$Z_j(\mathbf{d})$	the number of points that are marked in the j-th iteration of the loop in the deletion procedure, p. 151
$Z(k,c)$	a random variable used to define random walk that approximates the deletion procedure, p. 152
$Z^+(k,c,\xi)$	a random variable used to define a random walk that bounds the number of marked points in each step in the deletion procedure from above, p. 153
$Z^-(k,c,\xi)$	a random variable used to define a random walk that bounds the number of marked points in each step in the deletion procedure from below, p. 153

For the Differential Equation Method:

S_i	the number of points in heavy vertices after iteration i
T_i	the number of heavy vertices after iteration i
W_i	the number of points in light vertices after iteration i
y	$y(i/n)$ approximates S_i/n
z	$z(i/n)$ approximates T_i/n
T	$\min\{i : W_i = 0\}$, a stopping time indicating the deletion procedure has ended
T_D	$\min\{i : (i/n, S_i/n) \notin D\}$, a stopping time indicating the deletion procedure has left D

Chapter 6

Future directions

In this chapter, we summarise the results presented in this thesis and discuss future research directions.

In Chapter 3, we explored properties of random 2-cores to obtain an asymptotic formula for the number of 2-connected (n,m)-graphs in the sparse range $m-n\to\infty$ and $m=O(n\log n)$. (Recall that a (n,m)-graph is a graph with vertex set $[n]$ and m edges.) As we mentioned in the Introduction, less results are known for the enumeration of unlabelled graphs with some property of interest than for labelled graphs. In 1950 in the Gibbs Lecture at an American Mathematical Society meeting, G. E. Uhlenbeck cited the enumeration of 2-connected unlabelled graphs as one of the unsolved problems in statistical mechanics. The exact enumeration of 2-connected unlabelled graphs with fixed number of vertices and edges was addressed by Robinson [60]. From a result by Wright [66] combined with a result by Komlós and Szemerédi [41], we have that for $m>(1/2+\varepsilon)n\log n$, where $\varepsilon>0$ is a constant, the number of 2-connected unlabelled graphs with n vertices and m edges is asymptotic to the number of unlabelled graphs with n vertices and m edges. (We remark that the results in [66, 41] are about Hamilton cycles.) Our techniques for obtaining the asymptotic formula for 2-connected (n,m)-graphs provides information on the structure of 2-connected (n,m)-graphs. We intend to use this information and to further extend our techniques to find an asymptotic formula for the number of 2-connected unlabelled graphs with given number of vertices and edges in the sparse range.

In Chapter 4, we defined cores of 3-uniform hypergraphs and studied properties of random cores. From that, we obtained an asymptotic formula for the number of connected $(n,m,3)$-hypergraphs for the range $m=n/2+R$ with $R=o(n)$ and $R=\omega(n^{1/3}\ln^2 n)$. (Recall that a (n,m,k)-hypergraph is k-uniform hypergraph with vertex set $[n]$ and m edges.) As we mentioned in the Introduction, Andriamampianina and Ravelomanana [3] obtained an asymptotic formula for the number of connected (n,m,k)-hypergraphs for $m=n/(k-1)+o(n^{1/3})$, and Behrisch, Coja-Oghlan and Kang [4] provided an asymptotic formula for the case $m=n/(k-1)+\Theta(n)$.

No asymptotic formula is known for the number of connected (n,m,k)-hypergraphs for $k \geq 4$ in the range $m - n/(k-1) = \Omega(n^{1/3})$ and $m - n/(k-1) = o(n)$. The superlinear range $m - n/(k-1) = \omega(n)$ is open for any $k \geq 3$.

The reason why our method does not cover the whole range $R = o(n)$ and $R = \Omega(n^{1/3})$ is that, when computing the number of connected cores with given number of vertices and edges, we write this number as a summation $\sum_{x \in S} g(x)$, where $g(x)$ is the number of connected cores with some parameters $x = (n_1, k_0, k_1, k_2)$. We write a function $f(x)$ that approximates $g(x)$. We then find the point x^* achieving the maximum for f in S. By expanding the summation $\sum_x f(x)$ around x^*, we can determine the value of the summation $\sum_{x \in S} g(x)$ asymptotically. For doing this, it was important that x^* was 'reasonably' inside the interior of the set S. When $R = O(n^{1/3})$ the point achieving the maximum is 'too close' to the boundary of S. We intend to deal with this case by fixing some functions of the parameters that cause x^* to be 'too close' to the boundary of S and estimating the summation with these functions fixed. We expect to extend our results to obtain an asymptotic formula for the number of connected $(n,m,3)$-hypergraphs for $m = n/2 + R$ as long as $R = o(n)$ and $R = \Omega(n^{1/3})$. This way, we would completely close the gap between the case $R = o(n^{1/3})$ and the linear case $R = \Omega(n)$ in which no asymptotic formulae were found.

In [56], Pittel and Wormald obtained properties of the giant component of $G(n,m)$ by using the asymptotic formula they obtained for the number of connected $(n,m,2)$-cores (recall that an (n,m,k)-core is a graph with vertex set $[n]$, m edges and minimum degree at least k). They determined the limit joint distribution of X, Y and Z, where X is the number of vertices in the 2-core of the giant component, Y is the number of vertices in the giant component that are not in the 2-core and $Z = W - X$, where W is the number of edges in the 2-core. Since the number of vertices and of edges of the giant component can be written as linear combinations of X, Y and Z, [56] also determines the limit joint distribution of the number of vertices and edges. Even more than that, Pittel and Wormald determined the probability that $(X,Y,Z) = (x,y,z)$ under some constraints on (x,y,z) and conditioned upon a certain event B_n that imposes a lower bound in the number of vertices of the largest component and an upper bound in the number of vertices of the second-largest component. One of the main steps in the proof was to determine the number of (n,m)-graphs with one component with x vertices in its 2-core, y vertices not in its 2-core and $x + z$ edges. This is the same as

$$\binom{n}{x+y}\binom{x+y}{x} g(x, x+z) f(x+y, x) \binom{\binom{n-x-y}{2}}{m-x-y-z}, \qquad (6.1)$$

where $g(x, x+z)$ is the number of connected $(x, x+z, 2)$-cores and $f(x+y, x)$ is the number of rooted forests with vertex set $[x+y]$ and set of roots $[x]$. Pittel and Wormald [56] determined the value in (6.1) asymptotically by using their asymptotic formula for $g(x, x+z)$ and a known formula for $f(x+y, x)$. We intend to follow a similar strategy to find the limit joint distribution of X, Y and Z for 3-uniform hypergraphs since we have obtained an asymptotic formula for the number of connected cores with given number of vertices and edges. We remark that Bollobás and Riordan [14] determined the limit joint distribution of the number of vertices and of edges in the giant component for random k-uniform hypergraphs. As far as we know, their result does not provide point probabilities and it does not determine the limit distribution of the number of vertices in the core of the giant component.

In Chapter 5, we studied robustness properties of the random graph $G_k(n,m)$, which has uniform distribution on all (n,m,k)-cores. Our main result concerns the random graph $G_k(n,m)-e$ obtained by deleting an edge e chosen u.a.r. from the edges of $G_k(n,m)$. We defined a constant c'_k and analysed the k-core of $G_k(n,m)-e$ for $c = 2m/n$ above or below c'_k. We proved that, for $c < c'_k - \varepsilon$ and any $h(n) \to \infty$, the k-core of $G_k(n,m)-e$ is either empty or has at least $n - h(n)$ vertices a.a.s. Moreover, if $c \to k$, we showed that the k-core of $G_k(n,m)-e$ is empty a.a.s. For $c > c'_k + \psi(n)$ with $\psi(n) = \omega(n^{-1/4})$ and any $h(n) = \omega(\psi(n^{-1})$, we proved that the k-core of $G_k(n,m)-e$ has at least $n-h(n)$ vertices a.a.s. These results do not cover the range $c > c'_k + O(n^{-1/4})$. It would be interesting to obtain results for $c > c'_k + n^{-\varepsilon}$ where ε is a constant in $(0,1/2)$ since Pittel, Spencer and Wormald [54] showed that, when the average degree of $G(n,m)$ is above $c_k + n^{-\varepsilon}$, the k-core of $G(n,m)$ is nonempty. Naturally, results about the k-core when it is closer to the time of its emergence would be very interesting. Another range not covered is when c tends to c'_k from below.

For the range $k+\varepsilon < c < c'_k - \varepsilon$, we also proved that the k-core of $G_k(n,m)-e$ is empty with probability bounded away from zero. But this still leaves the possibility that there is a nonnegligible fraction of (n,m,k)-cores such that there is no edge (or very few edges) whose deletion would cause the whole k-core to be deleted. It would be interesting to find out if this is indeed the case. It is also natural to consider what happens when deleting more than one edge in the beginning. More specifically, given $h(n) \to \infty$, is it true that, if we delete $h(n)$ edges u.a.r., then the k-core of the new graph is empty a.a.s.? We intend to settle this question in the near future.

References

[1] D. Achlioptas and M. Molloy. The solution space geometry of random linear equations. *Random Structures Algorithms*, 2013.

[2] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.

[3] T. Andriamampianina and V. Ravelomanana. Enumeration of connected uniform hypergraphs. In *Proceedings of FP-SAC*, 2005.

[4] M. Behrisch, A. Coja-Oghlan, and M. Kang. Local limit theorems and number of connected hypergraphs. *ArXiv e-prints*, 2007.

[5] E. A. Bender and E. R. Canfield. The asymptotic number of labeled graphs with given degree sequences. *J. Combin. Theory Ser. A*, 24(3):296–307, 1978.

[6] E. A. Bender, E. R. Canfield, and B. D. McKay. The asymptotic number of labeled graphs with n vertices, q edges, and no isolated vertices. *J. Combin. Theory Ser. A*, 80(1):124–150, 1997.

[7] E. A. Bender, E. R. Canfield, and B.D. McKay. The asymptotic number of labeled connected graphs with a given number of vertices and edges. *Random Structures Algorithms*, 1(2):127–169, 1990.

[8] B. Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European J. Combin.*, 1(4):311–316, 1980.

[9] B. Bollobás. The evolution of random graphs. *Trans. Amer. Math. Soc.*, 286(1):257–274, 1984.

[10] B. Bollobás. The evolution of sparse graphs. In *Graph theory and combinatorics (Cambridge, 1983)*, pages 35–57. Academic Press, London, 1984.

[11] B. Bollobás. *Random graphs*, volume 73 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[12] B. Bollobás and A. M. Frieze. On matchings and Hamiltonian cycles in random graphs. In *Random graphs '83 (Poznań, 1983)*, volume 118 of *North-Holland Math. Stud.*, pages 23–46. North-Holland, Amsterdam, 1985.

[13] B. Bollobás, J. H. Kim, and J. Verstraëte. Regular subgraphs of random graphs. *Random Structures Algorithms*, 29(1):1–13, 2006.

[14] B. Bollobás and O. Riordan. Asymptotic normality of the size of the giant component in a random hypergraph. *Random Structures Algorithms*, 41(4):441–450, 2012.

[15] B. Bollobás and O. Riordan. Asymptotic normality of the size of the giant component via a random walk. *J. Combin. Theory Ser. B*, 102(1):53–61, 2012.

[16] B. Bollobás and A. Thomason. Threshold functions. *Combinatorica*, 7(1):35–38, 1987.

[17] J. Cain and N. Wormald. Encores on cores. *Electron. J. Combin.*, 13(1):Research Paper 81, 13 pp. (electronic), 2006.

[18] S. O. Chan and M. Molloy. $(k+1)$-cores have k-factors. *Combin. Probab. Comput.*, 21(6):882–896, 2012.

[19] V. Chvátal. Almost all graphs with $1.44n$ edges are 3-colorable. *Random Structures Algorithms*, 2(1):11–28, 1991.

[20] C. Cooper. The cores of random hypergraphs with a given degree sequence. *Random Structures Algorithms*, 25(4):353–375, 2004.

[21] J. Ding, J. H. Kim, E. Lubetzky, and Y. Peres. Anatomy of a young giant component in the random graph. *Random Structures Algorithms*, 39(2):139–178, 2011.

[22] P. Erdős. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, 53:292–294, 1947.

[23] P. Erdős and A. Rényi. On random graphs. I. *Publ. Math. Debrecen*, 6:290–297, 1959.

[24] P. Erdős and A. Rényi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 5:17–61, 1960.

[25] P. Erdős and A. Rényi. On the strength of connectedness of a random graph. *Acta Math. Acad. Sci. Hungar.*, 12:261–267, 1961.

[26] W. Feller. *An introduction to probability theory and its applications. Vol. I.* Third edition. John Wiley & Sons Inc., New York, 1968.

[27] W. Feller. *An introduction to probability theory and its applications. Vol. I.* Third edition. John Wiley & Sons Inc., New York, 1968.

[28] D. Fernholz and V. Ramachandaram. The giant k-core of a random graph with specified degree sequence. Manuscript available at `http://www.cs.utexas.edu/~fernholz/`, November 2003.

[29] E. Friedgut. Sharp thresholds of graph properties, and the k-sat problem. *J. Amer. Math. Soc.*, 12(4):1017–1054, 1999. With an appendix by Jean Bourgain.

[30] P. Gao. The first k-regular subgraph is large. *ArXiv e-prints*, 2013.

[31] O. Güler. *Foundations of optimization*, volume 258 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[32] F. Harary and E. M. Palmer. *Graphical enumeration.* Academic Press, New York, 1973.

[33] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963.

[34] S. Janson. Poisson convergence and Poisson processes with applications to random graphs. *Stochastic Process. Appl.*, 26(1):1–30, 1987.

[35] S. Janson and M. J. Luczak. A simple solution to the k-core problem. *Random Structures Algorithms*, 30(1-2):50–62, 2007.

[36] S. Janson and M. J. Luczak. Asymptotic normality of the k-core in random graphs. *Ann. Appl. Probab.*, 18(3):1085–1137, 2008.

[37] S. Janson, T. Łuczak, and A. Rucinski. *Random graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[38] M. Karoński and T. Łuczak. The number of connected sparsely edged uniform hypergraphs. *Discrete Math.*, 171(1-3):153–167, 1997.

[39] G. Kemkes, C. M. Sato, and N. Wormald. Asymptotic enumeration of sparse 2-connected graphs. *Random Structures Algorithms*, 43(3):354–376, 2013.

[40] J. H. Kim. Poisson cloning model for random graphs. In *International Congress of Mathematicians. Vol. III*, pages 873–897. Eur. Math. Soc., Zürich, 2006.

[41] J. Komlós and E. Szemerédi. Hamilton cycles in random graphs. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 1003–1010. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[42] A. D. Korshunov. On the number of graphs with a fixed number of vertices, edges, and isolated vertices [translation of *discrete analysis (russian)*, 43–93, Izdat. Ross. Akad. Nauk Sibirsk. Otdel. Inst. Mat., Novosibirsk, 1994]. *Siberian Adv. Math.*, 5(4):50–112, 1995. Siberian Advances in Mathematics.

[43] C. Lavault. Prüfer-like coding and counting forests of uniform hypertrees. In *8th International Conference on Computer Science and Information Technologies (CSIT 2011)*, pages 82–85, 2011.

[44] T. Łuczak. Component behavior near the critical point of the random graph process. *Random Structures Algorithms*, 1(3):287–310, 1990.

[45] T. Łuczak. Size and connectivity of the k-core of a random graph. *Discrete Math.*, 91(1):61–68, 1991.

[46] T. Łuczak. Sparse random graphs with a given degree sequence. In *Random graphs, Vol. 2 (Poznań, 1989)*, Wiley-Intersci. Publ., pages 165–182. Wiley, New York, 1992.

[47] B. D. McKay. Subgraphs of random graphs with specified degrees. In *Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing, Vol. II (Baton Rouge, La., 1981)*, volume 33, pages 213–223, 1981.

[48] B. D. McKay. Asymptotics for symmetric 0-1 matrices with prescribed row sums. *Ars Combin.*, 19(A):15–25, 1985.

[49] B. D. McKay and N. C. Wormald. Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$. *Combinatorica*, 11(4):369–382, 1991.

[50] M. Molloy. M. Math Thesis, University of Waterloo, 1992.

[51] M. Molloy. Cores in random hypergraphs and Boolean formulas. *Random Structures Algorithms*, 27(1):124–135, 2005.

[52] B. Pittel. On tree census and the giant component in sparse random graphs. *Random Structures Algorithms*, 1(3):311–342, 1990.

[53] B. Pittel. On the largest component of the random graph at a nearcritical stage. *J. Combin. Theory Ser. B*, 82(2):237–269, 2001.

[54] B. Pittel, J. Spencer, and N. Wormald. Sudden emergence of a giant k-core in a random graph. *J. Combin. Theory Ser. B*, 67(1):111–151, 1996.

[55] B. Pittel and N. C. Wormald. Asymptotic enumeration of sparse graphs with a minimum degree constraint. *J. Combin. Theory Ser. A*, 101(2):249–263, 2003.

[56] B. Pittel and N. C. Wormald. Counting connected graphs inside-out. *J. Combin. Theory Ser. B*, 93(2):127–172, 2005.

[57] M. Pretti and M. Weigt. Sudden emergence of q-regular subgraphs in random graphs. *Europhys. Lett.*, 75(1):8–14, 2006.

[58] V. Ravelomanana and L. Thimonier. Forbidden subgraphs in connected graphs. *Theoret. Comput. Sci.*, 314(1-2):121–171, 2004.

[59] O. Riordan. The k-core and branching processes. *Combin. Probab. Comput.*, 17(1):111–136, 2008.

[60] R. W. Robinson. Enumeration of non-separable graphs. *J. Combinatorial Theory*, 9:327–356, 1970.

[61] H. N. V. Temperley. On the enumeration of Mayer cluster integrals. *Proceedings of the Physical Society*, 72(6):1141, 1958.

[62] V. A. Voblyĭ. Wright and Stepanov-Wright coefficients. *Mat. Zametki*, 42(6):854–862, 911, 1987.

[63] N. Wormald and E. M. Wright. The exponential generating function of labelled blocks. *Discrete Math.*, 25(1):93–96, 1979.

[64] N. C. Wormald. Models of random regular graphs. In *Surveys in combinatorics, 1999 (Canterbury)*, volume 267 of *London Math. Soc. Lecture Note Ser.*, pages 239–298. Cambridge Univ. Press, Cambridge, 1999.

[65] N.C. Wormald. The differential equation method for random graph processes and greedy algorithms. In M. Karonski and H.J. Proemel, editors, *Lectures on Approximation and Randomized Algorithms*, pages 73–155. PWN, Warsaw, 1999.

[66] E. M. Wright. The proportion of unlabelled graphs which are Hamiltonian. *Bull. London Math. Soc.*, 8(3):241–244, 1976.

[67] E. M. Wright. The number of connected sparsely edged graphs. *J. Graph Theory*, 1(4):317–330, 1977.

[68] E. M. Wright. The number of connected sparsely edged graphs. II. Smooth graphs and blocks. *J. Graph Theory*, 2(4):299–305, 1978.

[69] E. M. Wright. The number of connected sparsely edged graphs. III. Asymptotic results. *J. Graph Theory*, 4(4):393–407, 1980.

[70] E. M. Wright. The number of connected sparsely edged graphs. IV. Large nonseparable graphs. *J. Graph Theory*, 7(2):219–229, 1983.

Appendix

Appendix A

Maple spreadsheets

In this appendix we include some Maple spreadsheets for computations in Chapter 4.

A.1 Spreadsheet for Section 4.2

In this section we include some computations used in Section 4.2.

```
> zetasol := lambda*(exp(2*lambda)+exp(lambda)+1)/(exp(2*lambda)-1);
> Phi1 := r/(1-r)*ln(r)+(1-zeta)*ln(1-r)+zeta/3*ln(1-r^3);
> Phi2:= 1/(1/r-1)*ln(r)+(1-zeta)*ln((1/r-1)*r)+zeta/3*ln((1/r^3-1)*r^3);
> is(simplify(Phi1-Phi2,symbolic),0);
```

$$zetasol := \frac{\lambda\left(e^{2\lambda}+e^{\lambda}+1\right)}{e^{2\lambda}-1}$$

$$Phi1 := \frac{r\ln(r)}{1-r}+(1-\zeta)\ln(1-r)+1/3\ (\zeta)\ \ln\left(1-r^3\right)$$

$$Phi2 := \frac{\ln(r)}{r^{-1}-1}+(1-\zeta)\ln\left(\left(r^{-1}-1\right)r\right)+1/3\ (\zeta)\ \ln\left(\left(r^{-3}-1\right)r^3\right)$$

true

```
> Philambda:= subs(r=1/exp(lambda),zeta=zetasol,Phi2);
```

$$Philambda := \frac{\ln\left(\left(e^{\lambda}\right)^{-1}\right)}{e^{\lambda}-1} + \left(1 - \frac{\lambda\left(e^{2\lambda}+e^{\lambda}+1\right)}{e^{2\lambda}-1}\right)\ln\left(\frac{e^{\lambda}-1}{e^{\lambda}}\right)$$
$$+ 1/3\,\lambda\left(e^{2\lambda}+e^{\lambda}+1\right)\ln\left(\frac{\left(e^{\lambda}\right)^3-1}{\left(e^{\lambda}\right)^3}\right)\left(e^{2\lambda}-1\right)^{-1}$$

```
> # Their formula
> DNM := M-M*ln(M)+3*M*ln(N)-M*ln(6)+N*Philambda;
```

$$DNM := M - M\ln(M) + 3\,M\ln(N) - M\ln(6)$$
$$+N\left(\frac{\ln\left(\left(e^{\lambda}\right)^{-1}\right)}{e^{\lambda}-1} + \left(1 - \frac{\lambda\left(e^{2\lambda}+e^{\lambda}+1\right)}{e^{2\lambda}-1}\right)\ln\left(\frac{e^{\lambda}-1}{e^{\lambda}}\right) +\right.$$
$$\left. 1/3\,\lambda\left(e^{2\lambda}+e^{\lambda}+1\right)\ln\left(\frac{\left(e^{\lambda}\right)^3-1}{\left(e^{\lambda}\right)^3}\right)\left(e^{2\lambda}-1\right)^{-1}\right)$$

```
> phi := -(1-x)/2*ln(1-x) + (1-x)/2 + 2*(R/N)*ln(N) - (ln(2)+2)*R/N
> -1/2*ln(2)*x + R/N*ln((exp(lambda)+1)/(lambda*(exp(lambda)-1)))
> +1/2*x*ln((exp(2*lambda)-1)/lambda);
```

$$\phi := -1/2\,(1-x)\ln(1-x) + 1/2 - 1/2\,x + 2\,\frac{R\ln(N)}{N} - \frac{(\ln(2)+2)\,R}{N}$$
$$- 1/2\,\ln(2)\,x + R\ln\left(\frac{e^{\lambda}+1}{\lambda\left(e^{\lambda}-1\right)}\right)N^{-1} + 1/2\,x\ln\left(\frac{e^{2\lambda}-1}{\lambda}\right)$$

```
> philambda := subs(x = (exp(2*lambda)-1-2*lambda)/(exp(2*lambda)-1),phi);
```

