On Causal Video Coding with Possible Loss of the First Encoded Frame

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Multiple Description Coding (MDC) was first formulated by A. Gersho and H. Witsenhausen as a way to improve the robustness of telephony links to outages. Lots of studies have been done in this area up to now. Another application of MDC is the transmission of an image in different descriptions. If because of the link outage during transmission, any one of the descriptions fails, the image could still be reconstructed with some quality at the decoder side. In video coding, inter prediction is a way to reduce temporal redundancy. From an information theoretical point of view, one can model inter prediction with Causal Video Coding (CVC). If because of link outage, we lose any I-frame, how can we reconstruct the corresponding P- or B-frames at the decoder? In this thesis, we are interested in answering this question and we call this scenario as causal video coding with possible loss of the first encoded frame and we denote it by CVC-PL as PL stands for possible loss.

In this thesis for the first time, CVC-PL is investigated. Although, due to lack of time, we mostly study two-frame CVC-PL, we extend the problem to M-frame CVC-PL as well. To provide more insight into two-frame CVC-PL, we derive an outer-bound to the achievable rate-distortion sets to show that CVC-PL is a subset of the region combining CVC and peer-to-peer coding. In addition, we propose and prove a new achievable region to highlight the fact that two-frame CVC-PL could be viewed as MDC followed by CVC. Afterwards, we present the main theorem of this thesis, which is the minimum total rate of CVC-PL with two jointly Gaussian distributed sources, i.e. X_1 and X_2 with normalized correlation coefficient r, for different distortion profiles (D_1, D_2, D_3). Defining $D_r = r^2(D_1 - 1) + 1$, we show that for small D_3, i.e. $D_3 \leq D_r + D_2 - 1$, CVC-PL could be treated as CVC with two jointly Gaussian distributed sources; for large D_3, i.e. $D_3 \geq \frac{D_r D_2}{D_r + D_2 - D_r D_2}$, CVC-PL could be treated as two parallel peer-to-peer networks with distortion constraints D_1 and D_2; and for the other cases of D_3, the minimum total rate is $\frac{1}{2} \log \frac{(1+\lambda)(D_3+\lambda)}{(D_r+\lambda)(D_2+\lambda)} + \frac{1}{2} \log \frac{D_r}{D_1 D_3}$ where $\lambda = \frac{D_3 - D_r D_2 + r\sqrt{(1-D_1)(1-D_2)(D_3-D_r)(D_3-D_2)}}{D_r + D_2 - (D_3+1)}$.

We also determine the optimal coding scheme which achieves the minimum total rate. We conclude the thesis by comparing the scenario of CVC-PL with two frames with a coding scheme, in which both of the sources are available at the encoders, i.e. distributed source coding versus centralized source coding. We show that for small D_2 or large D_3, the distributed source coding can perform as good as the centralized source coding. Finally, we talk about future work and extend and formulate the problem for M sources.

Acknowledgements

I would like to express my most sincere thanks and deepest gratitude to my mother, Narges Nateghi and to my brother, Behrad Eslamifar for their infinite love and continuous support and dedication. I would never have had the chance to follow my dreams without their support. To them, I owe my entire achievements. I would like to thank my supervisor, Prof. En-Hui Yang, for his guidance, kind support and encouragement throughout my graduate studies. This thesis would have not been possible without his help and support. My thanks and appreciation also go to Dr. Xiang Yu for his invaluable help throughout this research. I would also like to thank my committee members, Prof. Liang-Liang Xie and Prof. Zhou Wang for their valuable suggestions on my thesis.

Last but not least, I would like to thank all of my dear friends, Nasim Ramezani, Farhad Haghighizadeh and Yueming Gao who enriched my life beyond my studies. I have been lucky to have such a good friends by my side.

Dedication

To my mother, *Narges Nateghi*, for her unconditional love, support and encouragement.

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Chapter 1

Introduction

1.1 Motivations and Objectives

In Multiple Description Coding (MDC), as shown in Fig. (1.1), two descriptions of a given source, i.e. R_1 and R_2, are sent out through three channels. The first decoder with distortion constraint D_1 receives R_1, the second decoder with distortion constraint D_2 receives R_2 and the third decoder reproduces a reconstructed frame given R_1 and R_2, with distortion constraint D_3. MDC has different applications, but it is more interesting to us when we compare it with progressive coding, in which we send out M encoded packets of a given source. In this case, we cannot reproduce the reconstructed source unless we receive all the packets in the specific order. What if because of network conditions we lose one or more packets? If real-time decoding is not a necessary condition, we can just wait for the lost packets to be retransmitted, but what if we would like to have real-time communication. Here is when MDC comes into mind, in which having received any non-empty subset of encoded packets, we can reproduce a reconstructed frame [6].

From an information theoretical point of view, lots of studies have been done considering different scenarios facing MDC. In 1979, El-Gamal and Cover studied the achievable region for MDC and presented it in a Shannon Theory workshop. Although Witsenhausen's achievable region in [16] is a subspace of El-Gamal and Cover's region, Ozarow in [8] proved that this distortion-rate region for Gaussian distributed source is tight. MDC with Binary Symmetric Source (BSS) has been of the interest for researchers from an information theoretical point of view as well. Wolf, Wyner and Ziv, studied the achievable region for BSS [17] and in [20], Zhang and Berger elaborated on these results and derived tight lower bounds for some portions of the El-Gamal and Cover's achievable rate-distortion region.

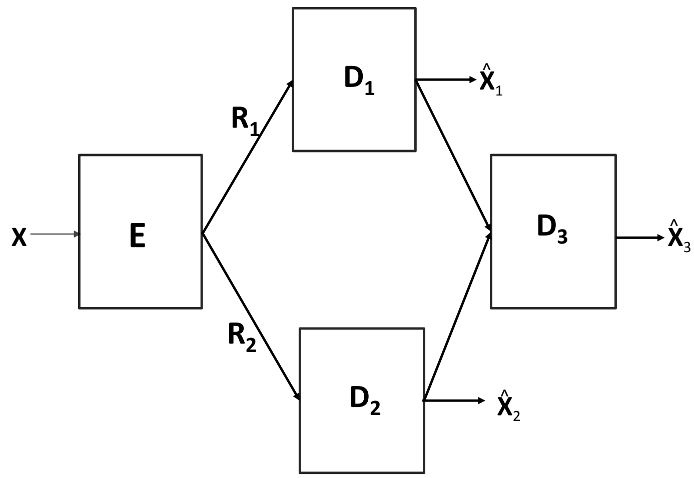

Figure 1.1: Mutliple description coding.

Researchers have studied different configurations of MDC as well. In [19], they considered a configuration in which all the decoders and the encoder have the same side information. In [15], the Wyner-Ziv setting of MDC was posed. A special case of MDC is two-layer Scalable Video Coding (SVC), in which we just consider distortion constarints D_1 and D_3 and since D_2 is not of the interest, we remove the second decoder. In [4], a two-layer SVC was considered and the necessary and sufficient conditions to have no excess rate condition at each layer were derived. In [9], Rimoldi derived the tight achievable region of two-layer SVC. It is easy to see that El-Gamal and Cover's achievable region in [5] for MDC could be reduced to Rimoldi's achievable region in [9].

In Causal Video Coding (CVC), we encode a given frame using all the previous frames and the previous encoded frames and we reproduce a reconstructed frame using all the encoded frames at the current and the previous layers. In Fig. (1.2), we sketch a CVC with three correlated frames. For example, one can see that at the second layer, X_2 is encoded using X_1 and the encoded frame at the first layer, and the reconstructed frame $\hat{X}_2$ is reproduced using the encoded frames at the first and the second layers.

An extension of CVC, is Scalable Causal Video Coding (SCVC), in which at each layer we have two pairs of encoders and decoders. One pair uses all of the previous encoded

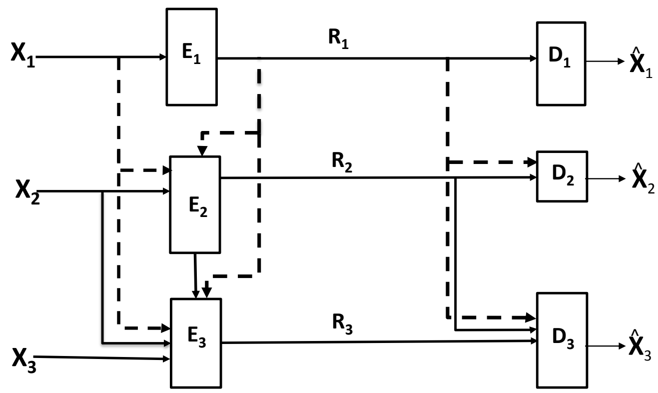

Figure 1.2: A three-frame causal video coding. One can see that at each layer, X_k is encoded using all the previous frames and all the previous encoded frames.

frames (the same as CVC) and the other pair uses just a subset of the previous encoded frames, which is a set of the encoded frames having been encoded by the first encoders at each layer. In Fig (1.3), as an example, we show a SCVC with three frames. If we just have one single frame, SCVC is reduced into SVC.

CVC is a general case in video communication [18]. If because of network conditions, we lose the first encoded frame, the second decoder should be able to reconstruct the second frame with a given distortion constraint. This scenario is not only very interesting but also could happen a lot in telecommunication. We refer to the scenario of causal video coding with possible loss of the first encoded frame by CVC-PL as PL stands for possible loss. In Fig. (1.4), we show a two-frame CVC-PL, in which the first encoded frame is available at the third decoder but not at the second decoder. One can see that two-frame CVC-PL is a combination of CVC and MDC. In [14], the authors presented an achievable region for multi-stage sequential coding with correlated sources. In this thesis for the first time, CVC-PL is investigated. Although, due to lack of time, we mostly study two-frame CVC-PL, we extend the problem to M-frame CVC-PL as well.