$$philambda := -1/2\left(1 - \frac{e^{2\lambda}-1-2\lambda}{e^{2\lambda}-1}\right)\ln\left(1 - \frac{e^{2\lambda}-1-2\lambda}{e^{2\lambda}-1}\right)$$
$$+ 1/2 - 1/2\,\frac{e^{2\lambda}-1-2\lambda}{e^{2\lambda}-1} + 2\,\frac{R\ln(N)}{N} - \frac{(\ln(2)+2)\,R}{N} - 1/2\,\frac{\ln(2)\left(e^{2\lambda}-1-2\lambda\right)}{e^{2\lambda}-1}$$
$$+ R\ln\left(\frac{e^{\lambda}+1}{\lambda\left(e^{\lambda}-1\right)}\right)N^{-1} + 1/2\left(e^{2\lambda}-1-2\lambda\right)\ln\left(\frac{e^{2\lambda}-1}{\lambda}\right)\left(e^{2\lambda}-1\right)^{-1}$$

```
> # Our formula
> CNM := N*ln(N)-N+N*philambda;
```

$$CNM := N\ln(N) - N + N\Bigg(-1/2\left(1 - \frac{e^{2\lambda}-1-2\lambda}{e^{2\lambda}-1}\right)\ln\left(1 - \frac{e^{2\lambda}-1-2\lambda}{e^{2\lambda}-1}\right)$$
$$+ 1/2 - 1/2\,\frac{e^{2\lambda}-1-2\lambda}{e^{2\lambda}-1} + 2\,\frac{R\ln(N)}{N} - \frac{(\ln(2)+2)R}{N} - 1/2\,\frac{\ln(2)\left(e^{2\lambda}-1-2\lambda\right)}{e^{2\lambda}-1}$$
$$+ R\ln\left(\frac{e^{\lambda}+1}{\lambda\left(e^{\lambda}-1\right)}\right)N^{-1} + 1/2\left(e^{2\lambda}-1-2\lambda\right)\ln\left(\frac{e^{2\lambda}-1}{\lambda}\right)\left(e^{2\lambda}-1\right)^{-1}\Bigg)$$

```
> # Their formula minus ours
> simplify(subs(R=M-N/2,M=zetasol*N/3,CNM-DNM),symbolic);
> num:= numer(%);
```

$$-\frac{1}{6}\frac{N}{e^{2\lambda}-1}\Big(3\,e^{2\lambda}\ln\left(e^{\lambda}+1\right) - 3\,e^{2\lambda}\ln\left(e^{2\lambda}-1\right) + 2\,e^{2\lambda}\lambda\ln\left(e^{2\lambda}-1\right)$$
$$- 2\,e^{2\lambda}\lambda\ln\left(e^{2\lambda}+e^{\lambda}+1\right) - 4\,\lambda\,e^{2\lambda}\ln\left(e^{\lambda}-1\right) + 2\,e^{2\lambda}\lambda\ln\left(e^{3\lambda}-1\right) - 2\,\lambda\,e^{2\lambda}\ln\left(e^{\lambda}+1\right)$$
$$+ 3\,e^{2\lambda}\ln\left(e^{\lambda}-1\right) + 2\,e^{\lambda}\lambda\ln\left(e^{2\lambda}-1\right) - 2\,e^{\lambda}\lambda\ln\left(e^{2\lambda}+e^{\lambda}+1\right) - 4\,\lambda\,e^{\lambda}\ln\left(e^{\lambda}-1\right)$$
$$+ 2\,e^{\lambda}\lambda\ln\left(e^{3\lambda}-1\right) - 2\,\lambda\,e^{\lambda}\ln\left(e^{\lambda}+1\right) - 3\ln\left(e^{\lambda}+1\right) - 3\ln\left(e^{\lambda}-1\right) + 3\ln\left(e^{2\lambda}-1\right)$$
$$+ 2\,\lambda\ln\left(e^{2\lambda}-1\right) - 2\,\lambda\ln\left(e^{2\lambda}+e^{\lambda}+1\right) - 4\,\lambda\ln\left(e^{\lambda}-1\right) + 2\,\lambda\ln\left(e^{3\lambda}-1\right) - 2\,\lambda\ln\left(e^{\lambda}+1\right)\Big)$$

$$num := -N\Big(3\,e^{2\lambda}\ln\left(e^{\lambda}+1\right) - 3\,e^{2\lambda}\ln\left(e^{2\lambda}-1\right) + 2\,e^{2\lambda}\lambda\ln\left(e^{2\lambda}-1\right)$$
$$- 2\,e^{2\lambda}\lambda\ln\left(e^{2\lambda}+e^{\lambda}+1\right) - 4\,\lambda\,e^{2\lambda}\ln\left(e^{\lambda}-1\right) + 2\,e^{2\lambda}\lambda\ln\left(e^{3\lambda}-1\right) - 2\,\lambda\,e^{2\lambda}\ln\left(e^{\lambda}+1\right)$$
$$+ 3\,e^{2\lambda}\ln\left(e^{\lambda}-1\right) + 2\,e^{\lambda}\lambda\ln\left(e^{2\lambda}-1\right) - 2\,e^{\lambda}\lambda\ln\left(e^{2\lambda}+e^{\lambda}+1\right) - 4\,\lambda\,e^{\lambda}\ln\left(e^{\lambda}-1\right)$$
$$+ 2\,e^{\lambda}\lambda\ln\left(e^{3\lambda}-1\right) - 2\,\lambda\,e^{\lambda}\ln\left(e^{\lambda}+1\right) - 3\ln\left(e^{\lambda}+1\right) - 3\ln\left(e^{\lambda}-1\right) + 3\ln\left(e^{2\lambda}-1\right)$$
$$+ 2\,\lambda\ln\left(e^{2\lambda}-1\right) - 2\,\lambda\ln\left(e^{2\lambda}+e^{\lambda}+1\right) - 4\,\lambda\ln\left(e^{\lambda}-1\right) + 2\,\lambda\ln\left(e^{3\lambda}-1\right) - 2\,\lambda\ln\left(e^{\lambda}+1\right)\Big)$$

```
> num2:= simplify(exp(num/N),symbolic);
```

$$num2 := \left(e^{\lambda}+1\right)^{-3\,e^{2\lambda}+2\,e^{2\lambda}\lambda+2\,e^{\lambda}\lambda+3}\left(e^{2\lambda}-1\right)^{3\,e^{2\lambda}-2\,e^{2\lambda}\lambda-2\,e^{\lambda}\lambda-3}\left(e^{2\lambda}+e^{\lambda}+1\right)^{2\lambda\left(e^{2\lambda}+e^{\lambda}+1\right)}$$
$$\left(e^{\lambda}-1\right)^{4\,e^{2\lambda}\lambda-3\,e^{2\lambda}+4\,e^{\lambda}\lambda+3+2\lambda}\left(e^{3\lambda}-1\right)^{-2\lambda\left(e^{2\lambda}+e^{\lambda}+1\right)}$$

```
> a := -2*lambda*(exp(2*lambda)+exp(lambda)+1);
> num3 := num2/(exp(3*lambda)-1)^a;
```

$$a := -2\,\lambda\left(e^{2\,\lambda}+e^{\lambda}+1\right)$$

$$\mathit{num3} := \left(e^{\lambda}+1\right)^{-3\,e^{2\,\lambda}+2\,e^{2\,\lambda}\lambda+2\,e^{\lambda}\lambda+3}\left(e^{2\,\lambda}-1\right)^{3\,e^{2\,\lambda}-2\,e^{2\,\lambda}\lambda-2\,e^{\lambda}\lambda-3}\left(e^{2\,\lambda}+e^{\lambda}+1\right)^{2\,\lambda\left(e^{2\,\lambda}+e^{\lambda}+1\right)}\left(e^{\lambda}-1\right)^{4\,e^{2\,\lambda}\lambda-3\,e^{2\,\lambda}+4\,e^{\lambda}\lambda+3+2\,\lambda}$$

```
> num4:= num3*(exp(lambda)-1)^a*(exp(2*lambda)+exp(lambda)+1)^a:
> num5:= simplify(num4,symbolic);
> b := 3*exp(2*lambda)-2*lambda*exp(2*lambda)-2*lambda*exp(lambda);
> (exp(lambda)+1)^(-b)*(exp(lambda)-1)^(-b)*(exp(2*lambda)-1)^b;
> is(simplify(%/num5),1);
```

$$\mathit{num5} := \left(e^{\lambda}+1\right)^{-3\,e^{2\,\lambda}+2\,e^{2\,\lambda}\lambda+2\,e^{\lambda}\lambda+3}\left(e^{2\,\lambda}-1\right)^{3\,e^{2\,\lambda}-2\,e^{2\,\lambda}\lambda-2\,e^{\lambda}\lambda-3}\left(e^{\lambda}-1\right)^{-3\,e^{2\,\lambda}+2\,e^{2\,\lambda}\lambda+2\,e^{\lambda}\lambda+3}$$

$$b := 3\,e^{2\,\lambda}-2\,e^{2\,\lambda}\lambda-2\,e^{\lambda}\lambda$$

$$\left(e^{\lambda}+1\right)^{-3\,e^{2\,\lambda}+2\,e^{2\,\lambda}\lambda+2\,e^{\lambda}\lambda}\left(e^{\lambda}-1\right)^{-3\,e^{2\,\lambda}+2\,e^{2\,\lambda}\lambda+2\,e^{\lambda}\lambda}\left(e^{2\,\lambda}-1\right)^{3\,e^{2\,\lambda}-2\,e^{2\,\lambda}\lambda-2\,e^{\lambda}\lambda}$$

true

```
> # Conclusion:  exp(numerator) = 1 and so numerator = 0
```

A.2 Spreadsheet for Lemma 4.7.4

```
> x*f1*g2/F2;
> f := subs(f1=exp(x)-1,g2=exp(x)+2,F2=exp(2*x)-1-2*x,%);
```

$$\frac{xf1\ g2}{F2}$$

$$f := \frac{x\,(e^x - 1)\,(e^x + 2)}{e^{2x} - 1 - 2x}$$

```
> series(f,x=0);
```

$$3/2 + 1/4\,x + 1/12\,x^2 + \frac{11}{720}\,x^3 + O\left(x^4\right)$$

```
> simplify(diff(f,x));
```

$$-\frac{-2 - e^{4x} - e^{3x} + 3\,e^{2x} + e^x - 2\,e^{2x}x + e^x x + 4\,e^{2x}x^2 + e^{3x}x + 2\,e^x x^2}{\left(e^{2x} - 1 - 2x\right)^2}$$

```
> F := numer(%);
```

$$F := 2 + e^{4x} + e^{3x} - 3\,e^{2x} - e^x + 2\,e^{2x}x - e^x x - 4\,e^{2x}x^2 - e^{3x}x - 2\,e^x x^2$$

```
> simplify(subs(x=0,F));
> F1p:= diff(F,x);
> F1p:= simplify(F1p/exp(x));
```

$$0$$

$$F1p := 4\,e^{4x} + 2\,e^{3x} - 4\,e^{2x} - 2\,e^x - 4\,e^{2x}x - 5\,e^x x - 8\,e^{2x}x^2 - 3\,e^{3x}x - 2\,e^x x^2$$

$$F1p := 4\,e^{3x} + 2\,e^{2x} - 4\,e^x - 2 - 4\,e^x x - 5\,x - 8\,e^x x^2 - 3\,e^{2x}x - 2\,x^2$$

```
> simplify(subs(x=0,F1p));
> F2p:= diff(F1p,x);
```

$$0$$

$$F2p := 12\,e^{3x} + e^{2x} - 8\,e^x - 20\,e^x x - 5 - 8\,e^x x^2 - 6\,e^{2x}x - 4\,x$$

```
> simplify(subs(x=0,F2p));
> F3p:= diff(F2p,x);
```

$$0$$

$$F3p := 36\,e^{3x} - 4\,e^{2x} - 28\,e^x - 36\,e^x x - 8\,e^x x^2 - 12\,e^{2x}x - 4$$

```
> simplify(subs(x=0,F3p));
> F4p:= diff(F3p,x);
> F4p:= simplify(F4p/exp(x));
```

$$0$$

$$F4p := 108\,e^{3x} - 20\,e^{2x} - 64\,e^x - 52\,e^x x - 8\,e^x x^2 - 24\,e^{2x}x$$

$$F4p := 108\,e^{2\,x} - 20\,e^{x} - 64 - 52\,x - 8\,x^2 - 24\,e^{x}x$$

```
> simplify(subs(x=0,F4p));
> F5p:= diff(F4p,x);
```

$$24$$

$$F5p := 216\,e^{2\,x} - 44\,e^{x} - 52 - 16\,x - 24\,e^{x}x$$

```
> 261-44-52-16-24;
```

$$125$$

A.3 Spreadsheet for Setion 4.7.3

```
> h := x -> x*n*ln(x)+x*n*ln(n)-x*n;
```

$$h := x \mapsto xn \ln(x) + xn \ln(n) - xn$$

```
> # setting functions of lambda
> g1def := exp(lambda)+1;
> g2def := exp(lambda)+2;
> f1def := exp(lambda)-1;
> f2def := exp(lambda)-1-lambda;
> F2def := exp(2*lambda)-1-2*lambda;
> F1def := exp(2*lambda)-1;
> eladef:= exp(lambda);
> c2lambdaval := lambda*f1/f2;
> # derivative of lambda
> dlambdaval := lambda*(-1+c2)/((3*m-n1)*(1+eta-c2));
> c2def := (3*m-n1)/(1-n1);
> etalambdaval := lambda*ela/f1;
> # More definitions
> Q2def := 3*m-n1;
> n2def := 1-n1;
> m3def := m-n1;
> subsf := x -> subs(eta=etalambdaval,c2=c2lambdaval,g1=g1def,g2=g2def,
f1=f1def,f2=f2def,F1=F1def,F2=F2def,ela=eladef,x);
> subsf2 := x -> subs(Q2=Q2def,n2=n2def,m3=m3def, x);
```

$$g1def := e^{\lambda} + 1$$

$$g2def := e^{\lambda} + 2$$

$$f1def := e^{\lambda} - 1$$

$$f2def := e^{\lambda} - 1 - \lambda$$

$$F2def := e^{2\lambda} - 1 - 2\lambda$$

$$F1def := e^{2\lambda} - 1$$

$$eladef := e^{\lambda}$$

$$c2lambdaval := \frac{\lambda\, f1}{f2}$$

$$dlambdaval := \frac{\lambda\,(-1 + c2)}{(3\,m - n1)\,(1 + \eta - c2)}$$

$$c2def := \frac{3\,m - n1}{1 - n1}$$

$$etalambdaval := \frac{\lambda\, ela}{f1}$$

$$Q2def := 3\,m - n1$$

$$n2def := 1 - n1$$

$$m3def := m - n1$$

$$subsf := x \mapsto x$$

$$subsf2 := x \mapsto x$$

```
> # fcore without lambda part
> f:=(n1)->expand(subsf2(h(Q2)-h(n2)-h(n1)-h(m3)-n1*n*ln(2)-m3*n*ln(6))/n):
> # First derivative
> d1:= simplify(diff(f(n1),n1),symbolic);
> d1simple := -ln(n1)+ln(m3)+ln(n2)+ln(3)-ln(Q2);
> is(simplify(subsf2(d1simple)-d1),0);
```

$$d1 := \ln(3) - \ln(n1) + \ln(1 - n1) - \ln(3\,m - n1) + \ln(m - n1)$$

$$d1simple := -\ln(n1) + \ln(m3) + \ln(n2) + \ln(3) - \ln(Q2)$$

$$true$$

```
> # exp of derivative with lambda part
> d := simplify(exp(d1)*lambda/f2);
> # simplifying
> d := simplify(d*c2def/c2lambdaval);
```

$$d := -3\,\frac{(-1 + n1)\,(m - n1)\,\lambda}{n1\,(3\,m - n1)\,f2}$$

$$d := 3\,\frac{m - n1}{f1\;n1}$$

```
> # second derivative
> # lambda part
> # dlambda is used to indicate the derivative of lambda
> # wrt n1
> diff(ln(lambda)-ln(f2def),lambda);
> dd := diff(diff(f(n1),n1),n1)+dlambda*(1/lambda)-dlambda*f1*(1/f2);
> ddsimple := 1/Q2-1/n2-1/n1-1/m3+dlambda/lambda-dlambda*f1/f2;
> is(simplify(subsf2(ddsimple)-dd),0);
```

$$\lambda^{-1} - \frac{e^{\lambda} - 1}{e^{\lambda} - 1 - \lambda}$$

$$dd := -3\,\frac{m}{(3\,m - n1)^2} + \frac{n1}{(3\,m - n1)^2} + 2\,(3\,m - n1)^{-1} + (1 - n1)^{-2}$$
$$-\frac{n1}{(1 - n1)^2} - 2\,(1 - n1)^{-1} - n1^{-1} + \frac{m}{(m - n1)^2} - \frac{n1}{(m - n1)^2}$$
$$-2\,(m - n1)^{-1} + \frac{dlambda}{\lambda} - \frac{dlambda\,f1}{f2}$$

$$ddsimple := Q2^{-1} - n2^{-1} - n1^{-1} - m3^{-1} + \frac{dlambda}{\lambda} - \frac{dlambda\,f1}{f2}$$

true

```
> # finding local max
> solve(numer(d)=denom(d),n1);
> n1solm := 3*m/g2;
> # checking if n1solm is correct sol
> is(simplify(subsf(n1solm-solve(numer(d)=denom(d),n1))),0);
```

$$3\,\frac{m}{f1 + 3}$$

$$n1solm := 3\,\frac{m}{g2}$$

true

```
> # Equation for m
> eqmax := simplify(subs(n1=n1solm,c2def=c2lambdaval));
> # Solution for m
> msol := (1/3)*lambda*f1*g2/F2;
> # checking if msol is correct sol
> is(simplify(subsf(msol-solve(eqmax,m))),0);
```

$$eqmax := 3\,\frac{m\,(g2-1)}{g2-3\,m}=\frac{\lambda\,f1}{f2}$$

$$msol := 1/3\,\frac{\lambda\,f1\;g2}{F2}$$

true

```
> # Solution for n1 in terms of lambda
> n1sol := subs(m=msol,n1solm);
```

$$n1sol := \frac{\lambda\,f1}{F2}$$

```
> # Series at optimal
> ssl := x -> series(subsf(x),lambda=0);
> mseries := ssl(msol);
> n1series := ssl(n1sol);
> rseries := ssl(msol-1/2);
> ddseries := ssl(subs(dlambda=dlambdaval, n1=n1sol,m=msol,dd));
> n2series := ssl(1-n1sol);
> m3series:= ssl(msol-n1sol);
> Q2series:= ssl(3*msol-n1sol);
> ssl(c2lambdaval-2);
> ssl(1+etalambdaval-c2lambdaval);
> ssl((1-c2lambdaval)^2/((3*msol-n1sol)*(1+etalambdaval-c2lambdaval)^2));
```

$$ssl := x\mapsto(subsf\,(x))$$

$$mseries := 1/2+1/12\,\lambda+1/36\,\lambda^2+\frac{11}{2160}\,\lambda^3+O\left(\lambda^4\right)$$

$$n1series := 1/2-1/12\,\lambda-1/36\,\lambda^2+\frac{1}{2160}\,\lambda^3+\frac{13}{6480}\,\lambda^4+\frac{71}{272160}\,\lambda^5+O\left(\lambda^6\right)$$

$$rseries := 1/12\,\lambda+1/36\,\lambda^2+\frac{11}{2160}\,\lambda^3+O\left(\lambda^4\right)$$

$$ddseries := -12\,\lambda^{-1} - 2 - 4/5\,\lambda + O\left(\lambda^2\right)$$

$$n2series := 1/2 + 1/12\,\lambda + 1/36\,\lambda^2 - \frac{1}{2160}\,\lambda^3 - \frac{13}{6480}\,\lambda^4 + O\left(\lambda^5\right)$$

$$m3series := 1/6\,\lambda + 1/18\,\lambda^2 + \frac{1}{216}\,\lambda^3 - \frac{7}{3240}\,\lambda^4 - \frac{11}{19440}\,\lambda^5 + O\left(\lambda^6\right)$$

$$Q2series := 1 + 1/3\,\lambda + 1/9\,\lambda^2 + \frac{2}{135}\,\lambda^3 - \frac{1}{405}\,\lambda^4 + O\left(\lambda^5\right)$$

$$1/3\,\lambda + 1/18\,\lambda^2 + \frac{1}{270}\,\lambda^3 + O\left(\lambda^4\right)$$

$$1/6\,\lambda + 1/36\,\lambda^2 - \frac{1}{270}\,\lambda^3 - \frac{7}{6480}\,\lambda^4 + \frac{1}{13608}\,\lambda^5 + O\left(\lambda^6\right)$$

$$36\,\lambda^{-2} + 3/5 + \frac{4}{15}\,\lambda + O\left(\lambda^2\right)$$

```
> # For the third derivative
> series(x/(exp(x)-1)*(exp(x)+x*exp(x)-x*exp(2*x)/(exp(x)-1)),x=0);
```

$$1/2\,x + 1/6\,x^2 - \frac{1}{180}\,x^4 + O\left(x^5\right)$$

A.4 Spreadsheet for Section 4.8.7

```
> h := x -> x*n*ln(x)+ x*n*ln(n)- x*n;
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn$$

```
> # functions of lambda
> eladef := exp(lambda);
> f1def := exp(lambda)-1;
> f2def := exp(lambda)-1-lambda;
> f3def := exp(lambda)-1-lambda-lambda^2/2;
> g1def := exp(lambda)+1;
> g2def := exp(lambda)+2;
> F1def := exp(2*lambda)-1;
> F2def := exp(2*lambda)-1-2*lambda;
> etadef := lambda*f1/f2;
> subdef := x -> subs(eta3=etadef,ela=eladef,f1=f1def,f2=f2def,
f3=f3def,g1=g1def,g2=g2def,F1=F1def,F2=F2def, x);
```

$$eladef := e^{\lambda}$$

$$f1def := e^{\lambda} - 1$$

$$f2def := e^{\lambda} - 1 - \lambda$$

$$f3def := e^{\lambda} - 1 - \lambda - 1/2\,\lambda^2$$

$$g1def := e^{\lambda} + 1$$

$$g2def := e^{\lambda} + 2$$

$$F1def := e^{2\lambda} - 1$$

$$F2def := e^{2\lambda} - 1 - 2\lambda$$

$$etadef := \frac{\lambda\, f1}{f2}$$

$$subdef := x \mapsto x$$

```
> # Relations
> n3def := 1-n1-k0-k1-k2;
> Q3def := 3*m-n1-2*(k0+k1+k2);
> m2def  := n1;
> m2pdef := m2def-k0;
```

```
> m3def  := m-m2def;
> P2def  := 2*m2pdef;
> P3def  := 3*m3def;
> T2def := P2def-k1;
> T3def := 3*m3def-k1-2*k2;
> n2def  := k0+k1+k2;
> subdef2 := x -> subs(n3=n3def,Q3=Q3def,m2=m2def,m3=m3def,m2p=m2pdef,
P2=P2def,P3=P3def, T2=T2def, T3=T3def, n2 =n2def, x);
```

$$n3def := 1 - n1 - k0 - k1 - k2$$
$$Q3def := 3\,m - n1 - 2\,k0 - 2\,k1 - 2\,k2$$
$$m2def := n1$$
$$m2pdef := n1 - k0$$
$$m3def := m - n1$$
$$P2def := 2\,n1 - 2\,k0$$
$$P3def := 3\,m - 3\,n1$$
$$T2def := 2\,n1 - 2\,k0 - k1$$
$$T3def := 3\,m - 3\,n1 - k1 - 2\,k2$$
$$n2def := k0 + k1 + k2$$
$$subdef2 := x \mapsto x$$