In this thesis, we are interested in CVC-PL with two jointly Gaussian distributed

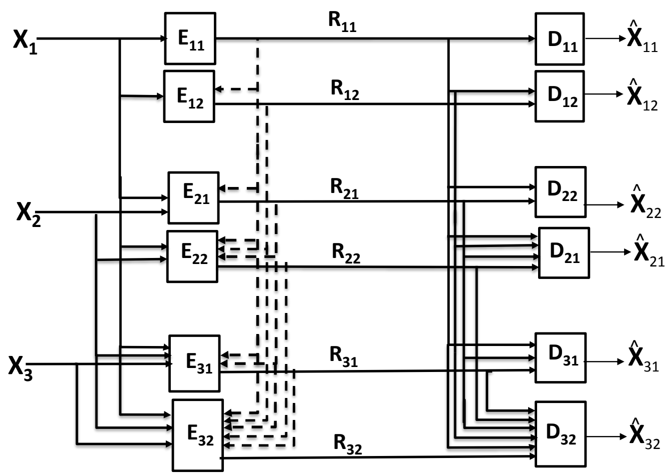

Figure 1.3: A three-frame scalable causal video coding. One can see that at each layer, there are two pairs of encoders and decoders. One pair uses all the previous encoded frames and the other pair uses just a subset of the previous encoded frames.

sources. The final goal is to derive the minimum total rate for different distortion profiles and to study the best coding scheme to achieve the minimum total rate. To find the best coding scheme, we refer to the achievable region in [14], which is not a tight region but we show that using this region, we could achieve the outer-bound on the achievable total rates. There are different approaches to show that an inner-bound and an outer-bound of the total rate are tight. In up-bottom approach, people present a coding scheme and show that the total rate using this coding scheme is equal to its outer-bound. In [13], Wang and Viswanath, used a bottom-up approach in which they built a coding scheme such that the inner-bound and the outer-bound became tight. A closed form expression for the minimum total rate gives us a very good insight into CVC-PL. For the purpose of comparison, we consider a centralized source coding in which both sources are available at both encoders. Without loss of generality, we present a coding scheme and derive its total rate to see the difference between the presented centralized source coding and CVC-PL as a distributed source coding.

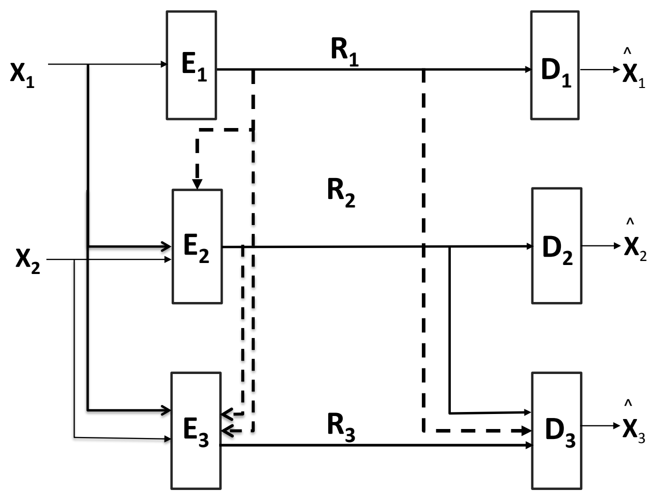

Figure 1.4: A two-frame CVC-PL. As it is shown, the second frame is encoded such that it can be decoded if the first encoded frame may or may not be available at the second decoder.

1.2 Thesis Organization

The thesis is organized as follows. Chapter 2 presents a literature survey on multiple description coding and causal video coding, followed by a review of the previous research proposed on two-frame CVC-PL. Chapter 3 presents and proves the minimum total rate of CVC-PL with two jointly Gaussian distributed sources. Finally, Chapter 4 summarizes and concludes the thesis and provides recommendations for future work.

Chapter 2

Background

The main components of this research are multiple description coding and causal video coding. In this chapter, background information required to understand later chapters is described. In the first section, we present some notions and definitions that will be used throughout this thesis. In the next section, we talk about rate-distortion region of a given peer-to-peer scenario and regarding this region, we study rate-distortion theorem. Afterwards, we consider multiple description coding and we talk about the previous research works accomplished in the area of multiple description coding. We explain causal video coding for M correlated sources in section 2.4 and we elaborate more on jointly Gaussian distributed sources. Finally, we present and formulate the scenario of two-frame CVC-PL.

2.1 Notions and Definitions

We show an n-order frame X as **X**, where

$$\mathbf{X} = \{X_1, ..., X_2\}$$

and X_i is a random variable from a sample space $\mathcal{X}_i$ at time i. We say **X** is an IID source if and only if,

$$p_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} p(X_i)$$

If source **X** is an IID source, then we use X as a general random variable at any time. Note that throughout this thesis, we always assume that all the sources are IID. The pdf of an IID, zero-mean Gaussian distributed source $\mathbf{X}$ is defined as follows

$$p_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma}} \exp -\frac{x_i^2}{2\sigma^2}$$

and is presented by $X \sim \mathbf{N}(0, \sigma^2)$. The distortion between X and $\hat{X}$ is measured by a single-letter distortion measure $\delta : \mathcal{X} \times \hat{\mathcal{X}} \to [0, \infty)$. We measure the distortion between $\mathbf{X}$ and $\hat{\mathbf{X}}$ as the arithmetic average distortion per symbol,

$$d(\mathbf{X}, \hat{\mathbf{X}}) = \frac{1}{n} \Sigma_{i=1}^{n} \delta(X_i, \hat{X}_i)$$

and we call d as a single-letter fidelity criterion. There are different single-letter distortion measures used in different problems. For a discrete source, a very well-used distortion measure is *hamming distance*,

$$\delta(X_i, \hat{X}_i) = \begin{cases} 1, & \text{if } X_i \neq \hat{X}_i \\ 0, & \text{if } X_i = \hat{X}_i \end{cases}$$

Considering a continuous source, *squared error* is the most-used single-letter distortion measure,

$$\delta(X_i, \hat{X}_i) = |X_i - \hat{X}_i|^2$$

From an information theoretical point of view, we are mostly interested in average distortion which is the expectation of single-letter fidelity criterion, i.e. $E_{(X,\hat{X})}[d(\mathbf{X}, \hat{\mathbf{X}})]$.

Encoding Function and Decoding Function

A $(|S|, n)$ code consists of an encoding function f which maps an n-order frame $\mathbf{X}$ to a finite-length sequence of 0s and 1s, i.e. S, and a decoding function g which maps S to an n-order reconstructed frame $\hat{\mathbf{X}}$, as follows

$$f(\mathbf{X}) = S$$
$$g(f(\mathbf{X})) = \hat{\mathbf{X}}$$

where $|S|$, is the length of sequence S.

Strongly Typical Sequence [3]

Let $p_X(x)$ be a probability distribution of an IID source X. For any $\epsilon \geq 0$, a sequence of $\mathbf{x}$ is said to be $\epsilon -$ *strongly typical* if

$$\begin{aligned} 1 \quad &. \quad |\frac{1}{n} N(a|\mathbf{x}) - p(a)| \leq \epsilon \\ 2 \quad &. \quad \forall a \in \mathcal{X} \ \text{ with } \ p(a) = 0, N(a|\mathbf{x}) = 0 \end{aligned} \tag{2.1}$$

where $N(a|\mathbf{x})$ is the number of occurrences of the symbol a in the sequence $\mathbf{x}$. The set of $\epsilon -$ *strongly typical* sequences $\mathbf{x}$ with respect to a distribution $p(x)$, is called the strongly typical set and is denoted by $A_{\epsilon}^{*(n)}(X)$ or $A_{\epsilon}^{*(n)}$.

Strongly Jointly Typical Pair of Sequences [3]
A pair of sequences $(\mathbf{x}, \mathbf{y})$ is said to be $\epsilon -$ *strongly jointly typical* with respect to a distribution $p(x, y)$ if

$$\begin{aligned} 1 \quad . \quad & |\frac{1}{n}N(a,b|\mathbf{x},\mathbf{y}) - p(a,b)| \leq \epsilon \\ 2 \quad . \quad & \forall (a,b) \in (\mathcal{X},\mathcal{Y}) \ \text{ with } \ p(a,b) = 0, N(a,b|\mathbf{x},\mathbf{y}) = 0 \end{aligned} \tag{2.2}$$

Where $N(a, b|\mathbf{x}, \mathbf{y})$ is the number of occurrences of the symbols (a, b) in the sequence $(\mathbf{x}, \mathbf{y})$. The set of $\epsilon-$*strongly jointly typical* sequences $(\mathbf{x}, \mathbf{y})$ with respect to a distribution $p(x, y)$, is called the strongly jointly typical set and is denoted by $A_{\epsilon}^{*(n)}(X, Y)$ or $A_{\epsilon}^{*(n)}$. From the definition in (2.2), it follows that if a pair of sequence is $\epsilon -$ *strongly jointly typical*, i.e. $(\mathbf{x}, \mathbf{y}) \in A_{\epsilon}^{*(n)}(X, Y)$, then $\mathbf{x} \in A_{\epsilon}^{*(n)}(X)$ and $\mathbf{y} \in A_{\epsilon}^{*(n)}(Y)$.

2.2 Classical Rate-Distortion Region

A classical peer-to-peer compression network is depicted in Fig. (2.1). Given an IID source X, we are interested in the minimum compression rate R that guarantees distortion constraint D, and we denote it by rate-distortion function, $R(D)$. We present the classical rate-distortion theory in the following but before that we present rate-distortion region.

A rate-distortion pair (R, D) is said to be achievable if and only if for any $\epsilon > 0$, with n sufficiently large, there exist encoding and decoding functions f and g such that

$$\log \frac{E|S|}{n} \leq R + \epsilon$$
$$E[d(\mathbf{X}, \hat{\mathbf{X}})] \leq D + \epsilon$$

where $|S|$ denotes the length of S. The rate-distortion region for a source is the set of all the achievable rate-distortion pairs, (R, D).

The rate-distortion function $R(D)$ is the infimum of rates R such that (R, D) is achievable. $R(D)$ specifies the minimum rate at which one must receive information about the source output in order to be able to reproduce it with an average distortion that does not exceed D [2]. Now we are ready to present rate-distortion theory,

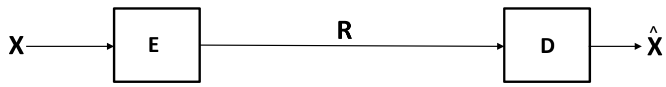

Figure 2.1: A classical peer-to-peer compression network.