```
> # fpre (without lambda part)
> f := (n1,k0,k1,k2) -> expand(subdef2( h(m2)+h(P3)+h(P2)+h(Q3)
> -h(k0)-h(k1)-h(k2)-h(n3)-h(m3)-h(T3)-h(T2)-2*h(m2p)
> -k2*n*ln(2)-m3*n*ln(6)-m2p*n*ln(2))/n):
> # fpre
> fpre := expand(f(n1,k0,k1,k2)+subdef2(n3*log(f3)- Q3*ln(lambda))):
> # Partial derivatives
> fn1 :=  simplify(exp(diff(f(n1,k0,k1,k2), n1))*lambda/f3);
> fk0 :=  simplify(exp(diff(f(n1,k0,k1,k2), k0)))*lambda^2/f3;
> fk1 :=  simplify(exp(diff(f(n1,k0,k1,k2), k1)))*lambda^2/f3;
> fk2 :=  simplify(exp(diff(f(n1,k0,k1,k2), k2)))*lambda^2/f3;
> # Simplifying
> fn1s := (4/9)*T3^3*n3*n1*lambda/(m3^2*Q3*T2^2*f3);
> is(simplify(subdef2(fn1/fn1s),symbolic),1);
```

```
> fk0s := (1/2)*n3*T2^2*lambda^2/(Q3^2*k0*f3);
> is(simplify(subdef2(fk0/fk0s),symbolic),1);
> fk1s := T3*n3*T2*lambda^2/(k1*Q3^2*f3);
> is(simplify(subdef2(fk1/fk1s),symbolic),1);
> fk2s := (1/2)*T3^2*n3*lambda^2/(k2*Q3^2*f3);
> is(simplify(subdef2(fk2/fk2s),symbolic),1);
```

$$fn1 := -4/9\,\frac{n1\,(-3\,m+3\,n1+k1+2\,k2)^3\,(-1+n1+k0+k1+k2)\,\lambda}{(-3\,m+n1+2\,k0+2\,k1+2\,k2)\,(m-n1)^2\,(-2\,n1+2\,k0+k1)^2\,f3}$$

$$fk0 := -1/2\,\frac{(-2\,n1+2\,k0+k1)^2\,(-1+n1+k0+k1+k2)\,\lambda^2}{k0\,(-3\,m+n1+2\,k0+2\,k1+2\,k2)^2\,f3}$$

$$fk1 := -\frac{(-2\,n1+2\,k0+k1)\,(-3\,m+3\,n1+k1+2\,k2)\,(-1+n1+k0+k1+k2)\,\lambda^2}{k1\,(-3\,m+n1+2\,k0+2\,k1+2\,k2)^2\,f3}$$

$$fk2 := -1/2\,\frac{(-3\,m+3\,n1+k1+2\,k2)^2\,(-1+n1+k0+k1+k2)\,\lambda^2}{k2\,(-3\,m+n1+2\,k0+2\,k1+2\,k2)^2\,f3}$$

$$fn1s := 4/9\,\frac{T3^3 n3\,n1\,\lambda}{m3^2\,Q3\,T2^2 f3}$$

true

$$fk0s := 1/2\,\frac{n3\,T2^2\lambda^2}{Q3^2 k0\,f3}$$

true

$$fk1s := \frac{T3\,n3\,T2\,\lambda^2}{k1\,Q3^2\,f3}$$

true

$$fk2s := 1/2\,\frac{T3^2 n3\,\lambda^2}{k2\,Q3^2 f3}$$

true

```
> # COMPUTATIONS FOR HESSIAN
> dln1n1 := (-1)*(c3-1)*(c3-1)/((1+eta3-c3)*Q3);
> dlkn1 :=(-1)*(c3-2)*(c3-1)/((1+eta3-c3)*Q3);
> dlkk := (-1)*(c3-2)*(c3-2)/((1+eta3-c3)*Q3);
> dd := (x,y) -> expand(diff(diff(f(n1,k0,k1,k2), x),y));
```

$$dln1n1 := -\frac{(c3-1)^2}{Q3\,(1+eta3-c3)}$$

$$dlkn1 := -\frac{(c3-2)(c3-1)}{Q3\ (1+eta3-c3)}$$

$$dlkk := -\frac{(c3-2)^2}{Q3\ (1+eta3-c3)}$$

$$dd := (x, y) \mapsto 0$$

```
> # n1
> dn1n1 := dd(n1,n1)+dln1n1:
> dn1k0 := dd(n1,k0)+dlkn1:
> dn1k1 := dd(n1,k1)+dlkn1:
> dn1k2 := dd(n1,k2)+dlkn1:
> dn1n1s := 9/P3 + 4/P2 - 9/T3 + 1/Q3 -1/n3 - 1/m3
> - 4/T2- 2/m2p +1/n1 +dln1n1;
> is(simplify(subdef2(dn1n1s-dn1n1)),0);
> dn1k0s := -4/P2 + 2/Q3 - 1/n3 + 4/T2 + 2/m2p + dlkn1;
> is(simplify(subdef2(dn1k0s-dn1k0)),0);
> dn1k1s := -3/T3 + 2/Q3 - 1/n3 + 2/T2 + dlkn1;
> is(simplify(subdef2(dn1k1s-dn1k1)),0);
> dn1k2s := - 6/T3 + 2/Q3 - 1/n3 + dlkn1;
> is(simplify(subdef2(dn1k2s-dn1k2)),0);
```

$$dn1n1s := 9\,P3^{-1} + 4\,P2^{-1} - 9\,T3^{-1} + Q3^{-1} - n3^{-1} - m3^{-1} - 4\,T2^{-1} - 2\,m2p^{-1} + n1^{-1} - \frac{(c3-1)^2}{Q3\ (1+eta3-c3)}$$

$$true$$

$$dn1k0s := -4\,P2^{-1} + 2\,Q3^{-1} - n3^{-1} + 4\,T2^{-1} + 2\,m2p^{-1} - \frac{(c3-2)(c3-1)}{Q3\ (1+eta3-c3)}$$

$$true$$

$$dn1k1s := -3\,T3^{-1} + 2\,Q3^{-1} - n3^{-1} + 2\,T2^{-1} - \frac{(c3-2)(c3-1)}{Q3\ (1+eta3-c3)}$$

$$true$$

$$dn1k2s := -6\,T3^{-1} + 2\,Q3^{-1} - n3^{-1} - \frac{(c3-2)(c3-1)}{Q3\ (1+eta3-c3)}$$

$$true$$

```
> # k0
> dk0n1 := dd(k0,n1)+dlkn1:
> dk0k0 := dd(k0,k0)+dlkk:
> dk0k1 := dd(k0,k1)+dlkk:
> dk0k2 := dd(k0,k2)+dlkk:
> dk0k0s := 4/P2 + 4/Q3 - 1/n3 - 4/T2 - 2/m2p -1/k0 + dlkk;
> is(simplify(subdef2(dk0k0s-dk0k0)),0);
> dk0k1s := 4/Q3 - 1/n3 -2/T2 + dlkk;
> is(simplify(subdef2(dk0k1s-dk0k1)),0);
> dk0k2s := 4/Q3 - 1/n3 + dlkk;
> is(simplify(subdef2(dk0k2s-dk0k2)),0);
```

$$dk0k0s := 4\,P2^{-1} + 4\,Q3^{-1} - n3^{-1} - 4\,T2^{-1} - 2\,m2p^{-1} - k0^{-1} - \frac{(c3-2)^2}{Q3\,(1+eta3-c3)}$$

$$true$$

$$dk0k1s := 4\,Q3^{-1} - n3^{-1} - 2\,T2^{-1} - \frac{(c3-2)^2}{Q3\,(1+eta3-c3)}$$

$$true$$

$$dk0k2s := 4\,Q3^{-1} - n3^{-1} - \frac{(c3-2)^2}{Q3\,(1+eta3-c3)}$$

$$true$$

```
> # k1
> dk1n1 := dd(k1,n1)+dlkn1:
> dk1k0 := dd(k1,k0)+dlkk:
> dk1k1 := dd(k1,k1)+dlkk:
> dk1k2 := dd(k1,k2)+dlkk:
> dk1k1s := - 1/k1 - 1/T3 + 4/Q3 - 1/n3 - 1/T2 + dlkk;
> is(simplify(subdef2(dk1k1s-dk1k1)),0);
> dk1k2s := - 2/T3 + 4/Q3 - 1/n3 + dlkk;
> is(simplify(subdef2(dk1k2s-dk1k2)),0);
```

$$dk1k1s := -k1^{-1} - T3^{-1} + 4\,Q3^{-1} - n3^{-1} - T2^{-1} - \frac{(c3-2)^2}{Q3\,(1+eta3-c3)}$$

$$true$$

$$dk1k2s := -2\,T3^{-1} + 4\,Q3^{-1} - n3^{-1} - \frac{(c3-2)^2}{Q3\,(1+eta3-c3)}$$

$$true$$

```
> # k2
> dk2n1 := dd(k2,n1)+dlkn1:
> dk2k0 := dd(k2,k0)+dlkk:
> dk2k1 := dd(k2,k1)+dlkk:
> dk2k2 := dd(k2,k2)+dlkk:
> dk2k2s := -1/k2 - 4/T3 + 4/Q3 - 1/n3 + dlkk;
> is(simplify(subdef2(dk2k2s-dk2k2)),0);
```

$$dk2k2s := -k2^{-1} - 4\,T3^{-1} + 4\,Q3^{-1} - n3^{-1} - \frac{(c3-2)^2}{Q3\,(1+eta3-c3)}$$

$$true$$

```
> msol := (1/3)*lambda*f1*g2/F2;
> n1sol := 3*m/g2;
> k0sol := 6*m*lambda/(g2*f1*g1);
> k1sol := 6*m*lambda/(g2*g1);
> k2sol := 3*m*lambda*f1/(2*g2*g1);
```

$$msol := 1/3\,\frac{\lambda\,f1\;g2}{F2}$$

$$n1sol := 3\,\frac{m}{g2}$$

$$k0sol := 6\,\frac{m\lambda}{g2\;f1\;g1}$$

$$k1sol := 6\,\frac{m\lambda}{g2\;g1}$$

$$k2sol := 3/2\,\frac{m\lambda\,f1}{g2\;g1}$$

```
> # For lambda at maximum
> ddl := x -> factor(subdef(subs(c3=lambda*f2/f3, k2=k2sol, k1=k1sol,
k0=k0sol, n1=n1sol, m=msol, subdef2(x)))):
```

```
> dn1n1la := ddl(dn1n1):
> dn1k0la := ddl(dn1k0):
> dn1k1la := ddl(dn1k1):
> dn1k2la := ddl(dn1k2):
> dk0n1la := ddl(dk0n1):
> dk0k0la := ddl(dk0k0):
> dk0k1la := ddl(dk0k1):
> dk0k2la := ddl(dk0k2):
> dk1n1la := ddl(dk1n1):
> dk1k0la := ddl(dk1k0):
> dk1k1la := ddl(dk1k1):
> dk1k2la := ddl(dk1k2):
> dk2n1la := ddl(dk2n1):
> dk2k0la := ddl(dk2k0):
> dk2k1la := ddl(dk2k1):
> dk2k2la := ddl(dk2k2):
> # Computing series for entries of Hessian as lambda -> 0 (at maximum)
> ss := x -> series(x,lambda=0,18);
> ss(dn1n1la);
> ss(dn1k0la);
> ss(dn1k1la);
> ss(dn1k2la);
> ss(dk0k0la);
> ss(dk0k1la);
> ss(dk0k2la);
> ss(dk1k1la);
> ss(dk1k2la);
> ss(dk2k2la);
```

$$ss := x \mapsto (x)$$

$$-132\,\lambda^{-2} + \frac{94}{5}\,\lambda^{-1} - \frac{176}{25} - \frac{3677}{5250}\,\lambda + O\left(\lambda^2\right)$$

$$-48\,\lambda^{-2}+\frac{32}{5}\,\lambda^{-1}+\frac{72}{25}+\frac{624}{875}\,\lambda+\frac{669}{4375}\,\lambda^{2}+O\left(\lambda^{3}\right)$$

$$-60\,\lambda^{-2}+\frac{22}{5}\,\lambda^{-1}+\frac{22}{25}+\frac{769}{5250}\,\lambda-\frac{1291}{157500}\,\lambda^{2}+O\left(\lambda^{3}\right)$$

$$-72\,\lambda^{-2}+\frac{12}{5}\,\lambda^{-1}-\frac{28}{25}-\frac{1103}{2625}\,\lambda-\frac{13333}{78750}\,\lambda^{2}+O\left(\lambda^{3}\right)$$

$$-24\,\lambda^{-2}-\frac{44}{5}\,\lambda^{-1}-\frac{134}{25}-\frac{9634}{2625}\,\lambda+O\left(\lambda^{2}\right)$$

$$-24\,\lambda^{-2}-\frac{24}{5}\,\lambda^{-1}-\frac{152}{75}-\frac{6152}{7875}\,\lambda-\frac{55747}{236250}\,\lambda^{2}+O\left(\lambda^{3}\right)$$

$$-24\,\lambda^{-2}-4/5\,\lambda^{-1}-\frac{52}{75}-\frac{1777}{7875}\,\lambda-\frac{15497}{236250}\,\lambda^{2}+\frac{22583}{7087500}\,\lambda^{3}+O\left(\lambda^{4}\right)$$

$$-28\,\lambda^{-2}-\frac{62}{15}\,\lambda^{-1}-\frac{656}{225}-\frac{63337}{47250}\,\lambda+O\left(\lambda^{2}\right)$$

$$-32\,\lambda^{-2}+\frac{8}{15}\,\lambda^{-1}-\frac{256}{225}-\frac{5506}{23625}\,\lambda-\frac{21058}{354375}\,\lambda^{2}+O\left(\lambda^{3}\right)$$

$$-48\,\lambda^{-2}+\frac{8}{15}\,\lambda^{-1}+O\left(1\right)$$

```
> # THIRD DERIVATIVE COMPUTATIONS
> ddd := (x,y,z) -> expand(diff(diff(diff(f(n1,k0,k1,k2), x),y),z));
> subss := (x,k) -> subdef(subs(c3=Q3/n3, Q3=Q3def, n3=n3def,n1=n1sol,
k0=k0sol, k1=k1sol,k2=k2sol,m=msol,x));
> sss := (x,k) -> series(subss(x),lambda=0,k);
```

$$ddd := (x,y,z)\mapsto 0$$

$$subss := (x,k)\mapsto x$$

```
> # NON LABMDA PART
> dn1n1n1 := ddd(n1,n1,n1):
> dn1n1k0 := ddd(n1,n1,k0):
> dn1n1k1 := ddd(n1,n1,k1):
> dn1n1k2 := ddd(n1,n1,k2):
> dn1k0k0 := ddd(n1,k0,k0):
> dn1k0k1 := ddd(n1,k0,k1):
> dn1k0k2 := ddd(n1,k0,k2):
> dn1k1k1 := ddd(n1,k1,k1):
```

```
> dn1k1k2 := ddd(n1,k1,k2):
> dn1k2k2 := ddd(n1,k2,k2):
> dk0k0k0 := ddd(k0,k0,k0):
> dk0k0k1 := ddd(k0,k0,k1):
> dk0k0k2 := ddd(k0,k0,k2):
> dk0k1k1 := ddd(k0,k1,k1):
> dk0k1k2 := ddd(k0,k1,k2):
> dk0k2k2 := ddd(k0,k2,k2):
> dk1k1k1 := ddd(k1,k1,k1):
> dk1k1k2 := ddd(k1,k1,k2):
> dk1k2k2 := ddd(k1,k2,k2):
> dk2k2k2 := ddd(k2,k2,k2):
> # Putting all terms together
> tder := (n1,k0,k1,k2) ->
> dn1n1n1*n1*n1*n1+ 3*dn1n1k0*n1*n1*k0+ 3*dn1n1k1*n1*n1*k1+
> 3*dn1n1k2*n1*n1*k2+ 3*dn1k0k0*n1*k0*k0+ 6*dn1k0k1*n1*k0*k1+
> 6*dn1k0k2*n1*k0*k2+ 3*dn1k1k1*n1*k1*k1+ 6*dn1k1k2*n1*k1*k2+
> 3*dn1k2k2*n1*k2*k2+ dk0k0k0*k0*k0*k0+ 3*dk0k0k1*k0*k0*k1+
> 3*dk0k0k2*k0*k0*k2+ 3*dk0k1k1*k0*k1*k1 + 6*dk0k1k2*k0*k1*k2+
> 3*dk0k2k2*k0*k2*k2+ dk1k1k1*k1*k1*k1+ 3*dk1k1k2*k1*k1*k2+
> 3*dk1k2k2*k1*k2*k2+ dk2k2k2*k2*k2*k2:
> # at a general term
> tder(a,a+b2,-3*a+b3,b4):
> s:= expand(%):
> #terms in s without a
> noa := subs(a=0,s):
```

```
> s1 := expand((s-noa)/a):
> # terms in s with one a
> onea:= subs(a=0,%):
> s2 := expand((s1-onea)/a):
> # terms in s with 2 a's
> twoa := subs(a=0,%):
> s3 := expand((s2-twoa)/a):
> # terms in s with 3 a's
> threea := subs(a=0,%):
> is(simplify(noa+a*onea+a^2*twoa+a^3*threea-s),0);
```

$$true$$

```
> # Three a's
> sss(threea,6);
```

$$-90\,\lambda^{-1}+O\left(1\right)$$

```
> # Two a's
> subs(b2=0,b3=0,twoa/b4):  expand(%):
> sss(%,3);
> subs(b2=0,b4=0,twoa/b3):expand(%):
> sss(%,3);
> subs(b3=0,b4=0,twoa/b2):expand(%):
> sss(%,3);
```

$$108\,\lambda^{-2}-54\,\lambda^{-1}-\frac{9}{20}+O\left(\lambda\right)$$

$$108\,\lambda^{-2}+18\,\lambda^{-1}+O\left(1\right)$$

$$-108\,\lambda^{-2}-198\,\lambda^{-1}-\frac{1449}{20}+O\left(\lambda\right)$$

```
> # One a
> subs(b2=0,b3=0,onea/b4^2):
> b42:= expand(%):
> sss(%,5);
> subs(b2=0,b4=0,onea/b3^2):
> b32:= expand(%):
> sss(%,3);
> subs(b3=0,b4=0,onea/b2^2):
> b22:=expand(%):
> sss(%,3);
```

$$-36\,\lambda^{-2} + 30\,\lambda^{-1} - \frac{31}{20} + O\left(\lambda\right)$$

$$-36\,\lambda^{-2} + 6\,\lambda^{-1} + O\left(1\right)$$

$$108\,\lambda^{-2} + 126\,\lambda^{-1} + \frac{1329}{20} + O\left(\lambda\right)$$

```
> # One a -- continuation
> subs(b2=0,onea-b42*b4^2-b32*b3^2):
> expand(%/(b4*b3)):
> sss(%,3);
> subs(b3=0,onea-b42*b4^2-b22*b2^2):
> expand(%/(b4*b2)):
> sss(%,3);
> subs(b4=0,onea-b32*b3^2-b22*b2^2):
> expand(%/(b2*b3)):
> sss(%,3);
```

$$-72\,\lambda^{-2} + 60\,\lambda^{-1} - \frac{31}{10} + O\left(\lambda\right)$$

$$-72\,\lambda^{-2} + 60\,\lambda^{-1} - \frac{31}{10} + O\left(\lambda\right)$$

$$72\,\lambda^{-2} + 156\,\lambda^{-1} + \frac{529}{10} + O\left(\lambda\right)$$

```
> # THIRD DERIVATIVE LAMDA PART
> ddd1:= -((2*c3-a-b)*(dQ3+c3)/n3-dQ3*(c3-a)*(c3-b)/Q3)/(Q3*(1+eta3-c3));
```

$$ddd1 := -\left(\frac{(2\,c3 - a - b)\,(dQ3 + c3)}{n3} - \frac{dQ3\ (c3 - a)\,(c3 - b)}{Q3}\right) Q3^{-1}\,(1 + eta3 - c3)^{-1}$$

```
> ddd2 := (dQ3+c3)/n3;
```

$$ddd2 := \frac{dQ3 + c3}{n3}$$

```
> dl := lambda*((c3+dQ3)/(Q3*(1+eta3-c3)));
```

$$dl := \frac{\lambda\ (dQ3 + c3)}{Q3\ (1 + eta3 - c3)}$$

```
> deta := dl*(1/lambda)*(eta3*(1+lambda*ela/f1-eta3));
```

$$deta := (dQ3 + c3)\,eta3\ \left(1 + \frac{\lambda\,ela}{f1} - eta3\right) Q3^{-1}\,(1 + eta3 - c3)^{-1}$$

```
> dddlambda := ddd1 + (ddd2+deta)*(c3-a)*(c3-b)/(Q3*(1+eta3-c3)^2);
```

$$dddlambda := -\left(\frac{(2\,c3 - a - b)\,(dQ3 + c3)}{n3} - \frac{dQ3\ (c3 - a)\,(c3 - b)}{Q3}\right) Q3^{-1}\,(1 + eta3 - c3)^{-1}$$
$$+\left(\frac{dQ3 + c3}{n3} + (dQ3 + c3)\,eta3\ \left(1 + \frac{\lambda\,ela}{f1} - eta3\right) Q3^{-1}\,(1 + eta3 - c3)^{-1}\right)\cdot$$
$$\cdot\,(c3 - a)\,(c3 - b)\,Q3^{-1}\,(1 + eta3 - c3)^{-2}$$

```
> dln1n1n1 := subs(a=1,b=1,dQ3=-1, dddlambda):
> dln1n1k := subs(a=1,b=1,dQ3=-2, dddlambda):
> dln1kk := subs(a=1,b=2,dQ3=-2, dddlambda):
> dlkkk := subs(a=2,b=2,dQ3=-2, dddlambda):
> tderlambda := (n1,k0,k1,k2) ->
> d111*n1*n1*n1 + 3*d11k*n1*n1*k0 + 3*d11k*n1*n1*k1 + 3*d11k*n1*n1*k2+
> 3*d1kk*n1*k0*k0 + 6*d1kk*n1*k0*k1 + 6*d1kk*n1*k0*k2 + 3*d1kk*n1*k1*k1+
> 6*d1kk*n1*k1*k2 + 3*d1kk*n1*k2*k2 + dkkk*k0*k0*k0 + 3*dkkk*k0*k0*k1+
> 3*dkkk*k0*k0*k2 + 3*dkkk*k0*k1*k1 + 6*dkkk*k0*k1*k2 + 3*dkkk*k0*k2*k2 +
> dkkk*k1*k1*k1 + 3*dkkk*k1*k1*k2 + 3*dkkk*k1*k2*k2 + dkkk*k2*k2*k2;
```

$$tderlambda := (n1, k0, k1, k2) \mapsto d111\ n1^3 + 3\ d11k\ n1^2 k0 + 3\ d11k\ n1^2 k1 + 3\ d11k\ n1^2 k2$$
$$+3\ d1kk\ n1\ k0^2 + 6\ d1kk\ n1\ k0\ k1 + 6\ d1kk\ n1\ k0\ k2 + 3\ d1kk\ n1\ k1^2$$
$$+6\ d1kk\ n1\ k1\ k2 + 3\ d1kk\ n1\ k2^2 + dkkk\ k0^3 + 3\ dkkk\ k0^2 k1$$
$$+3\ dkkk\ k0^2 k2 + 3\ dkkk\ k0\ k1^2 + 6\ dkkk\ k0\ k1\ k2 + 3\ dkkk\ k0\ k2^2$$
$$+dkkk\ k1^3 + 3\ dkkk\ k1^2 k2 + 3\ dkkk\ k1\ k2^2 + dkkk\ k2^3$$

```
> subsdddl := x->subs(d111=dln1n1n1,d11k=dln1n1k,d1kk=dln1kk,dkkk=dlkkk, x):
```

```
> # At general terminal
> tderlambda(a,a+b2,-3*a+b3,b4):
> sl:= expand(%):
> noal := subs(a=0,sl):
> sa:= expand(sl-noal):
> oneal:= subs(a=0, expand(sa/a)):
> saa := expand(sa-a*oneal):
> twoal:= subs(a=0, expand(saa/a^2)):
> saaa := expand(saa-a^2*twoal):
> threeal:= subs(a=0, expand(saaa/a^3)):
> is(simplify(noal+oneal*a+twoal*a^2+threeal*a^3-sl),0);
```

$$true$$

```
> # two a's
> subs(b2=0,b3=0,twoal):
> expand(%/b4):
> subsdddl(%):
> sss(%,9);
> subs(b2=0,b4=0,twoal):
> expand(%/b3):
> subsdddl(%):
> sss(%,9);
> subs(b3=0,b4=0,twoal):
> expand(%/b2):
> subsdddl(%):
> sss(%,9);
```

$$540\,\lambda^{-2} - 90\,\lambda^{-1} - \frac{1101}{100} + O\left(\lambda\right)$$

$$540\,\lambda^{-2} - 90\,\lambda^{-1} - \frac{1101}{100} + O\left(\lambda\right)$$

$$540\,\lambda^{-2} - 90\,\lambda^{-1} - \frac{1101}{100} + O\left(\lambda\right)$$

```
> # One a
> subs(b2=0,b3=0,oneal):
> b42 := expand(%/b4^2):
> subsdddl(%):
> sss(%,9);
> subs(b2=0,b4=0,oneal):
> b32 := expand(%/b3^2):
> subsdddl(%):
> sss(%,9);
> subs(b3=0,b4=0,oneal):
> b22 := expand(%/b2^2):
> subsdddl(%):
> sss(%,9);
> onea2:= expand(oneal-b22*b2^2-b32*b3^2-b42*b4^2);
> subs(b2=0,onea2):
> expand(%/(b3*b4)):
> subsdddl(%):
> sss(%,9);
> subs(b3=0,onea2):
> expand(%/(b2*b4)):
> subsdddl(%):
> sss(%,9);
> subs(b4=0,onea2):
> expand(%/(b3*b2)):
> subsdddl(%):
> sss(%,9);
```

$$-2592\,\lambda^{-3}+\frac{1188}{5}\,\lambda^{-2}+\frac{1764}{25}\,\lambda^{-1}-\frac{206847}{3500}+O\left(\lambda\right)$$

$$-2592\,\lambda^{-3}+\frac{1188}{5}\,\lambda^{-2}+\frac{1764}{25}\,\lambda^{-1}-\frac{206847}{3500}+O\left(\lambda\right)$$

$$-2592\,\lambda^{-3}+\frac{1188}{5}\,\lambda^{-2}+\frac{1764}{25}\,\lambda^{-1}-\frac{206847}{3500}+O\left(\lambda\right)$$

$$onea2 := 6\,d1kk\;b2\;b3 + 6\,d1kk\;b4\;b2 + 6\,d1kk\;b4\;b3 - 12\,dkkk\;b2\;b3 - 12\,dkkk\;b4\;b2 - 12\,dkkk\;b4\;b3$$

$$-5184\,\lambda^{-3} + \frac{2376}{5}\,\lambda^{-2} + \frac{3528}{25}\,\lambda^{-1} - \frac{206847}{1750} + O(\lambda)$$

$$-5184\,\lambda^{-3} + \frac{2376}{5}\,\lambda^{-2} + \frac{3528}{25}\,\lambda^{-1} - \frac{206847}{1750} + O(\lambda)$$

$$-5184\,\lambda^{-3} + \frac{2376}{5}\,\lambda^{-2} + \frac{3528}{25}\,\lambda^{-1} - \frac{206847}{1750} + O(\lambda)$$

```
> # No a's
> subs(b2=0,b3=0,noal):
> b43 := expand(%/b4^3):
> subsdddl(%):
> sss(%,9);
> subs(b2=0,b4=0,noal):
> b33:= expand(%/b3^3):
> subsdddl(%):
> sss(%,9);
> subs(b3=0,b4=0,noal):
> b23:= expand(%/b2^3):
> subsdddl(%):
> sss(%,9);
```

$$4032\,\lambda^{-4} - \frac{192}{5}\,\lambda^{-3} - \frac{2852}{25}\,\lambda^{-2} + \frac{70554}{875}\,\lambda^{-1} + \frac{16217}{500} + O(\lambda)$$

$$4032\,\lambda^{-4} - \frac{192}{5}\,\lambda^{-3} - \frac{2852}{25}\,\lambda^{-2} + \frac{70554}{875}\,\lambda^{-1} + \frac{16217}{500} + O(\lambda)$$

$$4032\,\lambda^{-4} - \frac{192}{5}\,\lambda^{-3} - \frac{2852}{25}\,\lambda^{-2} + \frac{70554}{875}\,\lambda^{-1} + \frac{16217}{500} + O(\lambda)$$

A.5 Spreadsheet for Section 4.8.8

```
> h := x -> x*n*ln(x)+ x*n*ln(n)- x*n;
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn$$

```
> eladef := exp(lambda);
> f1def := exp(lambda)-1;
> f2def := exp(lambda)-1-lambda;
> f3def := exp(lambda)-1-lambda-lambda^2/2;
> g1def := exp(lambda)+1;
> g2def := exp(lambda)+2;
> F1def := exp(2*lambda)-1;
> F2def := exp(2*lambda)-1-2*lambda;
> etadef := lambda*f1/f2;
> subdef := x -> subs(eta3=etadef,ela=eladef,f1=f1def,f2=f2def,
f3=f3def,g1=g1def,g2=g2def,F1=F1def,F2=F2def, x);
```

$$eladef := e^{\lambda}$$

$$f1def := e^{\lambda} - 1$$

$$f2def := e^{\lambda} - 1 - \lambda$$

$$f3def := e^{\lambda} - 1 - \lambda - 1/2\,\lambda^2$$

$$g1def := e^{\lambda} + 1$$

$$g2def := e^{\lambda} + 2$$

$$F1def := e^{2\lambda} - 1$$

$$F2def := e^{2\lambda} - 1 - 2\lambda$$

$$etadef := \frac{f1\,\lambda}{f2}$$

$$subdef := x \mapsto x$$

```
> # Relations
> n3def := 1-n1-k0-k1-k2;
> Q3def := 3*m-n1-2*(k0+k1+k2);
> m2def  := n1;
> m2pdef := m2def-k0;
> m3def  := m-m2def;
> P2def  := 2*m2pdef;
> P3def  := 3*m3def;
> T2def := P2def-k1;
> T3def := 3*m3def-k1-2*k2;
> n2def  := k0+k1+k2;
> subdef2 := x -> subs(n3=n3def,Q3=Q3def,m2=m2def,m3=m3def,m2p=m2pdef,
P2=P2def,P3=P3def, T2=T2def, T3=T3def, n2 =n2def, x);
```

$$n3def := 1 - n1 - k0 - k1 - k2$$

$$Q3def := 3\,m - n1 - 2\,k0 - 2\,k1 - 2\,k2$$

$$m2def := n1$$

$$m2pdef := n1 - k0$$

$$m3def := m - n1$$

$$P2def := 2\,n1 - 2\,k0$$

$$P3def := 3\,m - 3\,n1$$

$$T2def := 2\,n1 - 2\,k0 - k1$$

$$T3def := 3\,m - 3\,n1 - k1 - 2\,k2$$

$$n2def := k0 + k1 + k2$$

$$subdef2 := x \mapsto x$$

```
> # fpre (without lambda part)
> f := (n1,k0,k1,k2) -> expand(subdef2(
> h(m2)+h(P3)+h(P2)+h(Q3)
> -h(k0)-h(k1)-h(k2)-h(n3)-h(m3)-h(T3)-h(T2)-2*h(m2p)
> -k2*n*ln(2)-m3*n*ln(6)-m2p*n*ln(2))/n):
> # fpre
> fpre := expand(f(n1,k0,k1,k2)+subdef2(n3*log(f3)- Q3*ln(lambda))):
```

```
> fn1 := (4/9)*T3^3*n3*n1*lambda/(m3^2*Q3*T2^2*f3);
> fk0 := (1/2)*n3*T2^2*lambda^2/(Q3^2*k0*f3);
> fk1 := T3*n3*T2*lambda^2/(k1*Q3^2*f3);
> fk2 := (1/2)*T3^2*n3*lambda^2/(k2*Q3^2*f3);
```

$$fn1 := 4/9\,\frac{T3^3 n3\; n1\,\lambda}{m3^2 Q3\, T2^2 f3}$$

$$fk0 := 1/2\,\frac{n3\; T2^2\lambda^2}{Q3^2 k0\, f3}$$

$$fk1 := \frac{T3\; n3\; T2\,\lambda^2}{k1\; Q3^2\; f3}$$

$$fk2 := 1/2\,\frac{T3^2 n3\,\lambda^2}{k2\; Q3^2 f3}$$

```
> eqn1 := numer(fn1)-denom(fn1);
> eqk0 := numer(fk0)-denom(fk0);
> eqk1 := numer(fk1)-denom(fk1);
> eqk2 := numer(fk2)-denom(fk2);
> eqn1 := subdef2(eqn1):
> eqk0 := subdef2(eqk0):
> eqk1 := subdef2(eqk1):
> eqk2 := subdef2(eqk2):
```

$$eqn1 := 4\; T3^3 n3\; n1\,\lambda - 9\, m3^2 Q3\; T2^2 f3$$

$$eqk0 := n3\; T2^2\lambda^2 - 2\; Q3^2 k0\, f3$$

$$eqk1 := T3\; n3\; T2\,\lambda^2 - k1\; Q3^2 f3$$

$$eqk2 := T3^2 n3\,\lambda^2 - 2\,k2\; Q3^2 f3$$

```
> # Now I will start taking resultants of the equations above
> resultant(eqk0,eqk1,f3):
> eq1 := factor(%);
```

$$eq1 := \lambda^2\,(-2\,n1 + k1 + 2\,k0)\,(-1 + n1 + k0 + k1 + k2)\,\left(6\,n1\;k0 + 2\,n1\;k1 - k1^2 - 6\,mk0 + 4\,k0\;k2\right)(n1 + 2\,k0 + 2\,k1 + 2\,k2 - 3\,m)^2$$

```
> # Discarding all terms except the fourth
> eq1s := op(4, eq1);
> k2sola := solve(eq1s=0,k2);
```

$$eq1s := 6\,n1\,k0 + 2\,n1\,k1 - k1^2 - 6\,mk0 + 4\,k0\,k2$$

$$k2sola := 1/4\,\frac{-6\,n1\,k0 - 2\,n1\,k1 + k1^2 + 6\,mk0}{k0}$$

```
> resultant(eqk1,eqk2, lambda):
> eq2 := factor(%);
```

$$eq2 := f3^2\,(3\,n1 + k1 - 3\,m + 2\,k2)^2$$
$$(-1 + n1 + k0 + k1 + k2)^2\,\left(4\,k0\,k2 + 3\,mk1 - 3\,n1\,k1 - k1^2 - 4\,n1\,k2\right)^2$$
$$(n1 + 2\,k0 + 2\,k1 + 2\,k2 - 3\,m)^4$$

```
> eq2s := op(1,op(4, eq2));
```

$$eq2s := 4\,k0\,k2 + 3\,mk1 - 3\,n1\,k1 - k1^2 - 4\,n1\,k2$$

```
> eq3 := factor(resultant(eq1s,eq2s,k2));
```

$$eq3 := 4\,(-2\,n1 + k1 + 2\,k0)\,(3\,mk0 - 3\,n1\,k0 - n1\,k1)$$

```
> eq3s := op(3,eq3);
> k1sola := solve(eq3s, k1);
```

$$eq3s := 3\,mk0 - 3\,n1\,k0 - n1\,k1$$

$$k1sola := 3\,\frac{(m - n1)\,k0}{n1}$$

```
> eq4 := factor(resultant(eqn1,eq1s,k2));
```

$$eq4 := 32\,(-2\,n1 + k1 + 2\,k0)^3\,(-4\,\lambda\,k1^3 n1\,k0 - 2\,\lambda\,n1^2 k1^4 + \lambda\,n1\,k1^5$$
$$-2\,\lambda\,k1^3 n1^2 k0 + 6\,\lambda\,mn1\,k1^3 k0 + 4\,\lambda\,k1^4 n1\,k0 + 4\,\lambda\,n1\,k0^2 k1^3 + 36\,k0^3 f3\,k1\,n1^2$$
$$- 72\,k0^3 f3\,k1\,mn1 + 36\,k0^3 f3\,k1\,m^2 + 72\,f3\,n1^2 k0^4 - 144\,f3\,n1\,k0^4 m + 72\,f3\,m^2 k0^4)$$

```
> eq4s := op(3,eq4);
```

$$eq4s := -4\,\lambda\,k1^3 n1\,k0 - 2\,\lambda\,n1^2 k1^4 + \lambda\,n1\,k1^5 - 2\,\lambda\,k1^3 n1^2 k0 + 6\,\lambda\,mn1\,k1^3 k0$$
$$+ 4\,\lambda\,k1^4 n1\,k0 + 4\,\lambda\,n1\,k0^2 k1^3 + 36\,k0^3 f3\,k1\,n1^2 - 72\,k0^3 f3\,k1\,mn1$$
$$+36\,k0^3 f3\,k1\,m^2 + 72\,f3\,n1^2 k0^4 - 144\,f3\,n1\,k0^4 m + 72\,f3\,m^2 k0^4$$

```
> eq5 := factor(resultant(eq4s,eq3s,k1));
```

$$eq5 := 9\,n1\,k0^4\,(m - n1)^2\,(27\,k0\,\lambda\,m^3 - 45\,k0\,\lambda\,n1\,m^2 + 21\,k0\,\lambda\,mn1^2$$
$$-3\,\lambda\,n1^3 k0 + 12\,mn1^3 f3 + 12\,n1^3\lambda\,m - 12\,\lambda\,mn1^2 - 4\,f3\,n1^4$$
$$- 12\,n1^4\lambda + 12\,\lambda\,n1^3)$$

```
> eq5s := op(5,eq5);
```

$$eq5s := 27\,k0\,\lambda\,m^3 - 45\,k0\,\lambda\,n1\,m^2 + 21\,k0\,\lambda\,mn1^2 - 3\,\lambda\,n1^3k0 + 12\,mn1^3f3 + 12\,n1^3\lambda\,m - 12\,\lambda\,mn1^2 - 4\,f3\,n1^4 - 12\,n1^4\lambda + 12\,\lambda\,n1^3$$

```
> eq6 := factor(resultant(eqk0,eq1s,k2));
```

$$eq6 := -4\,k0\,(-2\,n1 + k1 + 2\,k0)^2\,(8\,k0^2f3 + 4\,\lambda^2k0^2 - 4\,k0\,\lambda^2 - 2\,\lambda^2n1\,k0 + 6\,\lambda^2mk0 + 4\,\lambda^2k0\,k1 + 8\,k1\,f3\,k0 + 2\,f3\,k1^2 - 2\,\lambda^2n1\,k1 + \lambda^2k1^2)$$

```
> eq6s:=op(4,eq6);
```

$$eq6s := 8\,k0^2f3 + 4\,\lambda^2k0^2 - 4\,k0\,\lambda^2 - 2\,\lambda^2n1\,k0 + 6\,\lambda^2mk0 + 4\,\lambda^2k0\,k1 + 8\,k1\,f3\,k0 + 2\,f3\,k1^2 - 2\,\lambda^2n1\,k1 + \lambda^2k1^2$$

```
> eq7 := factor(resultant(eq6s,eq3s,k1));
```

$$eq7 := k0\,(\,k0\,f3\,n1^2 + \lambda^2n1^2k0 - 6\,\lambda^2mn1\,k0 - 12\,k0\,f3\,mn1 + 18\,k0\,f3\,m^2 + 9\,\lambda^2k0\,m^2 - 4\,\lambda^2n1^2 + 4\,\lambda^2n1^3)$$

```
> eq7s:=op(2,eq7);
```

$$eq7s := 2\,k0\,f3\,n1^2 + \lambda^2n1^2k0 - 6\,\lambda^2mn1\,k0 - 12\,k0\,f3\,mn1 + 18\,k0\,f3\,m^2 + 9\,\lambda^2k0\,m^2 - 4\,\lambda^2n1^2 + 4\,\lambda^2n1^3$$

```
> k0sola := solve(eq7s=0,k0);
```

$$k0sola := -4\,\frac{\lambda^2n1^2\,(-1+n1)}{2\,f3\,n1^2 + \lambda^2n1^2 - 6\,\lambda^2mn1 - 12\,f3\,mn1 + 18\,m^2f3 + 9\,\lambda^2m^2}$$

```
> eq8 := factor(resultant(eq7s,eq5s,k0));
```

$$eq8 := 4\,f3\,n1^2\,(3\,m - n1)^2\,(6\,f3\,mn1 + 6\,\lambda\,mn1 + 3\,\lambda^2mn1 - 6\,m\lambda - 6\,\lambda\,n1^2 - \lambda^2n1^2 - 2\,f3\,n1^2 + 6\,n1\,\lambda)$$

```
> eq8s := op(5,eq8);
```

$$eq8s := 6\,f3\,mn1 + 6\,\lambda\,mn1 + 3\,\lambda^2mn1 - 6\,m\lambda - 6\,\lambda\,n1^2 - \lambda^2n1^2 - 2\,f3\,n1^2 + 6\,n1\,\lambda$$

```
> n2sola := factor(subs(k2=k2sola, k1=k1sola, k0=k0sola, k0+k1+k2));
```

$$n2sola := -\frac{(-1+n1)\,\lambda^2}{2\,f3 + \lambda^2}$$

```
> eq9s := lambda*f2*(1-n1-n2)-(3*m-n1-2*n2)*f3;
```

$$eq9s := \lambda\,f2\,(1 - n1 - n2) - (3\,m - n1 - 2\,n2)\,f3$$

```
> resultant(subs(n2=n2sola,f2=f2def,f3=f3def,eq9s),subs(n2=n2sol,
f2=f2def,f3=f3def,eq8s),n1):
> simplify(%);
> # eqsol relates m and lambda
> eqsol := op(4,%);
> solve(eqsol=0,m);
> msola := factor(%[2]);
> msol := (1/3)*lambda*g2*f1/F2;
> is(simplify(subs(f1=f1def,g2=g2def,F2=F2def,msol)-msola),0);
> rsol := msol-1/2;
```

$$\frac{1}{2(e^{\lambda}-1-\lambda)}\left(2\,e^{\lambda}-2-2\,\lambda-\lambda^2\right)^2\lambda\,(3\,me^{2\,\lambda}\lambda+3\,me^{2\,\lambda}-9\,m^2e^{2\,\lambda}$$
$$-\lambda\,e^{2\,\lambda}+3\,me^{\lambda}\lambda-e^{\lambda}\lambda+9\,m^2-3\,m-12\,m\lambda+18\,\lambda\,m^2+2\,\lambda)$$

$$eqsol := 3\,me^{2\,\lambda}\lambda+3\,me^{2\,\lambda}-9\,m^2e^{2\,\lambda}$$
$$-\lambda\,e^{2\,\lambda}+3\,me^{\lambda}\lambda-e^{\lambda}\lambda+9\,m^2-3\,m-12\,m\lambda+18\,\lambda\,m^2+2\,\lambda$$

$$1/3,\ -1/3\,\frac{\lambda\left(e^{2\,\lambda}+e^{\lambda}-2\right)}{-e^{2\,\lambda}+2\,\lambda+1}$$

$$msola := 1/3\,\frac{\lambda\left(e^{2\,\lambda}+e^{\lambda}-2\right)}{e^{2\,\lambda}-1-2\,\lambda}$$

$$msol := 1/3\,\frac{\lambda\,g2\,f1}{F2}$$

true

$$rsol := 1/3\,\frac{\lambda\,g2\,f1}{F2}-1/2$$

```
> subs(m=msol,n2=n2sola,eq9s):
> solve(%=0,n1);
> n1solb := simplify(subs(g2=g2def,f1=f1def,F2=F2def,f2=f2def,f3=f3def,%));
> n1sol := f1*lambda/F2;
> is(simplify(subs(f1=f1def,F2=F2def,n1sol)-n1solb),0);
```

$$\frac{\lambda\left(2\,f2\,F2-2\,g2\,f1\,f3-g2\,f1\,\lambda^2+2\,\lambda\,F2\right)}{F2\,\left(2\,\lambda\,f2-2\,f3+\lambda^2\right)}$$

$$n1solb := \frac{\left(e^{\lambda}-1\right)\lambda}{e^{2\,\lambda}-1-2\,\lambda}$$

$$n1sol := \frac{f1\,\lambda}{F2}$$

true

```
> k0sol := 2*lambda^2/(g1*F2);
```

```
> is(simplify(subs(n1=n1solb,m=msola,f3=f3def,g1=g1def,F2=F2def,
k0sol-k0sola)),0);
```

$$k0sol := 2\,\frac{\lambda^2}{g1\ F2}$$

true

```
> k1sol := 2*lambda^2*f1/(g1*F2);
```

```
> is(simplify(subs(k0=k0sola,n1=n1solb,m=msola,f1=f1def,f3=f3def,
g1=g1def,F2=F2def,k1sol-k1sola)),0);
```

$$k1sol := 2\,\frac{\lambda^2 f1}{g1\ F2}$$

true

```
> k2sol := lambda^2*f1^2/(2*g1*F2);
```

```
> is(simplify(subs(k1=k1sola,k0=k0sola,n1=n1solb,m=msola,f1=f1def,
f3=f3def,g1=g1def,F2=F2def,k2sol-k2sola)),0);
```

$$k2sol := 1/2\,\frac{\lambda^2 f1^2}{g1\ F2}$$

true

```
> n3sola := simplify(subs(n1=n1sol,k0=k0sol,k1=k1sol,k2=k2sol,n3def));
```

```
> n3sol := g1*f3/F2;
```

```
> is(simplify(subs(f3=f3def,g1=g1def,F2=F2def,f1=f1def,n3sola-n3sol)),0);
```

$$n3sola := 1/2\,\frac{2\,g1\ F2 - 2\,f1\ \lambda\,g1 - 4\,\lambda^2 - 4\,\lambda^2 f1 - \lambda^2 f1^2}{g1\ F2}$$

$$n3sol := \frac{g1\ f3}{F2}$$

true

```
> Q3sola := simplify(subs(m=msol,n1=n1sol,k0=k0sol,k1=k1sol,k2=k2sol,
f1=f1def,g1=g1def,g2=g2def,Q3def));
```

```
> Q3sol := lambda*g1*f2/F2;
```

```
> is(simplify(subs(f3=f3def,g1=g1def,F2=F2def,f2=f2def,Q3sola-Q3sol)),0);
```

$$Q3sola := -\frac{\left(-e^{2\,\lambda} + \lambda + 1 + e^{\lambda}\lambda\right)\lambda}{F2}$$

$$Q3sol := \frac{\lambda\,g1\ f2}{F2}$$

$$true$$

```
> T3sola := simplify(subs(m2=m2def,m=msol,n1=n1sol,k0=k0sol,k1=k1sol,
k2=k2sol,f1=f1def,g1=g1def,g2=g2def,D33def));
```

```
> T3sol := lambda*f2*f1/F2;
```

```
> is(simplify(subs(f3=f3def,g1=g1def,F2=F2def,f1=f1def,f2=f2def,
T3sola-T3sol)),0);
```