Theorem 1 *[2]*
Given an IID source $X \sim p_X(x)$, and distortion measure $\delta(X, \hat{X})$, the rate-distortion function $R(D)$ is,

$$R(D) = \inf_{p_{\hat{X}|X}(\hat{x}|x): E[d(X,\hat{X})] \leq D} I(X; \hat{X})$$

where the infimum is over all the conditional distributions $p_{\hat{X}|X}(\hat{x}|x)$ for which the corresponding joint distributions $p_{X,\hat{X}}(x, \hat{x}) = p_{\hat{X}|X}(\hat{x}|x)p_X(x)$ satisfy distortion constraint D.

Having talked about the classical rate-distortion theory, in the following we first derive $R(D)$ for a zero-mean Gaussian distributed source and then we talk about Multiple Description Coding (MDC) and Causal Video Coding (CVC).

Example 1
Given an IID source $X \sim \mathbf{N}(0, \sigma^2)$, we are interested in $R(D)$,

$$\begin{aligned}
I(X;\hat{X}) &= h(X) - h(X|\hat{X}) \\
&= h(X) - h(X - \hat{X}|\hat{X}) \\
&\overset{(a)}{\geq} h(X) - h(X - \hat{X}) \\
&\overset{(b)}{\geq} \frac{1}{2}\log(2\pi e \sigma^2) - \frac{1}{2}\log(2\pi e E[(X - \hat{X})^2]) \\
&\overset{(c)}{\geq} \frac{1}{2}\log(2\pi e \sigma^2) - \frac{1}{2}\log(2\pi e D) \\
&= \frac{1}{2}\log\frac{\sigma^2}{D}
\end{aligned}$$

where (a) is because of the fact that condition reduces entropy, (b) is because of the fact that Gaussian distributed random variable maximizes entropy for a given variance, and (c) comes from the problem assumption, i.e. $E[(X - \hat{X})^2] \leq D$ and it happens if $\hat{X}$ is Gaussian distributed such that $(X - \hat{X}) \sim \mathbf{N}(0, D)$. Hence, for all $D \leq \sigma^2$,

$$\inf_{p_{\hat{X}|X}(\hat{x}|x):E[d(X,\hat{X})] \leq D} I(X;\hat{X}) = \frac{1}{2}\log\frac{\sigma^2}{D}$$

If $D > \sigma^2$, we do not need to send any information to the decoder, and we could reconstruct $\hat{X} = E[X] = 0$. In this case, $E[(X - \hat{X})^2] = \sigma^2$. In conclusion, we could derive $R(D)$ as follows,

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D} & \text{if } \sigma^2 \geq D \\ 0 & \text{if } \sigma^2 < D \end{cases}$$

2.3 Multiple Description Coding (MDC)

In this section, we first formulate MDC. Afterwards, we talk about the main results in the literature for this scenario. Considering MDC as in Fig. (1.1), where X represents a video frame, S_k and $\hat{X}_k$ represent respectively its encoded frame and reconstructed frame at encoder and decoder k, there are two encoder functions (f_1, f_2) and three decoder functions (g_1, g_2, g_3) as follows,

$$\begin{aligned}
f_1(\mathbf{X}) &= S_1 \\
f_2(\mathbf{X}) &= S_2
\end{aligned}$$

and

$$\begin{aligned} g_1(f_1(\mathbf{X})) &= \hat{\mathbf{X}}_\mathbf{1} \\ g_2(f_2(\mathbf{X})) &= \hat{\mathbf{X}}_\mathbf{2} \\ g_3(f_1(\mathbf{X}), f_2(\mathbf{X})) &= \hat{\mathbf{X}}_\mathbf{3} \end{aligned}$$

where S_1 and S_2 are the finite-length sequences of 0s and 1s. Having considered this formulation, El-Gamal and Cover in [5] derived an achievable region for MDC,

Theorem 2 *[5] Given an IID source X, the set of $(R_1, R_2, D_1, D_2, D_3)$ is achievable if there exist auxiliary random variables $(\hat{X}_1, \hat{X}_2, \hat{X}_3)$ jointly distributed with generic source random variable X such that*

$$\begin{aligned} R_1 &\geq I(X; \hat{X}_1) \\ R_1 &\geq I(X; \hat{X}_2) \\ R_1 + R_2 &\geq I(X; \hat{X}_1 \hat{X}_2 \hat{X}_3) + I(\hat{X}_1; \hat{X}_2) \end{aligned}$$

and

$$\begin{aligned} E[d(X, \hat{X}_1)] &\leq D_1 \\ E[d(X, \hat{X}_2)] &\leq D_2 \\ E[d(X, \hat{X}_3)] &\leq D_3 \end{aligned} \tag{2.3}$$

In [16], Witsenhausen attributed to El Gamal and Cover the following antecedent of the theorem in (2.3) [20].

Theorem 3 *[16]*
Given an IID source X, the set of $(R_1, R_2, D_1, D_2, D_3)$ is achievable if there exist auxiliary random variables (U_1, U_2) jointly distributed with generic source random variable X and functions (g_1, g_2, g_3) such that

$$\begin{aligned} R_1 &\geq I(X; U_1) \\ R_1 &\geq I(X; U_2) \\ R_1 + R_2 &\geq I(X; U_1 U_2) + I(U_1; U_2) \end{aligned}$$

and

$$\begin{aligned} E[d(X, f_1(U_1))] &\leq D_1 \\ E[d(X, f_2(U_2))] &\leq D_2 \\ E[d(X, f_3(U_1, U_2))] &\leq D_3 \end{aligned} \tag{2.4}$$

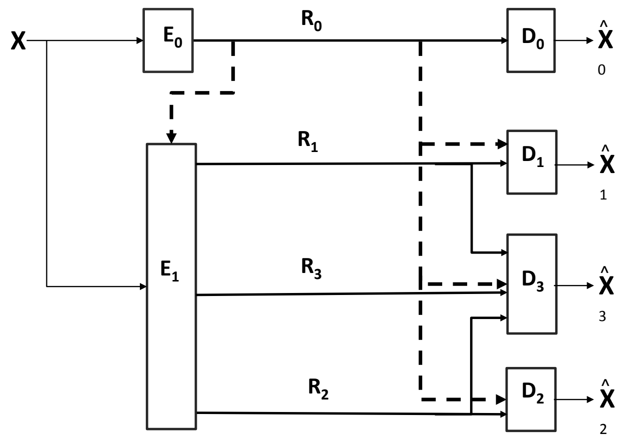

Figure 2.2: Multiple description coding with four layers.

It is easy to see that the region in (2.4) is the subspace of the region in (2.3). But in [8], Ozarow showed that the distortion-rate region in (2.4) for MDC with IID Gaussian distributed source X is tight. To get more elaborated on Ozarow's distortion-rate region, we extend the problem as in Fig. (2.2) in which we add one more layer on the top of the all the layers, i.e. (R_0, D_0), and we prove the following theorem. The proof is shown in Appendix A,

Theorem 4 *Given an IID source X, the set of $(R_0, R_1, R_2, R_3, D_0, D_1, D_2, D_3)$ is achievable if and only if*
CASE 1: *If $D_1 + D_2 < D_0(1 + \exp(-2(R_1 + R_2)))$,*

$$\begin{aligned} D_0 &\geq \exp(-2R_0) \\ D_1 &\geq \exp(-2(R_0 + R_1)) \\ D_2 &\geq \exp(-2(R_0 + R_2)) \\ D_3 &\geq \exp(-2(R_0 + R_1 + R_2 + R_3)) \frac{1}{1 - (\sqrt{\Pi} - \sqrt{\Delta})^2} \end{aligned}$$

$$\Delta = \bar{D}_1 \bar{D}_2 - \exp(-2(R_1 + R_2))$$

$$\Pi = (1 - \bar{D_1})(1 - \bar{D_2})$$

where $\bar{Y} = Y/\exp(-2R_0)$.
CASE 2: *If* $D_1 + D_2 \geq D_0(1 + \exp(-2(R_1 + R_2)))$,

$$\begin{aligned} D_0 &\geq \exp(-2R_0) \\ D_1 &\geq \exp(-2(R_0 + R_1)) \\ D_2 &\geq \exp(-2(R_0 + R_2)) \\ D_3 &\geq \exp(-2(R_0 + R_1 + R_2 + R_3)) \end{aligned}$$

Besides MDC with Gussian distributed source, another special case of MDC was considered in [1], where the author called it "no excess rate" condition by imposing $R_1 + R_2 = R(D_3)$. In this case, it was shown that the El-Gamal Cover's region in (2.3) is tight. But without "no excess rate" condition, Zhang and Berger in [21] showed that the region is not tight for the very special binary symmetric case. In [13], Wang and Viswanath, generalized the problem of MDC with a scalar Gaussian distributed source to the case with a vector Gaussian distributed source. They showed that in this general case, the optimum total rate could be achieved if and only if the auxiliary random variables introduced in [11] are jointly Gaussian distributed.

Researchers have studied different configurations of MDC as well. In [19], they considered a configuration in which all the decoders and the encoder have the same side information. In [15], the Wyner-Ziv setting of MDC was posed. A special case of MDC is two-layer Scalable Video Coding (SVC) shown in Fig. (2.3), in which we just consider distortion constarints D_1 and D_3 and since D_2 is not of the interest, we remove the second decoder. In [4], a two-layer SVC was considered and the necessary and sufficient conditions to have no excess rate condition at each layer, i.e. $R(D_1)$ and $R(D_2)$, were derived. In this case, SVC is called "successive refinement of information". In [9], Rimoldi derived the tight achievable region of two-layer SVC. It is easy to see that El-Gamal and Cover's achievable region in [5] for MDC could be reduced to Rimoldi's achievable region in [9]. M-layer SVC is an extension to the H.264 codec standard and is used by most of today's video conferencing devices. SVC allows video conferencing devices to send and receive multi-layered video streams composed of a small base layer and optional additional layers that enhance resolution, frame rate and quality [7]. Generally speaking, in SVC, at each layer we are interested in reconstructing the same source in different resolutions by slightly sending more bits to the decoder.