$$T3sola := D33def$$

$$T3sol := \frac{\lambda\, f2\, f1}{F2}$$

$$false$$

```
> T2sola := simplify(subs(m2=m2def,m=msol,n1=n1sol,k0=k0sol,k1=k1sol,
k2=k2sol,f1=f1def,g1=g1def,g2=g2def,D32def));
```

```
> T2sol := 2*lambda*f2/F2;
```

```
> is(simplify(subs(f3=f3def,g1=g1def,F2=F2def,f2=f2def,T2sola-T2sol)),0);
```

$$T2sola := D32def$$

$$T2sol := 2\,\frac{\lambda\, f2}{F2}$$

$$false$$

```
> m3sola := simplify(subs(m2=m2def,m=msol,n1=n1sol,k0=k0sol,k1=k1sol,
k2=k2sol,f1=f1def,g1=g1def,g2=g2def,m3def));
```

```
> m3sol := (1/3)*lambda*f1*f1/F2;
```

```
> is(simplify(subdef(m3sola-m3sol)),0);
```

$$m3sola := 1/3\,\frac{\lambda\,\left(e^{\lambda}-1\right)^{2}}{F2}$$

$$m3sol := 1/3\,\frac{\lambda\, f1^{2}}{F2}$$

$$true$$

```
> m2psola := simplify(subs(m2=m2def,m=msol,n1=n1sol,k0=k0sol,k1=k1sol,
k2=k2sol,f1=f1def,g1=g1def,g2=g2def,m2pdef));
```

```
> m2psol := lambda/g1;
```

```
> is(simplify(subdef(m2psola-m2psol)),0);
```

$$m2psola := \frac{\lambda\,\left(e^{2\,\lambda}-1-2\,\lambda\right)}{\left(e^{\lambda}+1\right) F2}$$

$$m2psol := \frac{\lambda}{g1}$$

$$true$$

```
> # Series
> seriesl:= (value,k) -> series(subdef(value),lambda=0,k);
> # This relates the parameters with lambda->0
> n1series := seriesl(n1sol,3);
> mseries := seriesl(msol,4);
> k0series := seriesl(k0sol,4);
> k1series := seriesl(k1sol,4);
> k2series := seriesl(k2sol,4);
> n3series := seriesl(n3sol,4);
> Q3series := seriesl(Q3sol,4);
> T3series := seriesl(T3sol,4);
> T2series := seriesl(T2sol,4);
> m3series := seriesl(m3sol,4);
> m2pseries := seriesl(m2psol,2);
> c3series := seriesl(Q3sol/n3sol,4);
> seriesl(Q3sol*(1+etadef-Q3sol/n3sol),5);
```

$$n1series := 1/2 - 1/12\,\lambda - 1/36\,\lambda^2 + O\left(\lambda^3\right)$$

$$mseries := 1/2 + 1/12\,\lambda + 1/36\,\lambda^2 + \frac{11}{2160}\,\lambda^3 + O\left(\lambda^4\right)$$

$$k0series := 1/2 - \frac{7}{12}\,\lambda + 2/9\,\lambda^2 + \frac{1}{2160}\,\lambda^3 + O\left(\lambda^4\right)$$

$$k1series := 1/2\,\lambda - 1/3\,\lambda^2 + \frac{1}{72}\,\lambda^3 + O\left(\lambda^4\right)$$

$$k2series := 1/8\,\lambda^2 - 1/48\,\lambda^3 + O\left(\lambda^4\right)$$

$$n3series := 1/6\,\lambda + O\left(\lambda^2\right)$$

$$Q3series := 1/2\,\lambda + O\left(\lambda^2\right)$$

$$T3series := O\left(\lambda^2\right)$$

$$T2series := 1/2\,\lambda + O\left(\lambda^2\right)$$

$$m3series := 1/6\,\lambda + O\left(\lambda^2\right)$$

$$m2pseries := 1/2\,\lambda + O\left(\lambda^2\right)$$

$$c3series := 3 + 1/4\,\lambda + O\left(\lambda^2\right)$$

$$1/24\,\lambda^2 + O\left(\lambda^3\right)$$

```
> fpre-(2*r*ln(n)-4*r*ln(r)):
> fl:= subs(m=msol, n1=n1sol,r=rsol,k0=k0sol,k1=k1sol,k2=k2sol,%):
> simplify(seriesl(fl,5),symbolic);
> evalf(-2/3*ln(2)-1/3*ln(3)+1/3);
> evalf(-(2/9)*ln(2)-(1/9)*ln(3)+7/36);
```

$$(-2/3\,\ln(2) - 1/3\,\ln(3) + 1/3)\,\lambda + \left(-2/9\,\ln(2) - 1/9\,\ln(3) + \frac{7}{36}\right)\lambda^2 + O\left(\lambda^3\right)$$

$$-0.4949688834$$

$$-0.0816562945$$

A.6 Case 1 in the proof of Lemma 4.8.18

```
> restart; h := x -> x*n*ln(x)+x*n*ln(n)-x*n;
```

$$h := x \mapsto xn \ln(x) + xn \ln(n) - xn$$

```
> # lambda
> eladef := exp(lambda);
> f1def := exp(lambda)-1;
> f2def := exp(lambda)-1-lambda;
> f3def := exp(lambda)-1-lambda-lambda^2/2;
> g1def := exp(lambda)+1;
> g2def := exp(lambda)+2;
> F1def := exp(2*lambda)-1;
> F2def := exp(2*lambda)-1-2*lambda;
> etadef := lambda*f1/f2;
> subdef := x -> subs(eta=etadef,ela=eladef,f1=f1def,f2=f2def,f3=f3def,
g1=g1def,g2=g2def,F1=F1def,F2=F2def, x);
> msolmax := (1/3)*lambda*g2*f1/F2;
> rsolmax := msolmax-1/2;
> series(subdef(rsolmax),lambda=0);
```

$$eladef := e^{\lambda}$$

$$f1def := e^{\lambda} - 1$$

$$f2def := e^{\lambda} - 1 - \lambda$$

$$f3def := e^{\lambda} - 1 - \lambda - 1/2\,\lambda^2$$

$$g1def := e^{\lambda} + 1$$

$$g2def := e^{\lambda} + 2$$

$$F1def := e^{2\lambda} - 1$$

$$F2def := e^{2\lambda} - 1 - 2\lambda$$

$$etadef := \frac{\lambda\, f1}{f2}$$

$$subdef := x \mapsto x$$

$$msolmax := 1/3\, \frac{\lambda\, g2\, f1}{F2}$$

$$rsolmax := 1/3\,\frac{\lambda\,g2\,f1}{F2} - 1/2$$

$$1/12\,\lambda + 1/36\,\lambda^2 + \frac{11}{2160}\,\lambda^3 + O\left(\lambda^4\right)$$

```
> n2rel := k0+k1+k2;
> n3rel := 1-n1-n2;
> P3rel := 3*m-3*n1;
> Q3rel := 3*m-n1-2*n2;
> P2rel := 2*(n1-k0);
> m2prel := n1-k0;
> m3rel := P3rel/3;
```

$$n2rel := k0 + k1 + k2$$
$$n3rel := 1 - n1 - n2$$
$$P3rel := 3\,m - 3\,n1$$
$$Q3rel := 3\,m - n1 - 2\,n2$$
$$P2rel := 2\,n1 - 2\,k0$$
$$m2prel := n1 - k0$$
$$m3rel := m - n1$$

```
> sol:= solve(subs(n2=n2rel,{n3rel=0, P2rel=k1, P3rel=k1+2*k2}),{k1,k2,n1});
> k1sol := rhs(sol[1]);
> k2sol := rhs(sol[2]);
> n1sol := rhs(sol[3]);
> subsfork0 := x -> subs(n1=n1sol,k1=k1sol,k2=k2sol,x);
> subsfork0(P3rel);
> subsfork0(P2rel);
> subsfork0(m2prel);
```

$$sol := \{k1 = -6\,m + 4 - 2\,k0, k2 = -5 + 9\,m + k0, n1 = -3\,m + 2\}$$
$$k1sol := -6\,m + 4 - 2\,k0$$
$$k2sol := -5 + 9\,m + k0$$
$$n1sol := -3\,m + 2$$
$$subsfork0 := x \mapsto x$$
$$12\,m - 6$$
$$-6\,m + 4 - 2\,k0$$
$$-3\,m + 2 - k0$$

```
> f := (k0) -> subs( m3 = m3rel, P3 = P3rel, P2 = P2rel, m2p = m2prel,
n2 = n2rel, k1=k1sol,k2 = k2sol, n1=n1sol, h(n1)+h(P3)+h(P2)-h(k0)-h(k1)
-h(k2)-h(m3)-2*h(m2p)-k2*n*ln(2)-m3*n*ln(6)-m2p*n*ln(2))/n:
> expand(f(k0));
```

$$1 - k0\ \ln(k0) + 5\ \ln(2) - 10\,m\ln(2) + 5\ \ln(-5+9\,m+k0) - 9\ \ln(-5+9\,m+k0)\,m$$
$$-\ln(-5+9\,m+k0)\,k0 - \ln(n) + 2\ \ln(n)\,m + 2\ \ln(-3\,m+2) - 3\ \ln(-3\,m+2)\,m$$
$$-4\ \ln(6) + 8\,m\ln(6) + 8\,m\ln(2\,m-1) - 4\ \ln(-3\,m+2-k0) + 6\ \ln(-3\,m+2-k0)\,m$$
$$+2\ \ln(-3\,m+2-k0)\,k0 - 4\ \ln(2\,m-1) - 2\,m$$

```
> # Partial derivatives
> fk0 :=  simplify(exp(diff(f(k0), k0)));
```

$$fk0 := \frac{(3\,m-2+k0)^2}{k0\ (-5+9\,m+k0)}$$

```
> fk00 :=  simplify(diff(diff(f(k0), k0),k0));
> numfk00:= simplify(subs(m=1/2+r,numer(-fk00)));
> simplify(subs(k0=+5-9*m,m=1/2+r,numfk00));
> simplify(subs(k0=+2-3*m,m=1/2+r,numfk00));
```

$$fk00 := \frac{33\,m - 10 - k0 - 27\,m^2 + 3\,mk0}{k0\ (-5+9\,m+k0)\,(3\,m-2+k0)}$$

$$numfk00 := 1/4 - 6\,r - 1/2\,k0 + 27\,r^2 - 3\,k0\,r$$

$$-3\,r + 54\,r^2$$

$$-6\,r + 36\,r^2$$

```
> eqk0 := numer(fk0)-denom(fk0);
```

$$eqk0 := (3\,m-2+k0)^2 - k0\ (-5+9\,m+k0)$$

```
> k0sol := solve(eqk0=0,k0); factor(%);
```

$$k0sol := \frac{9\,m^2 - 12\,m + 4}{3\,m-1}$$

$$\frac{(3\,m-2)^2}{3\,m-1}$$

```
> simplify(subs(k0=k0sol, f(k0sol))):
> subdef(subs(r=rsolmax, m=msolmax,w=exp(1),%-(2*r*ln(n)-4*r*ln(r)))):
> simplify(series(%,lambda=0),symbolic);
```

$$(1/3 - \ln(2) - 1/3\ \ln(3))\,\lambda + O\left(\lambda^2\right)$$

A.7 Case 2 in the proof of Lemma 4.8.18

```
> restart; h := x -> x*n*ln(x)+x*n*ln(n)-x*n;
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn$$

```
> # lambda functions
> eladef := exp(lambda);
> f1def := exp(lambda)-1;
> f2def := exp(lambda)-1-lambda;
> f3def := exp(lambda)-1-lambda-lambda^2/2;
> g1def := exp(lambda)+1;
> g2def := exp(lambda)+2;
> F1def := exp(2*lambda)-1;
> F2def := exp(2*lambda)-1-2*lambda;
> etadef := lambda*f1/f2;
> subdef := x -> subs(eta=etadef,ela=eladef,f1=f1def,f2=f2def,f3=f3def,
g1=g1def,g2=g2def,F1=F1def,F2=F2def, x);
> msolmax := (1/3)*lambda*g2*f1/F2;
> rsolmax := msolmax-1/2;
> series(subdef(rsolmax),lambda=0);
```

$$eladef := e^{\lambda}$$

$$f1def := e^{\lambda} - 1$$

$$f2def := e^{\lambda} - 1 - \lambda$$

$$f3def := e^{\lambda} - 1 - \lambda - 1/2\,\lambda^2$$

$$g1def := e^{\lambda} + 1$$

$$g2def := e^{\lambda} + 2$$

$$F1def := e^{2\lambda} - 1$$

$$F2def := e^{2\lambda} - 1 - 2\lambda$$

$$etadef := \frac{\lambda f1}{f2}$$

$$subdef := x \mapsto x$$

$$msolmax := 1/3\,\frac{\lambda\, g2\, f1}{F2}$$

$$rsolmax := 1/3\,{\frac{\lambda\,g2\,f1}{F2}}-1/2$$

$$1/12\,\lambda+1/36\,{\lambda}^{2}+{\frac{11}{2160}}\,{\lambda}^{3}+O\left({\lambda}^{4}\right)$$

```
> n2rel := k0+k1+k2;
> n3rel := 1-n1-n2;
> P3rel := 3*m-3*n1;
> Q3rel := 3*m-n1-2*n2;
> P2rel := 2*(n1-k0);
> m2prel := n1-k0;
> m3rel := P3rel/3;
```

$$n2rel := k0+k1+k2$$
$$n3rel := 1-n1-n2$$
$$P3rel := 3\,m-3\,n1$$
$$Q3rel := 3\,m-n1-2\,n2$$
$$P2rel := 2\,n1-2\,k0$$
$$m2prel := n1-k0$$
$$m3rel := m-n1$$

```
> Q3sol := 3*n3;
> n1sol := solve(P3rel=0,n1);
> k0sol := solve(subs(n3=n3rel,n2=n2rel,n1=n1sol,k1=0,k2=0,Q3rel=Q3sol),k0);
```

$$Q3sol := 3\,n3$$
$$n1sol := m$$
$$k0sol := 3-5\,m$$

```
> f := expand(subs( Q3=Q3sol, n3=n3rel, m3=m3rel, P3=P3rel, P2=P2rel,
m2p = m2prel, n2 = n2rel, k0=k0sol, n1=n1sol, k1=0,k2=0, (h(n1)+ h(Q3)
-h(k0)-h(n3)-h(m3) -2*h(m2p) -m3*n*ln(6)-m2p*n*ln(2))/n - n3*ln(6)));
```

$$f := 1+m\ln(m)+2\,m\ln(n)+8\,m\ln(6)-4\,m\ln(-1+2\,m)-\ln(n)-3\,\ln(3-5\,m)$$
$$+5\,\ln(3-5\,m)\,m-10\,m\ln(2)-12\,m\ln(3)+5\,\ln(2)+6\,\ln(3)-2\,m-4\,\ln(6)+2\,\ln(-1+2\,m)$$

```
> simplify(series(subs(m=msolmax,r=rsolmax,g2=g2def,f1=f1def,
F2=F2def,w=exp(1),f-(2*r*ln(n)-4*r*ln(r))),lambda=0, 4));
```

$$(-\ln(2)-1/3\,\ln(3)+1/3)\,\lambda+O\left({\lambda}^{2}\right)$$

A.8 Case 3 in the proof of Lemma 4.8.18

```
> restart; h := x -> x*n*ln(x)+x*n*ln(n)-x*n;
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn$$

```
> n2rel := k0+k1+k2;
> n3rel := 1-n1-n2;
> P3rel := 3*m-3*n1;
> Q3rel := 3*m-n1-2*n2;
> P2rel := 2*(n1-k0);
> m2rel := n1-k0;
> m3rel := P3rel/3;
```

$$n2rel := k0 + k1 + k2$$
$$n3rel := 1 - n1 - n2$$
$$P3rel := 3\,m - 3\,n1$$
$$Q3rel := 3\,m - n1 - 2\,n2$$
$$P2rel := 2\,n1 - 2\,k0$$
$$m2rel := n1 - k0$$
$$m3rel := m - n1$$

```
> Q3sol := 3*n3;
> k0sol := n1;
> k1sol := 0;
> k2sol := solve(subs(n3=n3rel,n2=n2rel,k1=0,k0=k0sol,Q3rel = Q3sol),k2);
```

$$Q3sol := 3\,n3$$
$$k0sol := n1$$
$$k1sol := 0$$
$$k2sol := 3 - 3\,n1 - 3\,m$$

```
> f := (n1) -> subs( Q3=Q3sol, n3=n3rel, m3=m3rel, P3=P3rel, P2=P2rel,
m2=m2rel, n2=n2rel,  k0=k0sol, k1=k1sol, k2=k2sol, (h(n1)+h(P3)+h(Q3)
-h(k0)-h(k2)-h(n3)-h(m3)-h(P3-k1-2*k2) -k2*n*ln(2)-m3*n*ln(6))/n - n3*ln(6)):
> expand(f(n1));
```

$$1 - 3\,\ln(2) + 3\,\ln(2)\,n1 + 3\,\ln(2)\,m - 4\,\ln(6)\,m + 6\,m\ln(3) + 2\,m\ln(m - n1)$$
$$-2\,n1\,\ln(m - n1) - \ln(n) + 2\,\ln(n)\,m + 3\,n1\,\ln(1 - n1 - m) + 3\,m\ln(1 - n1 - m)$$
$$+ 2\,\ln(-2 + n1 + 3\,m) - n1\,\ln(-2 + n1 + 3\,m)$$
$$-3\,m\ln(-2 + n1 + 3\,m) - 2\,m - 3\,\ln(3) - 3\,\ln(1 - n1 - m)$$
$$+ 2\,\ln(6)$$

```
> # Partial derivatives
> fn1 :=  simplify(exp(diff(f(n1), n1)));
> fn1n1 :=  simplify(diff(diff(f(n1), n1),n1));
```

$$fn1 := -8\,\frac{(-1+n1+m)^3}{(m-n1)^2\,(-2+n1+3\,m)}$$

$$fn1n1 := \frac{4-n1-15\,m+2\,n1\,m+14\,m^2}{(-2+n1+3\,m)\,(m-n1)\,(-1+n1+m)}$$

```
> eqn1 := numer(fn1)-denom(fn1);
```

$$eqn1 := -8\,(-1+n1+m)^3-(m-n1)^2\,(-2+n1+3\,m)$$

```
> sol:= solve(subs(m=1/2+r,eqn1=0),n1):  n1s := sol[1]; n1s2 := sol[2]:
> n1s3 := sol[3]:
```

$$n1s := \frac{2}{27}\,\sqrt[3]{-1052\,r^3+108\,\sqrt{93}r^3}+\frac{56}{27}\,\frac{r^2}{\sqrt[3]{-1052\,r^3+108\,\sqrt{93}r^3}}+1/2-\frac{25}{27}\,r$$

```
> # n1s is the solution I am seeking
> series(n1s,r=0);
> cr := (-2/27)*(1052-108*sqrt(93))^(1/3)
-56/(27*(1052-108*sqrt(93))^(1/3))-25/27;
> evalf(cr);
```

$$1/2+\left(\frac{2}{27}\,\sqrt[3]{1052-108\,\sqrt{93}}\sqrt[3]{-1}-\frac{56}{27}\,\frac{(-1)^{2/3}}{\sqrt[3]{1052-108\,\sqrt{93}}}-\frac{25}{27}\right)r$$

$$cr := -\frac{2}{27}\,\sqrt[3]{1052-108\,\sqrt{93}}-\frac{56}{27}\,\frac{1}{\sqrt[3]{1052-108\,\sqrt{93}}}-\frac{25}{27}$$

$$-2.035656368$$

```
> simplify(subs(m=1/2+r, n1=1/2+b*r, f(n1)-(2*r*ln(n)+2*r*ln(r))),symbolic);
```

$$r\,(-2-\ln(2)+2\,\ln(3)+2\,\ln(1-b)+3\,\ln(-b-1)-3\,\ln(b+3)+3\,b\ln(-b-1)$$
$$-\,b\ln(b+3)+3\,\ln(2)\,b-2\,b\ln(1-b))$$

```
> evalf(subs(b=cr, simplify(%/r)));
```

$$1.938961708$$

A.9 Case 4 in the proof of Lemma 4.8.18

```
> restart; h := x -> x*n*ln(x)+x*n*ln(n)-x*n;
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn$$

```
> # lambda functions
> eladef := exp(lambda);
> f1def := exp(lambda)-1;
> f2def := exp(lambda)-1-lambda;
> f3def := exp(lambda)-1-lambda-lambda^2/2;
> g1def := exp(lambda)+1;
> g2def := exp(lambda)+2;
> F1def := exp(2*lambda)-1;
> F2def := exp(2*lambda)-1-2*lambda;
> etadef := lambda*f1/f2;
> subdef := x -> subs(eta=etadef,ela=eladef,f1=f1def,f2=f2def,f3=f3def,
g1=g1def,g2=g2def,F1=F1def,F2=F2def, x);
> msolmax := (1/3)*lambda*g2*f1/F2;
> rsolmax := msolmax-1/2;
```

$$eladef := e^{\lambda}$$

$$f1def := e^{\lambda} - 1$$

$$f2def := e^{\lambda} - 1 - \lambda$$

$$f3def := e^{\lambda} - 1 - \lambda - 1/2\,\lambda^2$$

$$g1def := e^{\lambda} + 1$$

$$g2def := e^{\lambda} + 2$$

$$F1def := e^{2\,\lambda} - 1$$

$$F2def := e^{2\,\lambda} - 1 - 2\,\lambda$$

$$etadef := \frac{\lambda\, f1}{f2}$$

$$subdef := x \mapsto x$$

$$msolmax := 1/3\,\frac{\lambda\, g2\, f1}{F2}$$

$$rsolmax := 1/3\,\frac{\lambda\, g2\, f1}{F2} - 1/2$$

```
> n2rel := k0+k1+k2;
> n3rel := 1-n1-n2;
> P3rel := 3*m-3*n1;
> Q3rel := 3*m-n1-2*n2;
> P2rel := 2*(n1-k0);
> m2prel := n1-k0;
> m3rel := P3rel/3;
```

$$n2rel := k0 + k1 + k2$$
$$n3rel := 1 - n1 - n2$$
$$P3rel := 3\,m - 3\,n1$$
$$Q3rel := 3\,m - n1 - 2\,n2$$
$$P2rel := 2\,n1 - 2\,k0$$
$$m2prel := n1 - k0$$
$$m3rel := m - n1$$

```
> Q3sol := 3*n3;
> k0sol := solve(subs(n3=n3rel,n2=n2rel,Q3rel = Q3sol),k0);
> subs(n3=n3rel,n2=n2rel,k0=k0sol,Q3rel);
> subs(n3=n3rel,n2=n2rel,k0=k0sol,P3rel-k1-2*k2);
> subs(n3=n3rel,n2=n2rel,k0=k0sol,P3rel);
> subs(n3=n3rel,n2=n2rel,k0=k0sol,P2rel-k1);
> subs(n3=n3rel,n2=n2rel,k0=k0sol,P2rel);
> subs(n3=n3rel,n2=n2rel,k0=k0sol,n3rel);
```

$$Q3sol := 3\,n3$$
$$k0sol := 3 - 2\,n1 - k1 - k2 - 3\,m$$
$$9\,m + 3\,n1 - 6$$
$$3\,m - 3\,n1 - k1 - 2\,k2$$
$$3\,m - 3\,n1$$
$$6\,n1 - 6 + k1 + 2\,k2 + 6\,m$$
$$6\,n1 - 6 + 2\,k1 + 2\,k2 + 6\,m$$
$$-2 + n1 + 3\,m$$