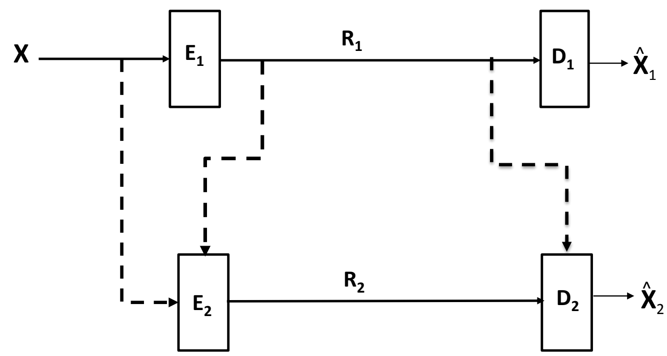

Figure 2.3: A two-layer scalable video coding.

2.4 Causal Video Coding for Correlated Sources

Causal Video Coding (CVC) for M correlated sources, shown in Fig. (1.2), consists of M encoders and M decoders, i.e. (f_i, g_i) for $i = 1, ..., M$. At layer $k = 1, ..., M$, encoding function f_k uses $(\mathbf{X_1}, ..., \mathbf{X_{k-1}}, S_1, ..., S_{k-1})$ as the side information to encode $\mathbf{X_k}$ to S_k,

$$f_k(\mathbf{X_1}, ..., \mathbf{X_k}, S_1, ..., S_{k-1}) = S_k$$

where S_k is a finite-length sequence of 0s and 1s. At decoder k, $(S_1, ..., S_k)$ are decoded to $\mathbf{\hat{X}_k}$ using decoding function g_k,

$$g_k(S_1, ..., S_{k-1}, S_k) = \mathbf{\hat{X}_k}$$

When $M = 2$, the causal coding model is the same as the sequential coding model of correlated source proposed in [12]. However, when $M > 2$, which is a typical case in MPEG-series and H-series video coding, the causal coding model considered here is quite different from sequential coding [18]. Visamanthan and Berger in [12] first studied sequential coding for two correlated sources, derived an achievable region and showed that the region is tight. In [18], the authors extended this achievable region for CVC of M correlated sources as follows,

Theorem 5 *[18] Given IID sources* $(X_1, ..., X_M)$*, the set of* $(R_1, ..., R_M, D_1, ..., D_M)$ *is achievable if and only if there exist auxiliary random variables* $(U_1, ..., U_M)$ *jointly distributed with sources* $(X_1, ..., X_M)$ *and functions* $(g_1, ..., g_M)$ *such that*

$$\begin{aligned} R_i &\geq I(X_1, ..., X_i; U_i | U_1, ..., U_{i-1}) \qquad i = 1, ..., M-1 \\ R_M &\geq I(X_M; U_M | U_1, ..., U_{M-1}) \end{aligned}$$

and

$$E[d(X_i, g_i(U_1, ..., U_i))] \leq D_i \qquad i = 1, ..., M$$

and

$$\begin{aligned} U_i &\rightarrow (X_1, ..., X_i, U_1, ..., U_{i-1}) \rightarrow (X_{i+1}, ..., X_M) \qquad i = 1, ..., M-1 \\ U_M &\rightarrow (X_M, U_1, ..., U_{M-1}) \rightarrow (X_1, ..., X_{M-1}) \end{aligned}$$

In addition, they derived the minimum total rate and presented an iterative algorithm to compute the minimum total rate.

Using the results in [18], the authors in [10], considered three jointly Gaussian distributed sources and derived a closed-form expression for the minimum total rate. Based on the results in [10], it is easy to show that the minimum total rate for a sequential coding of two jointly zero-mean Gaussian distributed sources, (X_1, X_2) is,

$$R_1 + R_2 \geq \frac{1}{2} \log \frac{1}{D_1} + \frac{1}{2} \log \frac{r^2 D_1 + \sigma_Z^2}{D_2} \tag{2.5}$$

where

$$X_2 = rX_1 + Z$$

and $X_1 \sim \mathbf{N}(0, 1)$, $X_2 \sim \mathbf{N}(0, 1)$, r is the normalized correlation coefficient of X_1 and X_2, $Z \sim \mathbf{N}(0, \sigma_Z^2)$ and is independent of X_1.

2.5 Causal Video Coding with Possible Loss of the First Encoded Frame (CVC-PL)

CVC is a general case in video communication [18]. If because of network conditions, we lose the first encoded frame, the second decoder should be able to reconstruct the second frame with a given distortion constraint. This scenario is not only very interesting but also could happen a lot in telecommunication. We refer to the scenario of causal video coding with possible loss of the first encoded frame by CVC-PL as PL stands for possible loss.

In Fig. (1.4), we show a two-frame CVC-PL, in which the first encoded frame is available at the third decoder but not at the second decoder. One can see that two-frame CVC-PL is a combination of CVC and MDC. In [14], the authors presented an achievable region for multi-stage sequential coding with correlated sources. In this thesis for the first time, CVC-PL is investigated. Although, due to lack of time, we mostly study two-frame CVC-PL, we extend the problem to M-frame CVC-PL as well. In the rest of this chapter, we first formulate a two-frame CVC-PL. In addition, we present and prove the achievable region derived in [14].

Considering CVC-PL with two frames as shown in Fig. (1.4), where $\mathbf{X}_1$ and $\mathbf{X}_2$ represent two video frames, S_k and $\hat{\mathbf{X}}_\mathbf{k}$ represent respectively encoded frame and reconstructed frame at layer k, an n-order two-frame CVC-PL is a frame-by-frame coding scheme, modeled as follows,

First Layer $\mathbf{X}_1$ is encoded using function f_1 to S_1 which is a finite-length sequence of 0s and 1s, i.e. $f_1(\mathbf{X}_1) = S_1$, and S_1 is decoded to $\hat{\mathbf{X}}_1$ using decoding function g_1, i.e. $g_1(S_1) = \hat{\mathbf{X}}_1$.

Second Layer Encoding function f_2 uses $(\mathbf{X}_1, S_1)$ as the side information to encode $\mathbf{X}_2$ to S_2,

$$f_1(\mathbf{X}_1, \mathbf{X}_2, S_1) = S_2$$

S_2 is decoded to $\hat{\mathbf{X}}_2$ using decoding function g_2,

$$g_2(S_2) = \hat{\mathbf{X}}_2$$

Third Layer In the last layer, encoding function f_3 uses all the previous encoded sequences and $\mathbf{X}_1$ as the side information to encode $\mathbf{X}_2$ to S_3,

$$f_3(\mathbf{X}_1, \mathbf{X}_2, S_1, S_2) = S_3$$

S_3 is decoded to $\hat{\mathbf{X}}_3$ using decoding function g_3,

$$g_3(S_1, S_2, S_3) = \hat{\mathbf{X}}_3$$

Achievability Definition
A set of $(R_1, R_2, R_3, D_1, D_2, D_3)$ is achievable if $\forall \epsilon > 0$ with n sufficiently large we have

$$\begin{aligned} \log \frac{E|S_i|}{n} &\leq R_i + \epsilon \qquad\qquad i = 1, 2, 3 \\ E[d(\mathbf{X_1}, \hat{\mathbf{X}}_\mathbf{1})] &\leq D_1 + \epsilon \\ E[d(\mathbf{X_2}, \hat{\mathbf{X}}_\mathbf{j})] &\leq D_j + \epsilon \qquad\qquad j = 2, 3 \end{aligned}$$

Theorem 6 *[14]*
Given IID sources X_1 and X_2, the set of $(R_1, R_2, R_3, D_1, D_2, D_3)$ is achievable if there exist auxiliary random variables (U_1, W, U_2) jointly distributed with (X_1, X_2) and functions (g_1, g_2, g_3) such that

$$\begin{aligned} R_1 &\geq I(X_1; U_1) \\ R_2 &\geq I(U_1 X_1 X_2; W) \\ R_3 &\geq I(X_2; U_2 | U_1, W) \end{aligned}$$

and

$$\begin{aligned} D_1 &\geq E[d(X_1, g_1(U_1))] \\ D_2 &\geq E[d(X_1, g_2(W))] \\ D_3 &\geq E[d(X_1, g_3(U_1, W, U_2))] \end{aligned}$$

and

$$\begin{aligned} U_1 &\rightarrow X_1 \rightarrow X_2 \\ U_2 &\rightarrow (X_2, U_1, W) \rightarrow X_1 \end{aligned} \tag{2.6}$$

The proof is shown in Appendix C.

Based on the above formulation for two-frame CVC-PL, the difference between two-layer SVC and two-frame CVC-PL is the fact that in CVC-PL, we send the same description to the second and third decoders while in SVC, at each layer we should send more bits compared to the previous layer.

Having talked about all the pre-requests, in the next chapter we present and prove the main theorem of the thesis, which is the minimum total rate of CVC-PL with two jointly Gaussian distributed sources.

Chapter 3

CVC-PL with Two Jointly Gaussian Distributed Sources

Considering CVC-PL with two frames as shown in Fig. (1.2) and discussed in Chapter 2, in this chapter, we derive the minimum total rate for CVC-PL with two jointly Gaussian distributed sources and we determine the optimal coding scheme which achieves the minimum total rate.

A quick look at Fig. (1.2), one can see that a two-frame CVC-PL consists of a two-layer causal video coding (the first and third decoders) and a peer-to-peer coding (the second decoder) as shown in Fig. (3.1). The following outer-bound on R_1 and R_2 shows this fact clearly. If $(R_1, R_2, R_3, D_1, D_2, D_3)$ is achievable then

$$\begin{aligned} n(R_1+\epsilon) &\geq H(S_1) \\ &\overset{(a)}{\geq} H(S_1) - H(S_1|\mathbf{X}_1) \\ &= I(S_1; \mathbf{X}_1) \\ &\overset{(b)}{=} \sum_{i=1}^{n} I(X_{1i}; S_1 | X_{1i}^{-}) \\ &\overset{(c)}{=} \sum_{i=1}^{n} I(X_{1i}; U_{1i}) \end{aligned}$$

where

(a). S_1 is a function of $\mathbf{X}_1$, i.e. $S_1 = f(\mathbf{X}_1)$, so $H(S_1|\mathbf{X}_1) = 0$

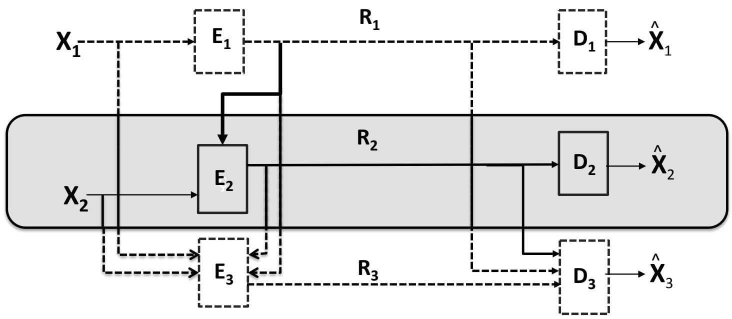

Figure 3.1: In CVC-PL, if the first encoded frame is available at the decoder, causal video coding and if it is not available at the decoder, peer-to-peer coding occurs.