```
> f := (n1,k1,k2) -> subs(Q3=Q3sol, n3=n3rel, m3=m3rel, P3=P3rel,
P2=P2rel, m2p=m2prel, n2=n2rel, k0=k0sol, (h(n1)+h(P3)+h(P2)+h(Q3)
-h(k0)-h(k1)-h(k2)-h(n3)-h(m3)-h(P3-k1-2*k2)-h(P2-k1)-2*h(m2p)
-k2*n*ln(2)-m3*n*ln(6)-m2p*n*ln(2))/n - n3*ln(6)):
> # Partial derivatives
```

```
> fn1 :=  simplify(exp(diff(f(n1,k1,k2), n1)));
```

$$fn1 := -8\,\frac{(-3\,m+3\,n1+k1+2\,k2)^3\,(-3+2\,n1+k1+k2+3\,m)^2\,(-2+n1+3\,m)^2\,n1}{(m-n1)^2\,(6\,n1-6+k1+2\,k2+6\,m)^6}$$

```
> fk1 :=  simplify(exp(diff(f(n1,k1,k2), k1)));
```

$$fk1 := 2\,\frac{(-3\,m+3\,n1+k1+2\,k2)\,(-3+2\,n1+k1+k2+3\,m)}{(6\,n1-6+k1+2\,k2+6\,m)\,k1}$$

```
> fk2:=  simplify(exp(diff(f(n1,k1,k2), k2)));
```

$$fk2 := -\frac{(-3\,m+3\,n1+k1+2\,k2)^2\,(-3+2\,n1+k1+k2+3\,m)}{(6\,n1-6+k1+2\,k2+6\,m)^2\,k2}$$

```
> eqn1 := numer(fn1)-denom(fn1);
```

$$eqn1 := -8\,(-3\,m+3\,n1+k1+2\,k2)^3\,(-3+2\,n1+k1+k2+3\,m)^2$$
$$(-2+n1+3\,m)^2\,n1-(m-n1)^2\,(6\,n1-6+k1+2\,k2+6\,m)^6$$

```
> eqk1 := numer(fk1)-denom(fk1);
```

$$eqk1 := 2\,(-3\,m+3\,n1+k1+2\,k2)\,(-3+2\,n1+k1+k2+3\,m)$$
$$-(6\,n1-6+k1+2\,k2+6\,m)\,k1$$

```
> eqk2 := numer(fk2)-denom(fk2);
```

$$eqk2 := -(-3\,m+3\,n1+k1+2\,k2)^2\,(-3+2\,n1+k1+k2+3\,m)$$
$$-(6\,n1-6+k1+2\,k2+6\,m)^2\,k2$$

```
> resultant(eqk1,eqk2,k1):
> eq1 := factor(%);
> k2solalt := 3-3*m-4*n1;
> subs(n3=n3rel,n2=n2rel,k0=k0sol,k2=k2solalt, P2rel);
```

$$eq1 := 9\,(-2+n1+3\,m)^2\,(k2-3+3\,m+4\,n1)$$
$$(9\,m^2k2-6\,mn1\,k2+n1^2k2+27\,m^3-27\,m^2-36\,m^2n1-9\,mn1^2+54\,mn1-27\,n1^2+18\,n1^3)$$

$$k2solalt := 3-3\,m-4\,n1$$
$$-2\,n1+2\,k1$$

```
> eq1s := op(4, eq1); k2sol := solve(%=0,k2); factor(%);
```

$$eq1s := 9\,m^2k2-6\,mn1\,k2+n1^2k2+27\,m^3-27\,m^2$$
$$-36\,m^2n1-9\,mn1^2+54\,mn1-27\,n1^2+18\,n1^3$$

$$k2sol := -9\,\frac{3\,m^3-3\,m^2-4\,m^2n1-mn1^2+6\,mn1-3\,n1^2+2\,n1^3}{9\,m^2-6\,mn1+n1^2}$$

$$-9\,{\frac {\left(3\,m-3+2\,{\it n1} \right) \left(m-{\it n1} \right) ^{2}}{\left(3\,m-{\it n1} \right) ^{2}}}$$

```
> resultant(eqk2,eqk1, k2):
> eq2 := factor(%);
```

$$\mathit{eq2} := 144\,\left(-2+{\it n1}+3\,m \right) ^{2} \left({\it k1}-2\,{\it n1} \right)$$
$$\left(9\,{m}^{2}{\it k1}-6\,m{\it n1}\,{\it k1}+{{\it n1}}^{2}{\it k1}+36\,{m}^{2}{\it n1}+36\,{{\it n1}}^{2}-24\,{{\it n1}}^{3}-36\,m{\it n1}-12\,m{{\it n1}}^{2} \right)$$

```
> eq2s := op(4,eq2); k1sol:= solve(%=0,k1);factor(%);
```

$$\mathit{eq2s} := 9\,{m}^{2}{\it k1}-6\,m{\it n1}\,{\it k1}+{{\it n1}}^{2}{\it k1}+36\,{m}^{2}{\it n1}+36\,{{\it n1}}^{2}-24\,{{\it n1}}^{3}-36\,m{\it n1}-12\,m{{\it n1}}^{2}$$

$$\mathit{k1sol} := -12\,{\frac {{\it n1}\,\left(3\,{m}^{2}-3\,m-m{\it n1}+3\,{\it n1}-2\,{{\it n1}}^{2} \right) }{9\,{m}^{2}-6\,m{\it n1}+{{\it n1}}^{2}}}$$

$$-12\,{\frac {{\it n1}\,\left(3\,m-3+2\,{\it n1} \right) \left(m-{\it n1} \right) }{\left(3\,m-{\it n1} \right) ^{2}}}$$

```
> eq3 := resultant(eq2s,eqn1,k1):
> resultant(eq3,eq1s,k2):
> factor(%);
> eqm := op(3,%);
```

$$46656\,{{\it n1}}^{5} \left(18\,m-36\,{m}^{2}+18\,{m}^{3}-18\,{\it n1}+18\,m{\it n1}-3\,{m}^{2}{\it n1}+22\,{{\it n1}}^{2}-16\,m{{\it n1}}^{2}-7\,{{\it n1}}^{3} \right)$$
$$\left(m-{\it n1} \right) ^{2} \left(-2+{\it n1}+3\,m \right) ^{5} \left(3\,m-{\it n1} \right) ^{17}$$

$$\mathit{eqm} := 18\,m-36\,{m}^{2}+18\,{m}^{3}-18\,{\it n1}+18\,m{\it n1}-3\,{m}^{2}{\it n1}+22\,{{\it n1}}^{2}-16\,m{{\it n1}}^{2}-7\,{{\it n1}}^{3}$$

```
> eq := simplify(subs(m=1/2+r,eqm));
> sol := solve(eq=0,n1):
> x1 := sol[1]:
> x2 := sol[2]:
> x3 := sol[3]:
> evalf(limit(x1,r=0));
> evalf(limit(x2,r=0));
> evalf(limit(x3,r=0));
```

$$\mathit{eq} := 9/4-9/2\,r-9\,{r}^{2}+18\,{r}^{3}-{\frac {39}{4}}\,{\it n1}+15\,{\it n1}\,r-3\,{\it n1}\,{r}^{2}+14\,{{\it n1}}^{2}-16\,{{\it n1}}^{2}r-7\,{{\it n1}}^{3}$$

$$0.5000000000$$
$$0.7499999995+0.2834733546\,i$$
$$0.7499999995-0.2834733546\,i$$

```
> n1sol := x1:
> sn1 := simplify(series(n1sol,r=0,4));
> series(subdef(rsolmax),lambda=0);
```

$$sn1 := 1/2 - r - 6\,r^2 + 60\,r^3 + O\left(r^4\right)$$

$$1/12\,\lambda + 1/36\,\lambda^2 + \frac{11}{2160}\,\lambda^3 + O\left(\lambda^4\right)$$

```
> expand(subs(k1=k1sol,k2=k2sol,m=1/2+r, f(n1,k1,k2)-(2*r*ln(n)-4*r*ln(r)))):
> simplify(%):
> simplify(series(simplify(subs(n1=sn1,%)),r=0,4),symbolic);
> series(subs(r=lambda/12+lambda^2/36,(4-8*ln(2)-4*ln(3))*r+6*r^2),lambda=0);
```

$$(4 - 8\,\ln(2) - 4\,\ln(3))\,r + 6\,r^2 + O\left(r^3\right)$$

$$(1/3 - 2/3\,\ln(2) - 1/3\,\ln(3))\,\lambda + \left(\frac{11}{72} - 2/9\,\ln(2) - 1/9\,\ln(3)\right)\lambda^2 + 1/36\,\lambda^3 + \frac{1}{216}\,\lambda^4$$

```
> evalf(11/72-(2/9)*ln(2)-(1/9)*ln(3));
```

$$-0.1233229611$$

A.10 Case 5 in the proof of Lemma 4.8.18

```
> restart; h := x -> x*n*ln(x)+ x*n*ln(n)- x*n;
> n2rel := k0+k1+k2;
> n3rel := 1-n1-n2;
> P3rel := 3*m-3*n1;
> Q3rel := 3*m-n1-2*n2;
> P2rel := 2*(n1-k0);
> m2rel := n1-k0;
> m3rel := P3rel/3;
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn$$
$$n2rel := k0 + k1 + k2$$
$$n3rel := 1 - n1 - n2$$
$$P3rel := 3\,m - 3\,n1$$
$$Q3rel := 3\,m - n1 - 2\,n2$$
$$P2rel := 2\,n1 - 2\,k0$$
$$m2rel := n1 - k0$$
$$m3rel := m - n1$$

```
> n1sol := m; k1sol := 0; k2sol:= 0; k0sol := 1-n1sol-n3;
```

$$n1sol := m$$
$$k1sol := 0$$
$$k2sol := 0$$
$$k0sol := 1 - m - n3$$

```
> f := (n3) -> subs(Q3 = 3*m-n1-2*n2, n2=k0sol, P2=2*m2, m2=n1-k0, M2=n1,
k0=k0sol, n1=n1sol, +h(M2)+h(P2)+h(Q3)-h(k0)-h(n3)-h(P2)-m2*n*ln(2)
-2*h(m2))/n:
> g := (n3, lambda) -> f(n3)+ subs(Q3 = 3*m - n1 - 2*(k0), k0 = 1-n1-n3,n1=m,
n3*log(f3)- Q3*ln(lambda)):
> simplify(subs(n2=1-n1-n3,n1=m, Q3rel));
```

$$4\,m - 2 + 2\,n3$$

```
> fn3 :=  simplify(exp(diff(f(n3), n3)))/lambda^2*f3;
```

$$fn3 := -2\,\frac{(-1 + m + n3)\,f3}{n3\,\lambda^2}$$

```
> eqn3 := numer(fn3)-denom(fn3);
```

$$eqn3 := -2\,(-1 + m + n3)\,f3 - n3\,\lambda^2$$

```
> n3sol := solve(eqn3=0,n3);
```

$$n3sol := -2\,\frac{(-1+m)\,f3}{2\,f3+\lambda^2}$$

```
> eq3 := simplify(n3*(Q3/n3) = n3sol*((2*f3+lambda^2)/f3)
*(n3*lambda/(2*Q3))*(Q3/n3));
```

$$eq3 := Q3 = -(-1+m)\,\lambda$$

```
> lambdasol := solve(eq3,lambda);
```

$$lambdasol := -\frac{Q3}{-1+m}$$

```
> n3sol2 := (simplify(solve(subs(Q3=Q3rel, n2=k0,k0=1-n1-n3,n1=m,eq3),n3)));
> simplify(subs(n3=n3sol2,m=1/2+r,rhs(eq3)/n3sol2));
> subs(f3=f2-lambda^2/2, f2=exp(lambda)-1-lambda,%= lambda*f2/f3);
> rsol := solve(%,r);
> rseries := series(rsol,lambda=0);
```

$$n3sol2 := 1/2\,\lambda - 1/2\,\lambda\,m - 2\,m + 1$$

$$2\,\frac{(-1+2\,r)\,\lambda}{-\lambda+2\,\lambda\,r+8\,r}$$

$$2\,\frac{(-1+2\,r)\,\lambda}{-\lambda+2\,\lambda\,r+8\,r} = \frac{\lambda\left(e^{\lambda}-1-\lambda\right)}{e^{\lambda}-1-\lambda-1/2\,\lambda^2}$$

$$rsol := 1/2\,\frac{-2\,e^{\lambda}+2+\lambda+e^{\lambda}\lambda}{2\,e^{\lambda}-2-3\,\lambda+e^{\lambda}\lambda}$$

$$rseries := 1/24\,\lambda + \frac{1}{288}\,\lambda^2 - \frac{7}{17280}\,\lambda^3 + O\left(\lambda^4\right)$$

```
> g2 := lambda -> expand(subs(f3=exp(lambda)-1-lambda-lambda^2/2,
n3=n3sol2,m=1/2+rsol,g(n3,lambda)-(2*r*ln(n)-4*r*ln(r)))):
> g2(lambda):  subs(m=1/2+r,r=rsol,%):
> simplify(series(%,lambda=0,3),symbolic);
```

$$(1/6 - 1/2\,\ln(2) - 1/6\,\ln(3))\,\lambda + \left(-1/24\,\ln(2) - \frac{1}{72}\,\ln(3) + \frac{1}{72}\right)\lambda^2 + O\left(\lambda^3\right)$$

A.11 Case 6 in the proof of Lemma 4.8.18

```
> restart; h := x -> x*n*ln(x)+x*n*ln(n)-x*n;
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn$$

```
> n2rel := k0+k1+k2;
> n3rel := 1-n1-n2;
> P3rel := 3*m-3*n1;
> Q3rel := 3*m-n1-2*n2;
> P2rel := 2*(n1-k0);
> m2rel := n1-k0;
> m3rel := P3rel/3;
```

$$n2rel := k0 + k1 + k2$$
$$n3rel := 1 - n1 - n2$$
$$P3rel := 3\,m - 3\,n1$$
$$Q3rel := 3\,m - n1 - 2\,n2$$
$$P2rel := 2\,n1 - 2\,k0$$
$$m2rel := n1 - k0$$
$$m3rel := m - n1$$

```
> k1sol := 0;
> k0sol := solve(P2rel=0, k0);
> Q3sol := solve(subs(k1=k1sol,k0=k0sol,Q3+2*k1+2*k2=P3rel+P2rel),Q3);
> n3sol := subs(n2=n2rel,k1=k1sol,k0=k0sol,1-n1-n2);
```

$$k1sol := 0$$
$$k0sol := n1$$
$$Q3sol := 3\,m - 3\,n1 - 2\,k2$$
$$n3sol := 1 - 2\,n1 - k2$$

```
> f := (n1,k2) -> subs(Q3=Q3rel, n3=n3rel, m3=m3rel, P3=P3rel, P2=P2rel,
m2=m2rel, n2=n2rel, k0=k0sol, k1=k1sol, +h(n1)+h(P3)+h(Q3)-h(k0)-h(k2)
-h(n3)-h(m3)-h(P3-k1-2*k2) -k2*n*ln(2)-m3*n*ln(6))/n;
> g := (lambda) -> f(n1,k2)+ subs(Q3 = 3*m - n1 - 2*(n1+k2), n3 = 1-2*n1-k2, n3*log(f3)-
Q3*ln(lambda)):
> # Partial derivatives
> fn1 := simplify(exp(diff(f(n1,k2), n1))*lambda^3/f3^2);
```

$$fn1 := 2/9\,\frac{\left(-1 + 2\,n1 + k2\right)^2 \lambda^3}{\left(m - n1\right)^2 \quad f3^2}$$

```
> fk2 := simplify(exp(diff(f(n1,k2), k2)))*lambda^2/f3;
```

$$fk2 := -1/2\,\frac{(-1+2\,n1+k2)\,\lambda^2}{k2\,f3}$$

```
> eqk2 := numer(fk2)-denom(fk2);solve(eqk2=0,n1);
```

$$eqk2 := -(-1+2\,n1+k2)\,\lambda^2-2\,k2\,f3$$

$$-1/2\,\frac{-\lambda^2+\lambda^2 k2+2\,k2\,f3}{\lambda^2}$$

```
> k2sol := solve(eqk2=0,k2); k2sol2 := k2sol*(lambda^2+2*f3)/(2*f2);
```

$$k2sol := -\frac{(-1+2\,n1)\,\lambda^2}{\lambda^2+2\,f3}$$

$$k2sol2 := -1/2\,\frac{(-1+2\,n1)\,\lambda^2}{f2}$$

```
> simplify(subs(k2=k2sol2, sqrt(fn1/fk2^2)),symbolic);
> %*(1-2*n1)/(2*n1-1);
> n1sol := simplify(solve(%=1,n1));
> y := simplify(%-1/2);
```

$$1/3\,\frac{\sqrt{2}\lambda^{3/2}\,(-1+2\,n1)}{(m-n1)\,f2}$$

$$1/3\,\frac{\sqrt{2}\lambda^{3/2}\,(1-2\,n1)}{(m-n1)\,f2}$$

$$n1sol := \frac{\sqrt{2}\lambda^{3/2}-3\,f2\,m}{2\,\sqrt{2}\lambda^{3/2}-3\,f2}$$

$$y := -3/2\,\frac{f2\,(2\,m-1)}{2\,\sqrt{2}\lambda^{3/2}-3\,f2}$$

```
> lambda*f2/f3-Q3/n3;
> simplify(subs(Q3=Q3rel, n3=n3rel,n2=n2rel,k1=k1sol,k0=k0sol, k2=k2sol2,
n1=n1sol,%),symbolic);
> subs(f3=f2-lambda^2/2,f2=exp(lambda)-1-lambda,%);
> solve(%=0,lambda);
> z := %[2];
```

$$\frac{\lambda\,f2}{f3}-\frac{Q3}{n3}$$

$$-\frac{\left(-2\,f2^2+\lambda^2 f2+2\,f3\,\sqrt{\lambda}\sqrt{2}-2\,f3\,\lambda\right)\lambda}{(2\,f2-\lambda^2)\,f3}$$

$$-\frac{\left(-2\,\left(\mathrm{e}^{\lambda}-1-\lambda\right)^2+\lambda^2\left(\mathrm{e}^{\lambda}-1-\lambda\right)+2\,\left(\mathrm{e}^{\lambda}-1-\lambda-1/2\,\lambda^2\right)\sqrt{\lambda}\sqrt{2}-2\,\left(\mathrm{e}^{\lambda}-1-\lambda-1/2\,\lambda^2\right)\lambda\right)\lambda}{\left(2\,\mathrm{e}^{\lambda}-2-2\,\lambda-\lambda^2\right)\left(\mathrm{e}^{\lambda}-1-\lambda-1/2\,\lambda^2\right)}$$

$$\mathit{RootOf}\left(2\,\_Z-\left(\mathrm{e}^{\_Z}\right)^2+2\,\mathrm{e}^{\_Z}-1, 0.0\right),\ \mathit{RootOf}\left(2\,\_Z-\left(\mathrm{e}^{\_Z}\right)^2+2\,\mathrm{e}^{\_Z}-1, 0.8267548776\right)$$

$$z := \mathit{RootOf}\left(2\,\_Z-\left(\mathrm{e}^{\_Z}\right)^2+2\,\mathrm{e}^{\_Z}-1, 0.8267548776\right)$$

```
> yy := (subs(f2=exp(lambda)-1-lambda,m=1/2+r,lambda=z,y)/r):
> bb := (subs(f2=exp(lambda)-1-lambda,m=1/2+r,lambda=z,
exp(lambda)-1-lambda-lambda^2/2)):
> simplify(subs(k2=(1-2*n1)*(b/2),m=1/2+r, n1=1/2+c*r,g(n1,k0)));
```

$$r(-2-2\,c\ln(2)-2\,c\ln(3)-2\,\ln(-r\,(-1+c))\,c+2\,\ln(cr\,(-2+b))\,c$$
$$+\,cb\ln(-crb)+3\,cb\ln(2)+2\,\ln(2)+2\,\ln(3)+2\,\ln(n)+2\,\ln(-r\,(-1+c))-\ln(cr\,(-2+b))\,cb$$
$$-\,2\,c\ln(\mathit{f3})+c\ln(\mathit{f3})\,b-3\,\ln(1+2\,cr)+3\,c\ln(1+2\,cr)-2\,cb\ln(1+2\,cr))$$

```
> 2+2*ln(1-c)*c+2*c*ln(2)+2*c*ln(3)-2*ln(2)-2*ln(3)-2*ln(1-c)-3*b*c*ln(2)
-2*ln(-2*c+b*c)*c+ln(-2*c+b*c)*b*c+2*c*ln(f3)-c*ln(f3)*b;
> evalf(subs(c=yy,b=bb,f3=exp(lambda)-1-lambda-lambda^2/2,lambda=z,%));
```

$$2+2\,\ln(1-c)\,c+2\,c\ln(2)+2\,c\ln(3)-2\,\ln(2)-2\,\ln(3)-2\,\ln(1-c)$$
$$-\,3\,cb\ln(2)-2\,\ln(-2\,c+cb)\,c+\ln(-2\,c+cb)\,bc+2\,c\ln(\mathit{f3})$$
$$-\,c\ln(\mathit{f3})\,b$$

$$-1.932194501$$

```
> -(2*ln(r)*c-2*ln(r)-2*ln(r)*c+ln(r)*b*c-c*b*ln(r)-4*ln(r));
```

$$6\,\ln(r)$$

A.12 Spreadsheet for Section 4.9: comparing g_{pre} and g_{core}