(b). Chaining rule and $X_{1i}^{-} = (X_{11}, ..., X_{1(i-1)})$

(c). $U_{1i} = (S_1, X_{1i}^{-})$

In addition, since $\mathbf{X}_1$ and $\mathbf{X}_2$ are IID sources,

$$U_{1i} \to X_{1i} \to X_{2i}$$

Also, we can derive an outer-bound on R_2 as follows,

$$
\begin{aligned}
n(R_2+\epsilon) &\geq H(S_2)\\
&= H(S_2, g_2(S_2))\\
&= H(g_2(S_2)) + H(S_2|g_2(S_2))\\
&\overset{(a)}{\geq} H(g_2(S_2)) - H(g_2(S_2)|\mathbf{X}_1,\mathbf{X}_2) + H(S_2|g_2(S_2),S_1) - H(S_2|g_2(S_2),S_1,\mathbf{X}_1,\mathbf{X}_2)\\
&= I(\mathbf{X}_1,\mathbf{X}_2; g_2(S_2)) + I(\mathbf{X}_1,\mathbf{X}_2; S_2|S_1, g_2(S_2))\\
&\overset{(b)}{=} \sum_{i=1}^{n} I(X_{1i},X_{2i}; g_2(S_2)|X_{2i}^{-}) + \sum_{i=1}^{n} I(X_{1i},X_{2i}; S_2|S_1, g_2(S_2), X_{1i}^{-}, X_{2i}^{-})\\
&\geq \sum_{i=1}^{n} H(X_{1i},X_{2i}|X_{1i}^{-},X_{2i}^{-}) - H(X_{1i},X_{2i}|g_2(S_2),X_{2i}^{-})\\
&+ \sum_{i=1}^{n} H(X_{1i},X_{2i}|S_1,g_2(S_2),X_{1i}^{-},X_{2i}^{-}) - H(X_{1i},X_{2i}|S_2,S_1,g_2(S_2),X_{1i}^{-},X_{2i}^{-})\\
&\overset{(c)}{=} \sum_{i=1}^{n} I(X_{1i},X_{2i}; W_i) + \sum_{i=1}^{n} I(X_{1i},X_{2i}; U_{2i}|W_i,U_{1i})
\end{aligned}
$$

where

(a). Since $g(S_2) = g(f(\mathbf{X}_1,\mathbf{X}_2))$ is a function of $(\mathbf{X}_1,\mathbf{X}_2)$, so $H(g_2(S_2)|\mathbf{X}_1,\mathbf{X}_2) = 0$ and $H(S_2|g_2(S_2)) \geq H(S_2|g_2(S_2),S_1) - H(S_2|g_2(S_2),S_1,\mathbf{X}_1,\mathbf{X}_2)$

(b). Chaining rule and $X_{2i}^{-} = (X_{21},...,X_{2(i-1)})$

(c). $W_i = (g_2(S_2), X_{2i}^{-})$ and $U_{2i} = (S_2, X_{2i}^{-})$

In addition, we can introduce $\hat{X}_{1k} = g_1(U_{1k})$, $\hat{X}_{2k} = g_2(W_k)$ and $\hat{X}_{3k} = g_3(U_{1k},W_k,U_{2k})$. Introducing uniformly distributed time-sharing random variable Q on the discrete set $\{1,...,n\}$and random variables $W = (W_Q, Q)$, $U_i = (U_{iQ}, Q)$, $X_i = (X_{iQ}, Q)$ and $\hat{X}_j = (\hat{X}_{jQ}, Q)$ for $i = 1,2$ and $j = 1,2,3$, one can write,

$$
\begin{aligned}
R_1+\epsilon &\geq I(X_1;U_1)\\
R_2+\epsilon &\geq I(X_1,X_2;W) + I(X_1,X_2;U_2|W,U_1)\\
U_1 &\to X_1 \to X_2
\end{aligned}
$$

and

$$D_1 + \epsilon \geq E[d(\mathbf{X}_1, \hat{\mathbf{X}}_1)] = \frac{1}{n}\sum_{k=1}^{n} E[d(X_{1k}, \hat{X}_{1k})] = E[d(X_1, \hat{X}_1)]$$
$$D_j + \epsilon \geq E[d(\mathbf{X}_2, \hat{\mathbf{X}}_j)] = \frac{1}{n}\sum_{k=1}^{n} E[d(X_{2k}, \hat{X}_{jk})] = E[d(X_2, \hat{X}_j)]$$

Letting $\epsilon- > 0$,

$$\begin{aligned} R_1 &\geq I(X_1; U_1) \\ R_2 &\geq I(X_1, X_2; W) + I(X_1, X_2; U_2|W, U_1) \\ U_1 &\to X_1 \to X_2 \end{aligned}$$

and

$$\begin{aligned} D_1 &\geq E[d(X_1, \hat{X}_1)] \\ D_j &\geq E[d(X_2, \hat{X}_j)] \end{aligned}$$

The above outer-bound gives us a hint to present the following achievable region. In terms of information theoretical point of view, we can say that if $U_1 \to (X_1, X_2) \to W$ then the above outer-bound would be an achievable region. Another way, is to generate an auxiliary random variable U_{21} which is jointly typical with U_1 and then generating U_{22} conditioned on U_{21}. Actually, in this approach, we consider CVC-PL as a combination of MDC and CVC and we derive the following achievable region,

Theorem 7 *For a two-frame CVC-PL, given IID sources X_1 and X_2, the set of $(R_1, R_2, R_3, D_1, D_2, D_3)$ is achievable if there exist auxiliary random variables (U_1, U_{21}, W, U_{22}) jointly distributed with generic random variables (X_1, X_2) and functions (g_1, g_2, g_3) such that*

$$\begin{aligned} R_1 &\geq I(X_1; U_1) \\ R_2 &\geq I(X_2; W|U_{21}) + I(X_2; U_{22}|W, U_{21}, U_1) \\ R_1 + R_2 &\geq I(X_1; U_1, U_{21}) + I(U_1; U_{21}) + I(X_1, X_2; W|U_{21}) \\ R_3 &\geq I(X_2; U_{22}|W, U_{21}, U_1) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} E[d(X_1, g_1(U_1))] &\leq D_1 \\ E[d(X_2, g_2(W, U_{21}))] &\leq D_2 \\ E[d(X_2, g_3(U_1, W, U_{21}, U_{22}))] &\leq D_3 \end{aligned}$$

and

$$\begin{aligned} (U_1, U_{21}) &\rightarrow X_1 \rightarrow X_2 \\ U_1 &\rightarrow (X_2, U_{21}) \rightarrow (W, U_{22}) \end{aligned}$$

The proof is shown in Appendix D. Having got some insight into CVC-PL, in the next section, we will present and prove a theorem on the minimum total rate of a CVC-PL with two jointly Gaussian distributed sources.

3.1 The Total rate of a CVC-PL with Two Jointly Gaussian Distributed Sources

The approach we use in this section, is the bottom-up approach discussed in Chapter 1. $\mathbf{X_1}$ and $\mathbf{X_2}$ are IID, zero-mean jointly Gaussian distributed sources. Without loss of generality, we assume unit-variance sources, i.e. $\sigma_1 = \sigma_2 = 1$. In addition, we consider MSE as the average distortion, and we are interested in minimizing MSE at the decoders. We formulate $\mathbf{X_1}$ and $\mathbf{X_2}$ as follows,

$$\begin{aligned} X_1 &= rX_2 + Z_2 \\ X_2 &= rX_1 + Z_1 \end{aligned}$$

where $Z_k \sim \mathbf{N}(0, \sigma_Z^2)$ for $k = 1, 2$ and r is the normalized correlation coefficient between X_1 and X_2. The main result of the thesis is in the following theorem,

Theorem 8 *For a given (D_1, D_2, D_3), the minimum total rate $R_T = R_1 + R_2 + R_3$ is*

$$R_T = \begin{cases} \frac{1}{2}\log\frac{D_r}{D_1 D_3} & D_3 \leq -1 + D_2 + D_r \\ f(D_1, D_2, D_3) & o.w. \\ \frac{1}{2}\log\frac{1}{D_1 D_2} & \frac{1}{D_3} \leq -1 + \frac{1}{D_2} + \frac{1}{D_r} \end{cases} \tag{3.2}$$

where

$$f(D_1, D_2, D_3) = \frac{1}{2}\log\frac{(1+\lambda)(D_3+\lambda)}{(D_r+\lambda)(D_2+\lambda)} + \frac{1}{2}\log\frac{D_r}{D_1 D_3} \tag{3.3}$$

and

$$\lambda = \frac{D_3 - D_r D_2 + r\sqrt{(1-D_1)(1-D_2)(D_3-D_r)(D_3-D_2)}}{D_r + D_2 - (D_3 + 1)}$$

To prove the theorem, we study the inner-bound and the outer-bound on the total rate R_T in three separate cases and we show that in all these three cases, the inner-bound and the outer-bound are equal, i.e. they are tight bounds to the total rate, hence the theorem is proved.

3.2 The Inner-Bound

CASE 1: $\max\{0, D_r + D_2 - 1\} \leq D_3 \leq \frac{D_r D_2}{D_r + D_2 - D_r D_2}$, $0 \leq (D_1, D_2) \leq 1$

Let's assume $U_2 = cte$ then $R_3 = 0$ and

1. Quantize $U_1 = X_1 + N_1$ where $N_1 \sim \mathbf{N}(0, \frac{D_1}{1-D_1})$, where N_1 is independent of X_1 and X_2

2. Quantize $W = X_2 + \gamma(N_1 + Z_2) + N'_W$ where N'_W is independent of (X_1, X_2, N_1). For simplicity we define $N_W = \gamma(N_1 + Z_2) + N'_W$, and $N'_1 = N_1 + Z_2$ such that

$$\begin{aligned}
\gamma &= \frac{\rho\sigma_{N'_1}\sigma_{N_W}(1-D_1)}{D_r} \\
\sigma^2_{N_W} &= \frac{D_2}{1-D_2} \\
\sigma^2_{N'_1} &= \frac{D_r}{1-D_1}
\end{aligned}$$

3. Reconstruct $\hat{X}_3 = MMSE(X_2|U_1, W)$, so $D_3 = E[X_2 - MMSE(X_2|U_1, W)]^2$

The covariance matrix of U_1 and W is

$$\sum_{U_1,W} = \begin{pmatrix} \frac{1}{1-D_1} & r + \rho\sigma_{N'_1}\sigma_{N_W} \\ r + \rho\sigma_{N'_1}\sigma_{N_W} & \frac{1}{1-D_2} \end{pmatrix}$$

So, using item 3, D_3 is

$$D_3 = \frac{\sigma^2_{N'_1}\sigma^2_{N_W}(1-\rho^2)}{\sigma^2_{N'_1} + r^2\sigma^2_{N_W} + \sigma^2_{N'_1}\sigma^2_{N_W}(1-\rho^2) - 2r\rho\sigma_{N'_1}\sigma_{N_W}}$$

Solving the equation of $\rho\sigma_{N'_1}\sigma_{N_W}$ as a function of D_3, we have

$$\rho\sigma_{N'_1}\sigma_{N_W} = \frac{rD_3 - \sqrt{\frac{(D_3-D_r)(D_3-D_2)}{(1-D_1)(1-D_2)}}}{1-D_3}$$

To have the expression under the squared-root non-negative, either $D_3 \leq \min\{D_r, D_2\}$ or $D_3 \geq \max\{D_r, D_2\}$. So, one sufficient condition for this coding scheme to be achievable is

$$(D_3 \leq \min\{D_r, D_2\}) \ \ || \ \ (D_3 \geq \max\{D_r, D_2\}) \tag{3.4}$$

For $D_3 \geq \max\{D_r, D_2\}$, we study the total rate in case 3. So, we just consider $D_3 \leq \min\{D_r, D_2\}$. According to the achievable region in (2.6) and selecting U_1 and W from items 1 and 2 respectively, and $U_2 = cte$ we set $R_1,\ R_2$ and R_3 as follows,

$$\begin{aligned} R_1 &= I(X_1; U_1) \\ R_2 &= I(X_1 X_2 U_1; W) \\ R_3 &= 0 \end{aligned}$$

Lets define $Y = X_2 + V$ where $V \sim \mathbf{N}(0, \alpha)$ and independent of (X_1, X_2, U_1, W, U_2). We have the following lemma,
Lemma 1
Given

$$\frac{1}{D_3} > -1 + \frac{1}{D_2} + \frac{1}{D_r} \tag{3.5}$$

and

$$D_3 > -1 + D_2 + D_r \tag{3.6}$$

If we define

$$\alpha = \frac{-\rho\sigma_{N_1'}\sigma_{N_W}}{r + \rho\sigma_{N_1'}\sigma_{N_W}}$$

then

$$h(U_1|Y) + h(W|Y) - h(U_1, W|Y) = 0$$

The proof is in AppendixB. Using this lemma, we study the summation of R_1, R_2 and R_3 as follows

$$
\begin{aligned}
& (\quad R_1 + R_2 \quad) \\
= \quad & h(U_1) + h(W) - h(U_1, W|X_1, X_2) + h(U_1, W) - h(U_1, W) \\
\stackrel{(a)}{=} \quad & h(U_1) + h(W) - h(U_1, W|X_1, X_2) + h(U_1, W) - h(U_1, W) \\
- \quad & [h(U_1|Y) + h(W|Y) - h(U_1, W|Y)] \\
= \quad & I(U_1; Y) + I(W; Y) + I(X_1, X_2; U_1, W) - I(Y; U_1, W) \\
\stackrel{(b)}{=} \quad & \frac{1}{2}\log\frac{1+\epsilon}{D_r+\alpha} + \frac{1}{2}\log\frac{1+\alpha}{D_2+\alpha} + h(X_1, X_2) - h(Y) + h(Y|U_1, W) - h(X_1, X_2|U_1, W) \\
\stackrel{(c)}{=} \quad & \frac{1}{2}\log\frac{2\pi e\sigma_Z^2(1+\alpha)}{(D_r+\alpha)(D_2+\alpha)} + I(Y; V|U_1, W) - h(X_1|X_2, U_1) \\
\stackrel{(d)}{=} \quad & \frac{1}{2}\log\frac{\sigma_Z^2(1+\alpha)}{(D_r+\alpha)(D_2+\alpha)} + h(V) - h(V|Y - MMSE(Y|U_1, W)) \\
- \quad & h(X_1 - MMSE(X_1|U_1)|r(X_1 - MMSE(X_1|U_1)) + Z) \\
= \quad & \frac{1}{2}\log\frac{\sigma_Z^2(1+\alpha)}{(D_r+\alpha)(D_2+\alpha)} + I(V; X_2 - MMSE(Y|U_1, W) + V) \\
- \quad & h(X_1 - MMSE(X_1|U_1)|r(X_1 - MMSE(X_1|U_1)) + Z) \\
= \quad & \frac{1}{2}\log\frac{\sigma_Z^2(1+\alpha)}{(D_r+\alpha)(D_2+\alpha)} + I(V; X_2 - MMSE(X_2|U_1, W) + V) \\
- \quad & h(Z|r(X_1 - MMSE(X_1|U_1)) + Z) \\
\stackrel{(e)}{=} \quad & \frac{1}{2}\log\frac{(1+\alpha)(D_3+\alpha)}{(D_r+\alpha)(D_2+\alpha)} + \frac{1}{2}\log\frac{D_r}{D_1 D_3}
\end{aligned}
$$

where

(a). Using *Lemma*, $h(U_1|Y) + h(W|Y) - h(U_1, W|Y) = 0$

(b). Since V is independent of (X_1, Z, U_1, W, U_2), $I(Y; W) = \frac{1}{2}\log\frac{1+\epsilon}{D_2+\epsilon}$ and $I(Y; U_1) = h(Y) - h(Y - MMSE(Y|U_1)|U_1) = h(Y) - h(Y - MMSE(Y|U_1)) = \frac{1}{2}\log\frac{1+\epsilon}{D_r+\epsilon}$

(c). (c) can be achieved using entropy chaining rule

(d). $h(A|B) = h(A|BC)$ if $A \to B \to C$

(e). Define $\tilde{X}_2 = X_2 - MMSE(X_2|U_1, W)$ where $\tilde{X}_2$ is independent of V. We can show that if $\tilde{X}_2$ is Gaussian distributed, then $E[(V - MMSE(V|\tilde{X}_2+V))^2] = \frac{D_3\epsilon}{D_3+\epsilon}$. Hence,

$$I(V; \tilde{X}_2 + V) = h(V) - h(V - MMSE(V|\tilde{X}_2 + V)) = \frac{1}{2}\log\frac{D_3 + \epsilon}{D_3}$$

So,

$$(R_1 + R_2 + R_3) = \frac{1}{2}\log\frac{(1+\alpha)(D_3+\alpha)}{(D_r+\alpha)(D_2+\alpha)} + \frac{1}{2}\log\frac{D_r}{D_1 D_3} \tag{3.7}$$

Intersecting conditions of (3.4),(3.5) and (3.6), we conclude the minimum total rate as in (3.3).

CASE 2: $D_3 \leq D_r + D_2 - 1, D_r + D_2 - 1 \geq 0$

First we study $D_3 = D_r + D_2 - 1$. In this case, we set $\rho\sigma_{N_1'}\sigma_{N_W} = -r$, so

$$\begin{aligned} U_1 &= X_1 + N_1 \\ W &= X_2 - r(Z_2 + N_1) + N_W' \end{aligned}$$

and

$$R_2 = \frac{1}{2}\log\frac{D_r}{D_3}$$

For $D_3 < -1 + D_2 + D_r$, we again set $\gamma = \frac{-r(1-D_1)}{D_r}$ and we send more rate R_3 to send random variable U_2 such that

$$U_2 = (X_2 - MMSE(X_2|\hat{X}_1, \hat{X}_2)) + N_2$$

such that $N_2 \sim \mathbf{N}(0, \frac{D_3}{1-D_3})$ Hence,

$$R_1 + R_2 + R_3 = \frac{1}{2}\log\frac{D_r}{D_1 D_3} \tag{3.8}$$

CASE 3: $D_3 \geq \frac{D_r D_2}{D_r + D_2 - D_r D_2}$

In this case we set $\rho\sigma_{N_1'}\sigma_{N_W} = 0$, so

$$R_2 = \frac{1}{2}\log\frac{1}{D_2}$$

and

$$\frac{1}{D_3} = -1 + \frac{1}{D_2} + \frac{1}{D_r}$$

3.3 The Outer-Bound

CASE 1: $\max\{0, D_r + D_2 - 1\} \leq D_3 \leq \frac{D_r D_2}{D_r + D_2 - 1}$**,** $0 \leq (D_1, D_2) \leq 1$