```
> restart; h := x -> x*n*ln(x)+ x*n*ln(n)- x*n;
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn$$

```
> eladef := exp(lambda);
> f1def := exp(lambda)-1;
> f2def := exp(lambda)-1-lambda;
> f3def := exp(lambda)-1-lambda-lambda^2/2;
> g1def := exp(lambda)+1;
> g2def := exp(lambda)+2;
> F1def := exp(2*lambda)-1;
> F2def := exp(2*lambda)-1-2*lambda;
> subdef := x -> subs(ela=eladef,f1=f1def,f2=f2def,f3=f3def,g1=g1def,
g2=g2def,F1=F1def,F2=F2def, x);
```

$$eladef := e^{\lambda}$$

$$f1def := e^{\lambda} - 1$$

$$f2def := e^{\lambda} - 1 - \lambda$$

$$f3def := e^{\lambda} - 1 - \lambda - 1/2\,\lambda^2$$

$$g1def := e^{\lambda} + 1$$

$$g2def := e^{\lambda} + 2$$

$$F1def := e^{2\lambda} - 1$$

$$F2def := e^{2\lambda} - 1 - 2\lambda$$

$$subdef := x \mapsto x$$

```
> # Relations
> pren3def := 1-n1-k0-k1-k2;
> preQ3def := 3*m-n1-2*(k0+k1+k2);
> prem2def  := n1;
> prem2pdef := prem2def-k0;
> prem3def  := m-prem2def;
> preP2def  := 2*prem2pdef;
> preP3def  := 3*prem3def;
> preT2def := preP2def-k1;
```

```
> preT3def := 3*prem3def-k1-2*
> pren2def  := k0+k1+k2;
> subdef2pre := x -> subs(n3=pren3def,Q3=preQ3def,m2=prem2def,m3=prem3def,
m2p=prem2pdef,P2=preP2def,P3=preP3def, T2=preT2def, T3=preT3def,
n2=pren2def, x);
> coreQ2def := 3*m-n1;
> coren2def := 1-n1;
> corem3def := m-n1;
> subdef2core := x -> subs(Q2=coreQ2def,n2=coren2def,m3=corem3def, x);
```

$$pren3def := 1 - n1 - k0 - k1 - k2$$
$$preQ3def := 3\,m - n1 - 2\,k0 - 2\,k1 - 2\,k2$$
$$prem2def := n1$$
$$prem2pdef := n1 - k0$$
$$prem3def := m - n1$$
$$preP2def := 2\,n1 - 2\,k0$$
$$preP3def := 3\,m - 3\,n1$$
$$preT2def := 2\,n1 - 2\,k0 - k1$$
$$preT3def := 3\,m - 3\,n1 - k1 - 2\,k2$$
$$pren2def := k0 + k1 + k2$$
$$subdef2pre := x \mapsto x$$
$$coreQ2def := 3\,m - n1$$
$$coren2def := 1 - n1$$
$$corem3def := m - n1$$
$$subdef2core := x \mapsto x$$

```
> msol := (1/3)*lambda*g2*f1/F2;
> n1sol:=  f1*lambda/F2;
> k0sol := 2*lambda^2/(g1*F2);
> k1sol := 2*lambda^2*f1/(g1*F2);
> k2sol := lambda^2*f1^2/(2*g1*F2);
```

$$msol := 1/3\,\frac{\lambda\,g2\,f1}{F2}$$
$$n1sol := \frac{f1\,\lambda}{F2}$$

$$k0sol := 2\,\frac{\lambda^2}{g1\ F2}$$

$$k1sol := 2\,\frac{\lambda^2 f1}{g1\ F2}$$

$$k2sol := 1/2\,\frac{\lambda^2 f1^2}{g1\ F2}$$

```
> # fpre (without lambda part)
> f := (n1,k0,k1,k2) -> expand(subdef2pre(
> h(m2)+h(P3)+h(P2)+h(Q3)
> -h(k0)-h(k1)-h(k2)-h(n3)-h(m3)-h(T3)-h(T2)-2*h(m2p)
> -k2*n*ln(2)-m3*n*ln(6)-m2p*n*ln(2))/n):
> # fpre
> fpre := expand(f(n1,k0,k1,k2)+subdef2pre(n3*log(f3)- Q3*ln(lambda))):
> # fcore without lambda part
> f:= (n1) -> expand(subdef2core(h(Q2)-h(n2)-h(n1)-h(m3)
-n1*n*ln(2)-m3*n*ln(6))/n):
> fcore := expand(f(n1)+subdef2core(n2*log(f2)- Q2*ln(lambda))):
> s:= simplify(subdef(subs(m=msol,n1=n1sol,k0=k0sol,k1=k1sol,k2=k2sol,
fpre-fcore)),symbolic);
```

$$\begin{aligned}
s := {} & \frac{1}{2(e^{2\lambda}-1-2\lambda)}\Big(2\ln\left(e^{\lambda}-1-\lambda\right)-2\lambda\ln\left(e^{2\lambda}-1\right)e^{2\lambda}+2\lambda\ln\left(e^{2\lambda}-1-\lambda-\lambda e^{\lambda}\right)e^{2\lambda} \\
& -2\lambda\ln\left(e^{2\lambda}-1-\lambda-\lambda e^{\lambda}\right)e^{\lambda}-2\lambda^2\ln\left(e^{2\lambda}-1-\lambda-\lambda e^{\lambda}\right)e^{\lambda}-2\lambda\ln\left(e^{\lambda}-1\right) \\
& -2\ln\left(2e^{\lambda}-2-2\lambda-\lambda^2\right)+\lambda^2 e^{\lambda}\ln\left(e^{\lambda}+1\right)+2\ln\left(2e^{2\lambda}-2-2\lambda e^{\lambda}-2\lambda-\lambda^2 e^{\lambda}-\lambda^2\right) \\
& -2\ln\left(e^{2\lambda}-1-\lambda-\lambda e^{\lambda}\right)-4\lambda\ln\left(e^{2\lambda}-1-\lambda-\lambda e^{\lambda}\right)+2\lambda^2\ln\left(e^{\lambda}-1-\lambda\right) \\
& +2\lambda\ln\left(2e^{2\lambda}-2-2\lambda e^{\lambda}-2\lambda-\lambda^2 e^{\lambda}-\lambda^2\right)+\lambda^2\ln\left(2e^{2\lambda}-2-2\lambda e^{\lambda}-2\lambda-\lambda^2 e^{\lambda}-\lambda^2\right) \\
& -2e^{2\lambda}\ln\left(e^{\lambda}-1-\lambda\right)+4\lambda\ln\left(e^{\lambda}-1-\lambda\right)-2e^{2\lambda}\ln\left(2e^{2\lambda}-2-2\lambda e^{\lambda}-2\lambda-\lambda^2 e^{\lambda}-\lambda^2\right) \\
& +\lambda^2\ln\left(e^{\lambda}+1\right)+2\ln\left(e^{2\lambda}-1-\lambda-\lambda e^{\lambda}\right)e^{2\lambda}-2\lambda^2\ln\left(e^{2\lambda}-1-\lambda-\lambda e^{\lambda}\right) \\
& +2\lambda\ln\left(e^{2\lambda}-1\right)+2\lambda^2 e^{\lambda}\ln\left(e^{\lambda}-1-\lambda\right)+2\lambda e^{\lambda}\ln\left(e^{\lambda}-1-\lambda\right)-2\lambda e^{2\lambda}\ln\left(e^{\lambda}-1-\lambda\right) \\
& +2\lambda e^{\lambda}\ln\left(2e^{2\lambda}-2-2\lambda e^{\lambda}-2\lambda-\lambda^2 e^{\lambda}-\lambda^2\right)+\lambda^2 e^{\lambda}\ln\left(2e^{2\lambda}-2-2\lambda e^{\lambda}-2\lambda-\lambda^2 e^{\lambda}-\lambda^2\right) \\
& +2\lambda e^{2\lambda}\ln\left(e^{\lambda}-1\right)-\lambda^2\ln\left(2e^{\lambda}-2-2\lambda-\lambda^2\right)-2\lambda\ln\left(2e^{\lambda}-2-2\lambda-\lambda^2\right) \\
& +2e^{2\lambda}\ln\left(2e^{\lambda}-2-2\lambda-\lambda^2\right)-2\lambda e^{\lambda}\ln\left(2e^{\lambda}-2-2\lambda-\lambda^2\right)-\lambda^2 e^{\lambda}\ln\left(2e^{\lambda}-2-2\lambda-\lambda^2\right)\Big)
\end{aligned}$$

```
> num := numer(s):
> snum := simplify(exp(num),symbolic);
> # show this is 1
```

$$snum := \left(e^{\lambda} - 1 - \lambda\right)^{2\,(1+\lambda)\left(-e^{2\,\lambda}+1+\lambda+\lambda\,e^{\lambda}\right)} \left(2\,e^{\lambda} - 2 - 2\,\lambda - \lambda^2\right)^{2\,e^{2\,\lambda}-2\,\lambda\,e^{\lambda}-\lambda^2 e^{\lambda}-2-\lambda^2}$$
$$\left(2\,e^{2\,\lambda} - 2 - 2\,\lambda\,e^{\lambda} - 2\,\lambda - \lambda^2 e^{\lambda} - \lambda^2\right)^{-2\,e^{2\,\lambda}+2\,\lambda\,e^{\lambda}+\lambda^2 e^{\lambda}+2+\lambda^2}$$
$$\left(e^{2\,\lambda} - 1 - \lambda - \lambda\,e^{\lambda}\right)^{-2\,(1+\lambda)\left(-e^{2\,\lambda}+1+\lambda+\lambda\,e^{\lambda}\right)} \left(e^{2\,\lambda} - 1\right)^{-2\,\lambda\,\left(e^{2\,\lambda}-1\right)} \left(e^{\lambda} - 1\right)^{2\,\lambda\,\left(e^{2\,\lambda}-1\right)}$$
$$\left(e^{\lambda} + 1\right)^{\lambda\,\left(\lambda\,e^{\lambda}+\lambda+2\right)}$$

```
> snum2 := a^x1 * (b*c)^x2 * (a*c)^(-x1) * b^(-x2) * (d*c)^x3
*c^(x1-x2-x3) * d^(-x3);
> simplify(snum2,symbolic);
> snum3 := a^x1 * bc^x2 * ac^(-x1) * b^(-x2) * dc^x3 * c^x4 * d^(-x3);
> simplify(subs(dc=d*c,bc=b*c,ac=a*c,x4=x1-x2-x3,snum2/snum3),symbolic);
```

$$snum2 := a^{x1}\,(bc)^{x2}\,(ac)^{-x1}\,b^{-x2}\,(dc)^{x3}\,c^{x1-x2-x3}\,d^{-x3}$$
$$1$$
$$snum3 := a^{x1}\,bc^{x2}\,ac^{-x1}\,b^{-x2}\,dc^{x3}\,c^{x4}\,d^{-x3}$$
$$1$$

```
> aa := 2*exp(lambda)-2-2*lambda-lambda^2;
> bb := exp(lambda)-1-lambda;
> cc := exp(lambda)+1;
> dd := exp(lambda)-1;
> xx1 := 2*exp(2*lambda)-2*lambda*exp(lambda)-lambda^2*exp(lambda)
-2-lambda^2;
> xx2 := (2*(1+lambda))*(exp(2*lambda)-1-lambda-lambda*exp(lambda));
> xx3 := -2*lambda*(exp(2*lambda)-1);
> xx4 := lambda*(lambda*exp(lambda)+lambda+2);
> simplify(xx4-(xx1-xx2-xx3));
> aacc := simplify(expand(aa*cc));
> ddcc := simplify(expand(dd*cc));
> bbcc := simplify(expand(bb*cc));
```

$$aa := 2\,e^{\lambda} - 2 - 2\,\lambda - \lambda^2$$
$$bb := e^{\lambda} - 1 - \lambda$$

$$cc := e^{\lambda} + 1$$

$$dd := e^{\lambda} - 1$$

$$xx1 := 2\,e^{2\,\lambda} - 2\,\lambda\,e^{\lambda} - \lambda^2 e^{\lambda} - 2 - \lambda^2$$

$$xx2 := 2\,(1+\lambda)\left(e^{2\,\lambda} - 1 - \lambda - \lambda\,e^{\lambda}\right)$$

$$xx3 := -2\,\lambda\,\left(e^{2\,\lambda} - 1\right)$$

$$xx4 := \lambda\,\left(\lambda\,e^{\lambda} + \lambda + 2\right)$$

$$0$$

$$aacc := 2\,e^{2\,\lambda} - 2 - 2\,\lambda\,e^{\lambda} - 2\,\lambda - \lambda^2 e^{\lambda} - \lambda^2$$

$$ddcc := e^{2\,\lambda} - 1$$

$$bbcc := e^{2\,\lambda} - 1 - \lambda - \lambda\,e^{\lambda}$$

```
> snum4:= subs(a=aa,b=bb,c=cc,d=dd,ac=aacc, bc=bbcc,dc = ddcc,
x1=xx1,x2=xx2,x3=xx3,x4=xx4,snum3);
```

$$snum4 := \left(2\,e^{\lambda} - 2 - 2\,\lambda - \lambda^2\right)^{2\,e^{2\,\lambda} - 2\,\lambda\,e^{\lambda} - \lambda^2 e^{\lambda} - 2 - \lambda^2}\,\left(e^{2\,\lambda} - 1 - \lambda - \lambda\,e^{\lambda}\right)^{2\,(1+\lambda)\left(e^{2\,\lambda} - 1 - \lambda - \lambda\,e^{\lambda}\right)}$$
$$\left(2\,e^{2\,\lambda} - 2 - 2\,\lambda\,e^{\lambda} - 2\,\lambda - \lambda^2 e^{\lambda} - \lambda^2\right)^{-2\,e^{2\,\lambda} + 2\,\lambda\,e^{\lambda} + \lambda^2 e^{\lambda} + 2 + \lambda^2}\,\left(e^{\lambda} - 1 - \lambda\right)^{-2\,(1+\lambda)\left(e^{2\,\lambda} - 1 - \lambda - \lambda\,e^{\lambda}\right)}$$
$$\left(e^{2\,\lambda} - 1\right)^{-2\,\lambda\,\left(e^{2\,\lambda} - 1\right)}\,\left(e^{\lambda} + 1\right)^{\lambda\,\left(\lambda\,e^{\lambda} + \lambda + 2\right)}\,\left(e^{\lambda} - 1\right)^{2\,\lambda\,\left(e^{2\,\lambda} - 1\right)}$$

```
> simplify(snum4/snum,symbolic);
```

$$1$$

A.13 Spreadsheet for Section 4.9: simplifying $t(\check{n})$

```
> h := x -> x*n*ln(x)+x*n*ln(n)-x*n;
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn$$

```
> # some function of lambda
> g1def := exp(lambda)+1;
> g2def := exp(lambda)+2;
> f1def := exp(lambda)-1;
> f2def := exp(lambda)-1-lambda;
> F2def := exp(2*lambda)-1-2*lambda;
> F1def := exp(2*lambda)-1;
> mrellambda := (1/3)*lambda*f1*g2/F2;
```

$$g1def := e^{\lambda} + 1$$

$$g2def := e^{\lambda} + 2$$

$$f1def := e^{\lambda} - 1$$

$$f2def := e^{\lambda} - 1 - \lambda$$

$$F2def := e^{2\lambda} - 1 - 2\lambda$$

$$F1def := e^{2\lambda} - 1$$

$$mrellambda := 1/3\,\frac{\lambda\, f1\; g2}{F2}$$

```
> f:=(n1)->(h(3*mn-n1)-h(1-n1)-h(n1)-h(mn-n1)-n1*n*ln(2)-(mn-n1)*n*ln(6))/n:
> n1sol := 3*mn/g2;
> # function fcore at n1sol
> fcore := expand(subs(n1=n1sol, f(n1)+(1-n1)*ln(f2)-(3*mn-n1)*ln(lambda))):
```

$$n1sol := 3\,\frac{mn}{g2}$$

```
> # We simplify fcore*n/N
> mdefn := 1/2+R/nu;
> mdefN := nu/2+R;
> rdef := R/nu;
> ndef := nu*N;
```

$$mdefn := 1/2 + \frac{R}{\nu}$$

$$mdefN := 1/2\,\nu + R$$

$$rdef := \frac{R}{\nu}$$

$$ndef := N\nu$$

```
> # First part with ln(n)
> x1 := limit(expand(fcore)/ln(n),n=infinity)*ln(n);
> s1:= expand(subs(mn=mdefn,n=ndef,%*n/N),symbolic);
> # scaled
> c1 := 2*R*ln(N);
> a := 2*R*ln(nu);
> is(simplify(c1+a-s1,symbolic),0);
```

$$x1 := (-1 + 2\,mn)\ln(n)$$

$$s1 := 2\,R\ln(N\nu)$$

$$c1 := 2\,R\ln(N)$$

$$a := 2\,R\ln(\nu)$$

$$true$$

```
> fx1:= expand(fcore-x1);
> # Getting the terms without functions of lambda
> x2 := 1+2*ln(3)*mn-ln(2)*mn-2*mn;
> # scaled
> s2 := simplify(subs(mn=mdefn,n=ndef,%*n/N));
> c2 := (2*ln(3)-ln(2)-2)*R;
> b := (ln(3)-(1/2)*ln(2))*nu;
> is(simplify(s2-c2-b,symbolic),0);
```

$$fx1 := 1 - 3\,\frac{mn\,\ln(2)}{g2} - 6\,\frac{mn\,\ln(3)}{g2} - 3\,mn\,\ln\left(mn - \frac{mn}{g2}\right)$$
$$g2^{-1} + 3\,\ln\left(1 - 3\,\frac{mn}{g2}\right) mn g2^{-1} - 3\,mn\,\ln\left(\frac{mn}{g2}\right) g2^{-1} + 3\,\ln\left(mn - 3\,\frac{mn}{g2}\right) mn g2^{-1}$$
$$+3\,\frac{\ln(6)\,mn}{g2} + 3\,mn\,\ln(3) + 3\,mn\,\ln\left(mn - \frac{mn}{g2}\right)$$
$$-\ln\left(1 - 3\,\frac{mn}{g2}\right) - \ln\left(mn - 3\,\frac{mn}{g2}\right) mn - \ln(6)\,mn - 2\,mn + \ln(f2) - 3\,\frac{\ln(f2)\,mn}{g2}$$
$$-3\,\ln(\lambda)\,mn + 3\,\frac{\ln(\lambda)\,mn}{g2}$$
$$x2 := 1 + 2\,mn\,\ln(3) - mn\,\ln(2) - 2\,mn$$
$$s2 := \ln(3)\,\nu + 2\,\ln(3)\,R - 1/2\,\ln(2)\,\nu - \ln(2)\,R - 2\,R$$
$$c2 := (2\,\ln(3) - \ln(2) - 2)\,R$$
$$b := (\ln(3) - 1/2\,\ln(2))\,\nu$$
$$true$$

```
> fx2 := expand(fx1-x2);
> #Terms of type ln(.)
> x3a := -ln((g2-3*mn)/g2)+ln(f2);
> simplify(subs(m=mrellambda,g2=g2def,f2=f2def,f1=f1def,F2=F2def,x3a),
> symbolic);
> x3 := ln(F2/g1);
> is(simplify( subs(mn=mrellambda,g1=g1def,g2=g2def,f2=f2def,f1=f1def,
> F2=F2def,exp(x3-x3a)),symbolic),1);
> s3 := simplify(subs(m=mdef,n=ndef,x3*n/N));
> c3 := 0;
> c := s3;
> is(simplify(subs(mn=mdefn,n=ndef, c-x3*n/N)),0);
```

$$fx2 := -3\,\frac{mn\,\ln(2)}{g2} - 6\,\frac{mn\,\ln(3)}{g2} - 3\,mn\,\ln\left(mn - \frac{mn}{g2}\right) g2^{-1}$$
$$+3\,\ln\left(1 - 3\,\frac{mn}{g2}\right) mn g2^{-1} - 3\,mn\,\ln\left(\frac{mn}{g2}\right) g2^{-1}$$
$$+3\,\ln\left(mn - 3\,\frac{mn}{g2}\right) mn g2^{-1} + 3\,\frac{\ln(6)\,mn}{g2} + mn\,\ln(3) + 3\,mn\,\ln\left(mn - \frac{mn}{g2}\right)$$
$$-\ln\left(1 - 3\,\frac{mn}{g2}\right) - \ln\left(mn - 3\,\frac{mn}{g2}\right) mn - \ln(6)\,mn + \ln(f2) - 3\,\frac{\ln(f2)\,mn}{g2}$$
$$-3\,\ln(\lambda)\,mn + 3\,\frac{\ln(\lambda)\,mn}{g2} + mn\,\ln(2)$$

$$x3a := -\ln\left(\frac{g2 - 3\,mn}{g2}\right) + \ln(f2)$$
$$-\ln\left(e^{\lambda} + 2 - 3\,mn\right) + \ln\left(e^{\lambda} + 2\right) + \ln\left(e^{\lambda} - 1 - \lambda\right)$$

$$x3 := \ln\left(\frac{F2}{g1}\right)$$

true

$$s3 := \ln\left(\frac{F2}{g1}\right)\nu$$

$$c3 := 0$$

$$c := \ln\left(\frac{F2}{g1}\right)\nu$$

true

```
> fx3 := fx2-x3a;
> # Terms of type mn*ln(.)
> x4a := 3*mn*ln(mn*(g2-1)/g2)-mn*ln(mn*(g2-3)/g2)-3*mn*ln(lambda);
> x4 := mn*ln((mn^2*g1^3)/(g2^2*f1*lambda^3));
> is(simplify(subs(mn=mrellambda,f1=f1def,g1=g1def,
> g2=g2def,exp(x4/mn-x4a/mn))),1);
> c4 := 0;
> d := simplify(subs(mn=mdefn,n=ndef,mn*n/N))
> *ln(mn^2*g1^3/(g2^2*f1*lambda^3));
> is(simplify(subs(mn=mdefn,n=ndef, d-x4*n/N)),0);
```

$$fx3 := -3\,\frac{\ln(2)\,mn}{g2} - 6\,\frac{mn\,\ln(3)}{g2} - 3\,mn\,\ln\left(mn - \frac{mn}{g2}\right) g2^{-1}$$

$$+3\,\ln\left(1 - 3\,\frac{mn}{g2}\right) mn g2^{-1} - 3\,mn\,\ln\left(\frac{mn}{g2}\right) g2^{-1}$$

$$+3\,\ln\left(mn - 3\,\frac{mn}{g2}\right) mn g2^{-1} + 3\,\frac{\ln(6)\,mn}{g2} + mn\,\ln(3) + 3\,mn\,\ln\left(mn - \frac{mn}{g2}\right)$$

$$-\ln\left(1 - 3\,\frac{mn}{g2}\right) - \ln\left(mn - 3\,\frac{mn}{g2}\right) mn - \ln(6)\,mn - 3\,\frac{\ln(f2)\,mn}{g2}$$

$$-3\,\ln(\lambda)\,mn + 3\,\frac{\ln(\lambda)\,mn}{g2} + \ln(2)\,mn + \ln\left(\frac{g2 - 3\,mn}{g2}\right)$$

$$x4a := 3\,mn\,\ln\left(\frac{mn\,(g2-1)}{g2}\right) - \ln\left(\frac{mn\,(g2-3)}{g2}\right) mn - 3\,\ln(\lambda)\,mn$$

$$x4 := mn\,\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right)$$

$$true$$

$$c4 := 0$$

$$d := (1/2\,\nu + R)\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right)$$

$$true$$

```
> # Now I will get the terms that are multiplied by m and divided by g2
> fx4 := expand(fx3-x4a);
> x5a := %:
> simplify(%/(3*mn)*g2,symbolic):
> simplify(exp(subs(mn=mrellambda,g2=g2def,f2=f2def,f1=f1def,
> F2=F2def,%)),symbolic);
```

$$fx4 := -3\,\frac{\ln(2)\,mn}{g2} - 6\,\frac{mn\,\ln(3)}{g2} - 3\,mn\,\ln\left(mn - \frac{mn}{g2}\right) g2^{-1}$$
$$+3\,\ln\left(1 - 3\,\frac{mn}{g2}\right) mn g2^{-1} - 3\,mn\,\ln\left(\frac{mn}{g2}\right) g2^{-1}$$
$$+3\,\ln\left(mn - 3\,\frac{mn}{g2}\right) mn g2^{-1} + 3\,\frac{\ln(6)\,mn}{g2} + mn\,\ln(3) + 3\,mn\,\ln\left(mn - \frac{mn}{g2}\right)$$
$$-\ln\left(1 - 3\,\frac{mn}{g2}\right) - \ln\left(mn - 3\,\frac{mn}{g2}\right) mn - \ln(6)\,mn - 3\,\frac{\ln(f2)\,mn}{g2}$$
$$+3\,\frac{\ln(\lambda)\,mn}{g2} + \ln(2)\,mn + \ln\left(\frac{g2 - 3\,mn}{g2}\right) - 3\,mn\,\ln\left(\frac{mn\,(g2-1)}{g2}\right) + \ln\left(\frac{mn\,(g2-3)}{g2}\right) mn$$

1