If $(R_1, R_2, R_3, D_1, D_2, D_3)$ is achievable then by introducing $\mathbf{Y} = \mathbf{X}_2 + \mathbf{V}$ where $V \sim \mathbf{N}(0, \epsilon)$ and independent of $(\mathbf{X}_1, \mathbf{Z}, S_1, S_2, S_3)$, we have

$$
\begin{aligned}
(\quad & R_1 + R_2 + R_3) \\
\geq \quad & \frac{1}{n}[H(S_1) + H(S_2) + H(S_3)] \\
\geq \quad & \frac{1}{n}[H(S_1) + H(S_2, S_3)] \\
= \quad & \frac{1}{n}[H(S_1) + H(S_2, S_3) - H(S_1, S_2, S_3|\mathbf{X}_1, \mathbf{X}_2) + H(S_1, S_2, S_3) - H(S_1, S_2, S_3)] \\
\geq \quad & \frac{1}{n}[H(S_1) + H(S_2, S_3) - H(S_1, S_2, S_3|\mathbf{X}_1, \mathbf{X}_2) + H(S_1, S_2, S_3) - H(S_1, S_2, S_3) \\
- \quad & (H(S_1|\mathbf{Y}) + H(S_2, S_3|\mathbf{Y}) - H(S_1, S_2, S_3|\mathbf{Y}))] \\
\geq \quad & \frac{1}{n}[I(S_1; \mathbf{Y}) + I(S_2; \mathbf{Y}) + I(\mathbf{X}_1, \mathbf{X}_2; S_1, S_2, S_3) - I(\mathbf{Y}; S_1, S_2, S_3)] \\
\geq \quad & \frac{1}{2}\log\frac{1+\epsilon}{r^2 D_1 + \sigma_Z^2 + \epsilon} + \frac{1}{2}\log\frac{1+\epsilon}{D_2+\epsilon} \\
+ \quad & \frac{1}{n}[h(\mathbf{X}_1, \mathbf{X}_2) - h(\mathbf{Y}) + h(\mathbf{Y}|S_1, S_2, S_3) - h(\mathbf{X}_1, \mathbf{X}_2|S_1, S_2, S_3)] \\
= \quad & \frac{1}{2}\log\frac{2\pi e\sigma_Z^2(1+\epsilon)}{(D_r+\epsilon)(D_2+\epsilon)} + \frac{1}{n}[I(\mathbf{Y}; \mathbf{V}|S_1, S_2, S_3) - h(\mathbf{X}_1|\mathbf{X}_2, S_1, S_2, S_3)] \\
\geq \quad & \frac{1}{2}\log\frac{2\pi e\sigma_Z^2(1+\epsilon)}{(D_r+\epsilon)(D_2+\epsilon)} + \frac{1}{n}[I(\mathbf{Y}; \mathbf{V}|S_1, S_2, S_3) - h(\mathbf{X}_1|\mathbf{X}_2, S_1)] \\
\overset{(a)}{\geq} \quad & \frac{1}{2}\log\frac{2\pi e\sigma_Z^2(1+\epsilon)}{(D_r+\epsilon)(d_2+\epsilon)} + \frac{1}{2}\log\frac{(D_3+\epsilon)}{D_3} - \frac{1}{n}h(\mathbf{X}_1 - MMSE(\mathbf{X}_1|S_1)|\mathbf{X}_2 - rMMSE(\mathbf{X}_1|S_1)) \\
= \quad & \frac{1}{2}\log\frac{2\pi e\sigma_Z^2(1+\epsilon)}{(D_r+\epsilon)(D_2+\epsilon)} + \frac{1}{2}\log\frac{(D_3+\epsilon)}{D_3} - \frac{1}{n}h(\mathbf{Z}|r(\mathbf{X}_1 - MMSE(\mathbf{X_1}|S_1)) + \mathbf{Z}) \\
\geq \quad & \frac{1}{2}\log\frac{(1+\epsilon)(D_3+\epsilon)}{(D_r+\epsilon)(d_2+\epsilon)} + \frac{1}{2}\log\frac{D_r}{D_3 D_1}
\end{aligned}
$$

Hence,

$$(R_1 + R_2 + R_3) \geq \frac{1}{2} \log \frac{(1+\epsilon)(D_3+\epsilon)}{(D_r+\epsilon)(D_2+\epsilon)} + \frac{1}{2} \log \frac{D_r}{D_1 D_3} \tag{3.9}$$

for any $\epsilon \geq 0$. Therefore, if we set $\epsilon = \alpha$, (3.7) and (3.9) are equal.

***R*emark 1**

Lets define $\mathbf{T} = \mathbf{Y} - MMSE(S_1, S_2, S_3)$, $\mathbf{T}^N$ as its correspondence Gaussian distributed, γ as the MMSE coefficient of $\mathbf{V}$ given $\mathbf{T}^N$, so (a) is derived as follows

$$\begin{aligned}
I(\mathbf{Y}; \mathbf{V}|S_1, S_2, S_3) &\geq h(\mathbf{V}) - h(\mathbf{V}|\mathbf{T}) \\
&\geq h(\mathbf{V}) - h(\mathbf{V} - MMSE(\mathbf{V}|\mathbf{T})) \\
&\geq \sum_{i=1}^{n} h(V_i) - \sum_{i=1}^{n} h(V_i - [MMSE(\mathbf{V}|\mathbf{T})]_i) \\
&\geq \sum_{i=1}^{n} h(V_i) - \sum_{i=1}^{n} h(V_i - \gamma T_i^N) \\
&\geq \frac{n}{2} \log \frac{(D_3+\epsilon)}{D_3}
\end{aligned}$$

***R*emark 2**

Maximizing (3.9) on ϵ, we have

$$\epsilon^* = \frac{D_3 - D_r D_2 + r\sqrt{(1-D_1)(1-D_2)(D_3-D_r)(D_3-D_2)}}{D_r + D_2 - (D_3+1)} \tag{3.10}$$

ϵ^* is positive if and only if

$$\frac{1}{D_3} > -1 + \frac{1}{D_2} + \frac{1}{D_r}$$

and

$$D_3 > -1 + D_2 + D_r$$

After some manipulation, one can show that, $\epsilon^* = \alpha$.

CASE 2: $D_3 \leq D_r + D_2 - 1, D_r + D_2 - 1 \geq 0$

If $(R_1, R_2, R_3, D_1, D_2, D_3)$ is achievable then

$$\begin{aligned}
(R_1 + R_2 + R_3) &\geq \frac{1}{n}(H(S_1) + H(S_2) + H(S_3)) \\
&\geq \frac{1}{n}(H(S_1, S_2, S_3) - H(S_1, S_2, S_3|\mathbf{X}_1, \mathbf{X}_2)) \\
&= \frac{1}{n} I(\mathbf{X}_1, \mathbf{X}_2; S_1, S_2, S_3) \\
&\geq \frac{1}{n} I(\mathbf{X}_1, \mathbf{X}_2; \hat{\mathbf{X}}_\mathbf{1}, \hat{\mathbf{X}}_\mathbf{3}) \\
&\overset{(a)}{\geq} \frac{1}{2}\log\frac{D_r}{D_1 D_3}
\end{aligned}$$

where (a) is from (2.5).

CASE 3: $D_3 \geq \frac{D_r D_2}{D_r + D_2 - D_r D_2}$

If $(R_1, R_2, R_3, D_1, D_2, D_3)$ is achievable then

$$\begin{aligned}
(R_1 + R_2 + R_3) &\geq \frac{1}{n}(H(S_1) + H(S_2) + H(S_3)) \\
&\geq \frac{1}{n}(H(S_1) + H(S_2) - H(S_1|\mathbf{X}_1) - H(S_2|\mathbf{X}_2)) \\
&= \frac{1}{n}(I(\mathbf{X}_1; S_1) + I(\mathbf{X}_2; S_2)) \\
&\geq \frac{1}{2}\log\frac{1}{D_1} + \frac{1}{2}\log\frac{1}{D_2}.
\end{aligned}$$

Chapter 4

Conclusion and Future Works

In this chapter, we first study the scenario in which both sources are available at the encoders as shown in Fig. (4.1). Considering this scenario as a centralized sources scenario and in the opposite, the scenario of CVC-PL as a distributed sources scenario, we compare the results in Chapter 3 with the derived total rate for the presented centralized sources. Afterwards, we conclude the thesis and talk about future work.

4.1 Centralized Sources vs. Distributed Sources

We model the scenario of centralized sources shown in Fig. (4.1) as follows,

First Layer (X_1, X_2) is encoded using function f_1 to S_1 which is a finite-length sequence of 0s and 1s,

$$f_1(\mathbf{X}_1, \mathbf{X}_2) = S_1$$

S_1 is decoded to $\hat{\mathbf{X}}_\mathbf{1}$ using decoding function g_1,

$$g_1(S_1) = \hat{\mathbf{X}}_\mathbf{1}$$

Second Layer Encoding function f_2 uses $(\mathbf{X}_1, S_1)$ as the side information to encode $\mathbf{X}_2$ to S_2,

$$f_2(\mathbf{X}_1, \mathbf{X}_2, S_1) = S_2$$

S_2 is decoded to $\hat{\mathbf{X}}_\mathbf{2}$ using decoding function g_2,

$$g_2(S_2) = \hat{\mathbf{X}}_2$$

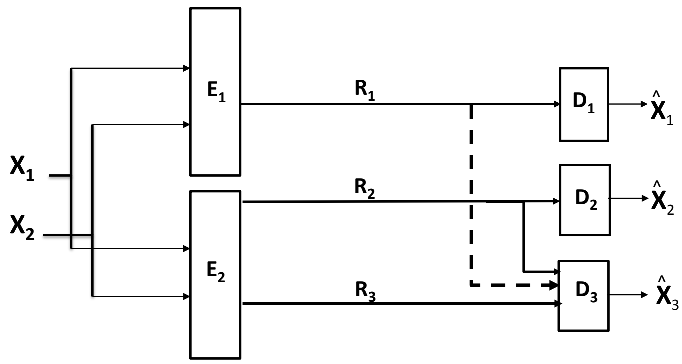

Figure 4.1: In this scenario, X_1 and X_2 are available at both the encoders.