```
> # Did we get all the terms?
> is(simplify(x1+x2+x3a+x4a+x5a-fcore,symbolic),0);
> # This is the part that does not depend on nu
> c1+c2+c3+c4;
> # This is the part that depends on nu
> a+b+c+d;
```

true

$$2\,R\ln(N) + (2\,\ln(3) - \ln(2) - 2)\,R$$

$$2\,R\ln(\nu) + (\ln(3) - 1/2\,\ln(2))\,\nu + \ln\left(\frac{F2}{g1}\right)\nu + (1/2\,\nu + R)\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right)$$

A.14 Spreadsheet for Lemma 4.9.2

```
> (2*x*f1*g2-3*F2)/(f1*g1);
> f:= subs(f1=exp(x)-1,g1=exp(x)+1, g2=exp(x)+2, F2=exp(2*x)-1-2*x,%);
```

$$\frac{2\,x f1\;g2 - 3\,F2}{f1\;g1}$$

$$f := \frac{2\,x\,(e^x - 1)\,(e^x + 2) - 3\,e^{2\,x} + 3 + 6\,x}{(e^x - 1)\,(e^x + 1)}$$

```
> series(f,x=0);
```

$$1/2\,x^2 - 1/40\,x^4 + O\left(x^5\right)$$

```
> simplify(diff(f,x));
```

$$-2\,\frac{-e^{4\,x} - e^{3\,x} + e^x + 1 + xe^{3\,x} + 4\,xe^{2\,x} + xe^x}{(e^x - 1)^2\,(e^x + 1)^2}$$

```
> g := numer(%);
> simplify(subs(x=0,g));
```

$$g := 2\,e^{4\,x} + 2\,e^{3\,x} - 2\,e^x - 2 - 2\,xe^{3\,x} - 8\,xe^{2\,x} - 2\,xe^x$$

$$0$$

```
> g1:= diff(g,x);
> simplify(subs(x=0,g1));
```

$$g1 := 8\,e^{4\,x} + 4\,e^{3\,x} - 4\,e^x - 6\,xe^{3\,x} - 8\,e^{2\,x} - 16\,xe^{2\,x} - 2\,xe^x$$

$$0$$

```
> g2:= diff(g1,x);
> simplify(subs(x=0,g2));
```

$$g2 := 32\,e^{4\,x} + 6\,e^{3\,x} - 6\,e^x - 18\,xe^{3\,x} - 32\,e^{2\,x} - 32\,xe^{2\,x} - 2\,xe^x$$

$$0$$

```
> g3:= diff(g2,x);
> simplify(subs(x=0,g3));
```

$$g3 := 128\,e^{4\,x} - 8\,e^x - 54\,xe^{3\,x} - 96\,e^{2\,x} - 64\,xe^{2\,x} - 2\,xe^x$$

$$24$$

```
> g4:= diff(g3,x);
> simplify(subs(x=0,g4));
```

$$g4 := 512\,e^{4\,x} - 10\,e^x - 54\,e^{3\,x} - 162\,xe^{3\,x} - 256\,e^{2\,x} - 128\,xe^{2\,x} - 2\,xe^x$$

$$192$$

```
> g5:= diff(g4,x);
> simplify(subs(x=0,g5));
```

$$g5 := 2048\,e^{4\,x} - 12\,e^{x} - 324\,e^{3\,x} - 486\,x e^{3\,x} - 640\,e^{2\,x} - 256\,x e^{2\,x} - 2\,x e^{x}$$

$$1072$$

A.15 Spreadsheet for Section 4.9.1

```
> restart; h := x -> x*n*ln(x)+x*n*ln(n)-x*n*ln(w);
```

$$h := x \mapsto xn\ln(x) + xn\ln(n) - xn\ln(w)$$

```
> # setting functions of lambda
> g1def := exp(lambda)+1;
> g2def := exp(lambda)+2;
> f1def := exp(lambda)-1;
> f2def := exp(lambda)-1-lambda;
> F2def := exp(2*lambda)-1-2*lambda;
> F1def := exp(2*lambda)-1;
> mrellambda := (1/3)*lambda*f1*g2/F2;
> subsf := x -> subs(ela=exp(lambda),g1=g1def,g2=g2def, f1=f1def,
f2=f2def,F1=F1def,F2=F2def,x);
```

$$g1def := e^{\lambda} + 1$$
$$g2def := e^{\lambda} + 2$$
$$f1def := e^{\lambda} - 1$$
$$f2def := e^{\lambda} - 1 - \lambda$$
$$F2def := e^{2\lambda} - 1 - 2\lambda$$
$$F1def := e^{2\lambda} - 1$$
$$mrellambda := 1/3\,\frac{\lambda\, f1\; g2}{F2}$$
$$subsf := x \mapsto x$$

```
> # setting scaled functions nu = n/N
> mdefn := 1/2+R/nu;
> mdefN := nu/2+R;
> rdef := R/nu;
> ndef := nu*N;
```

$$mdefn := 1/2 + \frac{R}{\nu}$$
$$mdefN := 1/2\,\nu + R$$
$$rdef := \frac{R}{\nu}$$

$$ndef := \nu N$$

```
> # fcore at maximum
> # only the part depending on nu
> a := 2*R*ln(nu);
> b := (ln(3)-(1/2)*ln(2))*nu;
> c := ln(F2/g1)*nu;
> d := ((1/2)*nu+R)*ln(mn^2*g1^3/(g2^2*f1*lambda^3));
> fcore := a+b+c+d;
```

$$a := 2\,R\ln(\nu)$$

$$b := (\ln(3) - 1/2\,\ln(2))\,\nu$$

$$c := \ln\left(\frac{F2}{g1}\right)\nu$$

$$d := (1/2\,\nu + R)\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right)$$

$$fcore := 2\,R\ln(\nu) + (\ln(3) - 1/2\,\ln(2))\,\nu + \ln\left(\frac{F2}{g1}\right)\nu + (1/2\,\nu + R)\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right)$$

```
> # cacti part
> cacti := -(1-nu)/2*ln(1-nu)+(1-nu)/2;
```

$$cacti := -1/2\,(1-\nu)\ln(1-\nu) + 1/2 - 1/2\,\nu$$

```
> # derivative for part that does not
> # depend on lambda
> dcacti := diff(cacti,nu);
> difa := diff(a,nu);
> difb := diff(b,nu);
> difsimple := dcacti+difa+difb;
```

$$dcacti := 1/2\,\ln(1-\nu)$$

$$difa := 2\,\frac{R}{\nu}$$

$$difb := \ln(3) - 1/2\,\ln(2)$$

$$difsimple := 1/2\,\ln(1-\nu) + 2\,\frac{R}{\nu} + \ln(3) - 1/2\,\ln(2)$$

```
> # implicit diffentiation for lambda
> dlambdaval:= diff(subs(mn=mdefn,mn),nu)/diff(subs(f1=f1def,F2=F2def,
g2=g2def, mrellambda),lambda);
> # simplifying
> dlambdasimple := -(R/nu^2)/(mn*(1/lambda+ela/f1+ela/g2-2*ela^2/F2+2/F2));
> #checking if simplification is correct
> is(simplify(subs(mn=mrellambda,f1=f1def,g2=g2def,F2=F2def,ela=exp(lambda),
dlambdaval-dlambdasimple)),0);
```

$$dlambdaval := -R\nu^{-2}\left(1/3\,\frac{\left(e^{\lambda}-1\right)\left(e^{\lambda}+2\right)}{e^{2\,\lambda}-1-2\,\lambda}+1/3\,\frac{\lambda\,e^{\lambda}\left(e^{\lambda}+2\right)}{e^{2\,\lambda}-1-2\,\lambda}+1/3\,\frac{\lambda\,\left(e^{\lambda}-1\right)e^{\lambda}}{e^{2\,\lambda}-1-2\,\lambda}-1/3\,\frac{\lambda\,\left(e^{\lambda}-1\right)\left(e^{\lambda}+2\right)\left(2\,e^{2\,\lambda}-2\right)}{\left(e^{2\,\lambda}-1-2\,\lambda\right)^{2}}\right)^{-1}$$

$$dlambdasimple := -R\nu^{-2}mn^{-1}\left(\lambda^{-1}+\frac{ela}{f1}+\frac{ela}{g2}-2\,\frac{ela^{2}}{F2}+2\,F2^{-1}\right)^{-1}$$

$$true$$

```
> # derivative for c
> c;
> cc := simplify(exp(c/nu),symbolic);
> difc := simplify(c/nu)+nu*dlc;
> dlcval := (1/cc)*dcc*dlambda;
> dccval := simplify(diff(subs(F2=F2def,f2=f2def,g1=g1def,cc),lambda));
> dccsimple := 2*F1/g1-F2*ela/g1^2;
> is(simplify(subs(F1=F1def, F2=F2def,f1=f1def,g1=g1def,ela=exp(lambda),
dccval-dccsimple)),0);
```

$$\ln\left(\frac{F2}{g1}\right)\nu$$

$$cc := \frac{F2}{g1}$$

$$difc := \ln\left(\frac{F2}{g1}\right)+\nu\,dlc$$

$$dlcval := \frac{g1\ dcc\ dlambda}{F2}$$

$$dccval := \frac{e^{3\lambda} + 2e^{2\lambda} - e^{\lambda} - 2 + 2e^{\lambda}\lambda}{\left(e^{\lambda}+1\right)^2}$$

$$dccsimple := 2\frac{F1}{g1} - \frac{F2\ ela}{g1^2}$$

$$true$$

```
> difcval:= subs(dlc = dlcval, dcc = dccsimple,dlambda=dlambdasimple,
ela=exp(lambda),difc);
```

```
> difcval2:= subs(dlc = dlcval, dcc = dccsimple,dlambda=dlambdasimple,
mn=mrellambda,difc);
```

$$difcval := \ln\left(\frac{F2}{g1}\right) - g1\left(2\frac{F1}{g1} - \frac{F2\,e^{\lambda}}{g1^2}\right) R\nu^{-1} F2^{-1} mn^{-1} \left(\lambda^{-1} + \frac{e^{\lambda}}{f1} + \frac{e^{\lambda}}{g2} - 2\frac{\left(e^{\lambda}\right)^2}{F2} + 2F2^{-1}\right)^{-1}$$

$$difcval2 := \ln\left(\frac{F2}{g1}\right) - 3\,g1\left(2\frac{F1}{g1} - \frac{F2\ ela}{g1^2}\right) R\nu^{-1}\lambda^{-1} f1^{-1} g2^{-1} \left(\lambda^{-1} + \frac{ela}{f1} + \frac{ela}{g2} - 2\frac{ela^2}{F2} + 2F2^{-1}\right)^{-1}$$

```
> # derivative for d
> d;
> dd := simplify(exp(d/(nu/2+R)),symbolic);
> dldval := (1/dd)*(ddd);
> #ddd = dd1val+dd2val
> ddd1val := 2*dd/mn*(diff(mdefn,nu));
> ddd2val := diff(subs(g1=g1def,g2=g2def,F2=F2def,f2=f2def,f1=f1def,dd),
lambda)*dlambda;
> ddd1simple := 2*mn*(diff(mdefn,nu))*(dd/mn^2);
> ddd2simple := -3*mn^2*(2*F1/g1-F2*ela/g1^2)*g1^4/(g2^3*f1^2*lambda^4)
*dlambda;
> is(simplify(subs(g1=g1def,g2=g2def,F2=F2def,f2=f2def,f1=f1def,
ddd1val-ddd1simple)),0);
> is(simplify(subs(m=mrellambda,g1=g1def,g2=g2def,F1=F1def,F2=F2def,
f2=f2def,f1=f1def, ela=exp(lambda),ddd2val-ddd2simple)),0);
```

$$(1/2\,\nu + R)\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right)$$

$$dd := \frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}$$

$$dldval := \frac{g2^2 f1\,\lambda^3\,ddd}{mn^2 g1^3}$$

$$ddd1val := -2\,\frac{mn\,g1^3R}{g2^2 f1\,\lambda^3\nu^2}$$

$$ddd2val := \left(3\,\frac{mn^2\left(e^\lambda+1\right)^2 e^\lambda}{\left(e^\lambda+2\right)^2\left(e^\lambda-1\right)\lambda^3} - 2\,\frac{mn^2\left(e^\lambda+1\right)^3 e^\lambda}{\left(e^\lambda+2\right)^3\left(e^\lambda-1\right)\lambda^3} - \frac{mn^2\left(e^\lambda+1\right)^3 e^\lambda}{\left(e^\lambda+2\right)^2\left(e^\lambda-1\right)^2\lambda^3} - 3\,\frac{mn^2\left(e^\lambda+1\right)^3}{\left(e^\lambda+2\right)^2\left(e^\lambda-1\right)\lambda^4}\right) dlambda$$

$$ddd1simple := -2\,\frac{mn\,g1^3R}{g2^2 f1\,\lambda^3\nu^2}$$

$$ddd2simple := -3\,mn^2\left(2\,\frac{F1}{g1} - \frac{F2\;ela}{g1^2}\right) g1^4\,dlambda\,g2^{-3} f1^{-2}\lambda^{-4}$$

$$true$$

$$true$$

```
> dldval1:= subs(ddd=ddd1simple,dldval);
> dldval2:= subs(ddd=ddd2simple,dlambda=dlambdasimple,dldval);
```

$$dldval1 := -2\,\frac{R}{mn\,\nu^2}$$

$$dldval2 := 3\,g1\left(2\,\frac{F1}{g1} - \frac{F2\;ela}{g1^2}\right) R g2^{-1} f1^{-1}\lambda^{-1}\nu^{-2} mn^{-1}\left(\lambda^{-1} + \frac{ela}{f1} + \frac{ela}{g2} - 2\,\frac{ela^2}{F2} + 2\,F2^{-1}\right)^{-1}$$

```
> difdval := diff(mdefN,nu)*ln(dd)+simplify(subs(mn=mdefn,mdefN*dldval1))
+(subs(mn=mdefn,mdefN*dldval2));
```

$$difdval := 1/2\,\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right) - 2\,\frac{R}{\nu} + 3\,(1/2\,\nu + R)\,g1\left(2\,\frac{F1}{g1} - \frac{F2\;ela}{g1^2}\right) R g2^{-1} f1^{-1} \lambda^{-1}\nu^{-2}\left(1/2 + \frac{R}{\nu}\right)^{-1}\left(\lambda^{-1} + \frac{ela}{f1} + \frac{ela}{g2} - 2\,\frac{ela^2}{F2} + 2\,F2^{-1}\right)^{-1}$$

```
> # first derivative
> eq1 := simplify(difsimple+difdval+difcval2);
```

$$eq1 := 1/2\,\ln(1-\nu) + \ln(3) - 1/2\,\ln(2) + 1/2\,\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right) + \ln\left(\frac{F2}{g1}\right)$$

```
> eq2 := subsf(subs(mn=mdefn,3*mn-3*mrellambda));
> nusol := solve(eq2,nu);
```

$$eq2 := 3/2 + 3\,\frac{R}{\nu} - \frac{\lambda\left(e^\lambda-1\right)\left(e^\lambda+2\right)}{e^{2\lambda} - 1 - 2\,\lambda}$$

$$nusol := 6\,\frac{R\left(e^{2\,\lambda}-1-2\,\lambda\right)}{3-3\,e^{2\,\lambda}+2\,\lambda+2\,\left(e^{\lambda}\right)^2\lambda+2\,e^{\lambda}\lambda}$$

```
> solve(eq1=0,nu);
> nulambda := F2/(f1*g1);
> is(subsf(nulambda) - simplify(subsf(subs(mn=mrellambda,%))),0);
> Rlambda := solve(subs(nu=nulambda,mdefn=mrellambda),R);
> Mlambda := simplify(subsf(1/2+Rlambda));
```

$$1/9\,\frac{9\,mn^2 g1\ F2^2-2\,g2^2 f1\ \lambda^3}{mn^2 g1\ F2^2}$$

$$nulambda := \frac{F2}{f1\ g1}$$

$$true$$

$$Rlambda := -1/6\,\frac{-2\,\lambda\,f1\ g2+3\,F2}{f1\ g1}$$

$$Mlambda := 1/3\,\frac{\left(1+e^{2\,\lambda}+e^{\lambda}\right)\lambda}{e^{2\,\lambda}-1}$$

```
> # Series at lambda=0
> ss:= (x,y) -> series(subsf(subs(dlambda=dlambdasimple,mn=mrellambda,
R=Rlambda,nu=nulambda,x)),lambda=0,y);
> nuseries:= ss(nu,3);
> Rseries := ss(Rlambda,3);
> rseries := ss(Rlambda/nulambda,3);
```

$$nuseries := \lambda+O\left(\lambda^2\right)$$

$$Rseries := O\left(\lambda^2\right)$$

$$rseries := O\left(\lambda\right)$$

```
> # Second derivative computations
> # decomposing first derivative in eq1
> eq1;
> dif1 := (1/2)*ln(1-nu)+ln(3)-(1/2)*ln(2);
> dif2 := +1/2*ln((mn^2*g1^3)/(g2^2*f1*lambda^3));
> dif3 := ln((F2)/(g1));
> is(simplify(eq1-dif1-dif2-dif3),0);
```

$$1/2\,\ln(1-\nu)+\ln(3)-1/2\,\ln(2)+1/2\,\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right)+\ln\left(\frac{F2}{g1}\right)$$

$$dif1 := 1/2\,\ln(1-\nu)+\ln(3)-1/2\,\ln(2)$$

$$dif2 := 1/2\,\ln\left(\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}\right)$$

$$dif3 := \ln\left(\frac{F2}{g1}\right)$$

$$true$$

```
> ddif1 := diff(dif1,nu);
```

$$ddif1 := -(2-2\,\nu)^{-1}$$

```
> # For dif3, use dlc
> ddif3 := subs(dcc=dccsimple,dlcval);
```

$$ddif3 := g1\,\left(2\,\frac{F1}{g1}-\frac{F2\;ela}{g1^2}\right) dlambda F2^{-1}$$

```
> # For dif2, use dld with ddd1simple and ddd2simple
> dd;
> ddif2a := subs(ddd=ddd1simple, (1/2)*dldval);
> ddif2b := subs(ddd=ddd2simple, (1/2)*dldval);
```

$$\frac{mn^2 g1^3}{g2^2 f1\,\lambda^3}$$

$$ddif2a := -\frac{R}{mn\,\nu^2}$$

$$ddif2b := -3/2\,g1\,\left(2\,\frac{F1}{g1}-\frac{F2\;ela}{g1^2}\right) dlambda \lambda^{-1} f1^{-1} g2^{-1}$$

```
> Bval := 2*F1-F2*ela/g1;
> Aval := 1/lambda+ela/f1+ela/g2-2*ela^2/F2+2/F2;
> ss(subsf(Aval),3);
> ss(subsf(Bval),3);
```

$$Bval := 2\,F1-\frac{F2\;ela}{g1}$$

$$Aval := \lambda^{-1}+\frac{ela}{f1}+\frac{ela}{g2}-2\,\frac{ela^2}{F2}+2\,F2^{-1}$$

$$O(\lambda)$$

$$4\,\lambda+3\,\lambda^{2}+O\left(\lambda^{3}\right)$$

```
> # Second derivative
> ddif := ddif1+ddif2a+ddif2b+ddif3;
> ddifsimple := dd1+dd2+dd3;
> dd2val := (3/2)*B*R/(g2*f1*lambda*nu^2*mn*A);
> dd1val := -1/(2*(1-nu))-R/(mn*nu^2);
> dd3val := - B*R/(F2*nu^2*mn*A);
> is(simplify(subs(dlambda=dlambdasimple, ddif)-subs(dd1=dd1val,dd2=dd2val,
dd3=dd3val, B=Bval, A=Aval,ddifsimple)),0);
> # Series at lambda=0
> ss(ddif,8);
```

$$ddif\ :=\ -\left(2-2\,\nu\right)^{-1}-\frac{R}{mn\,\nu^{2}}-3/2\,g1\,\left(2\,\frac{F1}{g1}-\frac{F2\ ela}{{g1}^{2}}\right)dlambda$$
$$\lambda^{-1}{f1}^{-1}{g2}^{-1}$$
$$+\,g1\,\left(2\,\frac{F1}{g1}-\frac{F2\ ela}{{g1}^{2}}\right)dlambda{F2}^{-1}$$

$$ddifsimple\ :=\ dd1+dd2+dd3$$

$$dd2val\ :=\ 3/2\,\frac{B\,R}{g2\,f1\,\lambda\,\nu^{2}mn\,A}$$

$$dd1val\ :=\ -\left(2-2\,\nu\right)^{-1}-\frac{R}{mn\,\nu^{2}}$$

$$dd3val\ :=\ -\frac{B\,R}{F2\,\nu^{2}mn\,A}$$

true

$$-1-2/3\,\lambda-\frac{23}{60}\,\lambda^{2}+O\left(\lambda^{3}\right)$$

```
> # Third derivative
> dd1val;
> subs(mn=mdefn,dd1val);
> ddd1 := diff(%,nu);
> ss(%,6);
```

$$-\left(2-2\,\nu\right)^{-1}-\frac{R}{mn\,\nu^{2}}$$

$$- (2 - 2\,\nu)^{-1} - R \left(1/2 + \frac{R}{\nu}\right)^{-1} \nu^{-2}$$

$$ddd1 := -2\,(2 - 2\,\nu)^{-2} - R^2 \left(1/2 + \frac{R}{\nu}\right)^{-2} \nu^{-4} + 2\,R \left(1/2 + \frac{R}{\nu}\right)^{-1} \nu^{-3}$$

$$1/3\,\lambda^{-1} + O(1)$$

```
> dd2val;
> dd2val2 := simplify(subs(mn=mdefn,%));
> dd2a := 3*R/(nu*(nu+2*R));
> dd2b := B/(g2*f1*lambda*A);
> is(simplify(dd2val2-dd2a*dd2b),0);
> ddd2 := diff(dd2a,nu)*subsf(subs(A=Aval, B=Bval,dd2b))+
dd2a*diff(subsf(subs(A=Aval,B=Bval,dd2b)),lambda)*dlambdaval:
> ss(ddd2,9);
```

$$3/2\,\frac{B R}{g2\,f1\,\lambda\,\nu^2 mn\,A}$$

$$dd2val2 := 3\,\frac{B R}{\nu\,g2\,f1\,\lambda\,(\nu + 2\,R)\,A}$$

$$dd2a := 3\,\frac{R}{\nu\,(\nu + 2\,R)}$$

$$dd2b := \frac{B}{g2\,f1\,\lambda\,A}$$

$$true$$

$$-2\,\lambda^{-2} + 1/3\,\lambda^{-1} + O(1)$$

```
> dd3val;
> dd3val2 := simplify(subs(mn=mdefn,%));
> dd3a := -2*R/(nu*(nu+2*R));
> dd3b := B/(F2*A);
> is(simplify(dd3val2-dd3a*dd3b),0);
> ddd3 := diff(dd3a,nu)*subsf(subs(A=Aval, B=Bval,dd3b))+
dd3a*diff(subsf(subs(A=Aval,B=Bval,dd3b)),lambda)*dlambdaval:
> ss(ddd3,9);
```

$$-\frac{B R}{F2\,\nu^2 mn\,A}$$

$$dd3val2 := -2\,\frac{B\,R}{\nu\,F2\,(\nu + 2\,R)\,A}$$

$$dd3a := -2\,\frac{R}{\nu\,(\nu + 2\,R)}$$

$$dd3b := \frac{B}{F2\,A}$$

$$true$$

$$2\,\lambda^{-2} + 1/3\,\lambda^{-1} + O\left(1\right)$$

```
> # Series for third derivative at maximum
> ss(ddd1+ddd2+ddd3,10);
```

$$\lambda^{-1} + O\left(\lambda\right)$$

Index