Third Layer In the last layer, encoding function f_3 uses all the previous encoded sequences and $\mathbf{X}_2$ as the side information to encode $\mathbf{X}_2$ to S_3,

$$f_3(\mathbf{X}_1, \mathbf{X}_2, S_1, S_2) = S_3$$

S_3 is decoded to $\hat{\mathbf{X}}_\mathbf{3}$ using decoding function g_3,

$$g_3(S_1, S_2, S_3) = \hat{\mathbf{X}}_3$$

Achievability Definition
A set of $(R_1, R_2, R_3, D_1, D_2, D_3)$ is achievable if $\forall \epsilon > 0$ with n sufficiently large we have

$$\begin{aligned} \log \frac{E|S_i|}{n} &\leq R_i + \epsilon & i = 1, 2, 3 \\ E[d(\mathbf{X_1}, \hat{\mathbf{X}}_\mathbf{1})] &\leq D_1 + \epsilon & \\ E[d(\mathbf{X_2}, \hat{\mathbf{X}}_\mathbf{j})] &\leq D_j + \epsilon & j = 2, 3 \end{aligned}$$

In this scenario, it is easy to show that the following region could be an achievable region,

Theorem 9 *Given IID sources X_1 and X_2, the set of $(R_1, R_2, R_3, D_1, D_2, D_3)$ is achievable if there exist auxiliary random variables (U_1, U_2, U_3) and functions (g_1, g_2, g_3) such that,*

$$
\begin{aligned}
R_1 &\geq I(X_1X_2; U_1) \\
R_2 &\geq I(X_1X_2; U_2) \\
R_1 + R_2 &\geq I(U_1; U_2) + I(X_1, X_2; U_1, U_2) \\
R_3 &\geq I(X_1X_2; U_3|U_1, U_2)
\end{aligned}
$$

and

$$
\begin{aligned}
D_1 &\geq E[d(X_1, g_1(U_1))] \\
D_2 &\geq E[d(X_2, g_2(U_2))] \\
D_3 &\geq E[d(X_2, g_3(U_1, U_2, U_3))]
\end{aligned}
$$

The proof is shown in Appendix E. Considering jointly Gaussian distributed sources X_1 and X_2 as the previous arguments, we present the following coding scheme based on the above achievable region,

$$
\begin{aligned}
U_1 &= X_1 + a_1 Z_1 + N_1 \\
U_2 &= X_2 + a_2 Z_2 + N_2 \\
U_3 &= (X_2 - MMSE(X_2|U_1, U_2)) + N_3
\end{aligned}
$$

where N_1, N_2 and N_3 are jointly Gaussian distributed independent of X_1 and X_2 with the following covariance matrix,

$$
\sum_{N_1,N_2,N_3} = \begin{pmatrix} \frac{D_1}{1-D_1} - a_1^2\sigma_Z^2 & \rho\sqrt{(\frac{D_1}{1-D_1} - a_1^2\sigma_Z^2)(\frac{D_2}{1-D_2} - a_2^2\sigma_Z^2)} & 0 \\ \rho\sqrt{(\frac{D_1}{1-D_1} - a_1^2\sigma_Z^2)(\frac{D_2}{1-D_2} - a_2^2\sigma_Z^2)} & \frac{D_2}{1-D_2} - a_2^2\sigma_Z^2 & 0 \\ 0 & 0 & D_3 \end{pmatrix}
$$

and

$$
\begin{aligned}
X_1 &= rX_2 + Z_2 \\
X_2 &= rX_1 + Z_1
\end{aligned}
$$

and $Z_i \sim \mathbf{N}(0, \sigma_Z^2)$ and independent of X_i, for $i = 1, 2$. We have an achievable set $(R_1, R_2, R_3, D_1, D_2, D_3)$ such that its summation is

$$
R_1 + R_2 + R_3 = I(U_1; U_2) + I(X_1, X_2; U_1, U_2, U_3) \tag{4.1}
$$

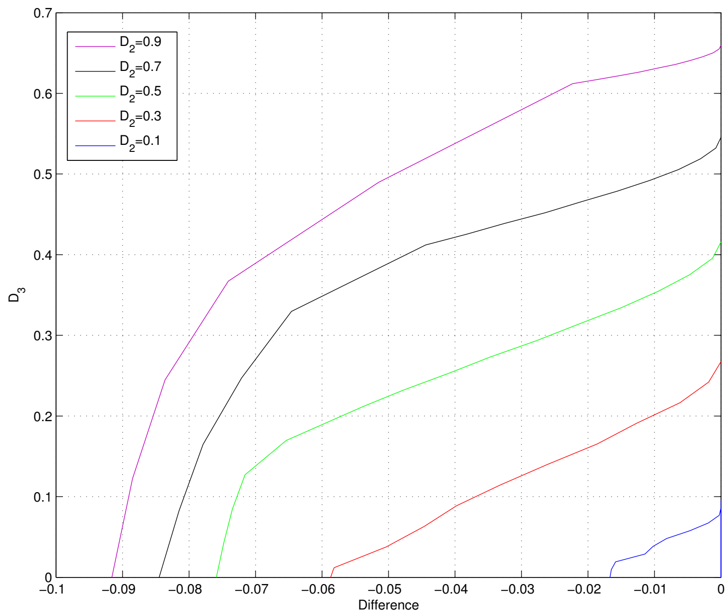

Figure 4.2: Numerical results on the difference of the summation rates, i.e. (4.1)-(3.2), for normalized correlation coefficient $r = 0.6$ and $D_1 = 0.2$

The numerical comparison between (3.2) and (4.1) is shown in Fig. (4.2). In this figure, it is shown that by increasing D_3, the absolute difference between (3.2) and (4.1), i.e. (4.1)-(3.2), is decreasing and finally the difference is zero. Having both of the sources at both of the encoders, i.e. centralized sources, could be a big condition. Based on the simulation results, one can see that if D_2 is small, the difference between the summation rates in (3.2) and (4.1) is not large. Therefore, the distributed sources scenario, does not change the result that much. In addition, one can have the same observation when D_3 is large enough.

4.2 Conclusion and Future Works

In this thesis, we considered two-frame causal video coding with possible loss of the first encoded frame (CVC-PL). This scenario could happen a lot in telecommunication, e.g. when one of the previous encoded packets is lost in a Causal Video Coding. If real-time communication is an issue, we cannot wait for the lost packet to be retransmitted. We studied this scenario from different angles. We showed that one possible achievable region is the combination of Causal Video Coding (CVC) and Multiple Description Coding (MDC). In addition, we derived an outer-bound to the achievable rate-distortion sets, and we clearly demonstrated that two-frame CVC-PL is a subspace of the region combining peer-to-peer coding and CVC. In summary, the contributions of the thesis are as follows,

1. Provided insight into CVC-PL with two frames by deriving an outer-bound on the achievable rate-distortion sets and by studying a new achievable region for this scenario.

2. Derived the minimum total rate of CVC-PL and demonstrated a coding scheme to achieve this total rate, in the case of two jointly Gaussian distributed sources.

3. Fully characterized the minimum total rate of CVC-PL with two jointly Gaussian distributed sources.

4. Showed that for small D_3, i.e. $D_3 \leq D_r + D_2 - 1$, CVC-PL could be treated as CVC with two jointly Gaussian distributed sources.

5. Showed that for large D_3, i.e. $D_3 \geq \frac{D_r D_2}{D_r + D_2 - D_r D_2}$, CVC-PL could be treated as two parallel peer-to-peer coding with distortion constraints D_1 and D_2.

6. Showed that if $D_r + D_2 - 1 < D_3 < \frac{D_r D_2}{D_r + D_2 - D_r D_2}$, the minimum total rate for CVC-PL with two jointly Gaussian distributed sources is $\frac{1}{2} \log \frac{(1+\lambda)(D_3+\lambda)}{(D_r+\lambda)(D_2+\lambda)} + \frac{1}{2} \log \frac{D_r}{D_1 D_3}$ where $\lambda = \frac{D_3 - D_r D_2 + r\sqrt{(1-D_1)(1-D_2)(D_3-D_r)(D_3-D_2)}}{D_r + D_2 - (D_3+1)}$.

In the future, we extend the scenario in Chapter 3 to M-jointly Gaussian distributed sources. In Fig. (4.3), we have sketched a CVC-PL with three correlated frames. One can consider this scenario as an example of a video coding when a packet lost but for the sake of simplification, as shown in Fig. (4.3), we just consider the scenario in which the first encoded frame, i.e. S_1, may or may not be available at the decoder sides. We formulate CVC-PL with M frames as follows,

First Layer X_1 is encoded using function f_1 to S_1 which is a finite-length sequence of 0s and 1s, *i*.e. $f_1(\mathbf{X_1}) = S_1$ and S_1 is decoded to $\hat{\mathbf{X}}_\mathbf{1}$ using decoding function g_1, *i*.e. $g_1(S_1) = \hat{\mathbf{X}}_\mathbf{1}$.

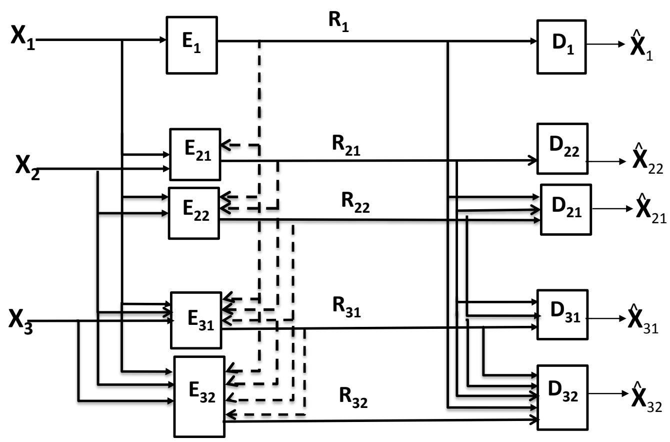

Figure 4.3: CVC-PL with three frames. The first encoded frame may or may not be available at the second and third layers.

M_1^{th} **Layer** Encoding function $f_{M,1}$ uses $(\mathbf{X_1}, ..., \mathbf{X_{M-1}}, S_{1,1}, S_{1,2}, ..., S_{M-1,1})$ as the side information to encode $\mathbf{X_M}$ to $S_{M,1}$,

$$f_{M,1}(\mathbf{X_1}, ..., \mathbf{X_M}, S_{1,1}, S_{2,1}, ..., S_{M-1,1}) = S_{M,1}$$

$S_{M,1}$ is decoded to $\hat{\mathbf{X}}_{\mathbf{M,1}}$ using decoding function $g_{M,1}$,

$$g_{M,1}(S_{1,1}, S_{2,1}, ..., S_{M-1,1}, S_{M,1}) = \hat{\mathbf{X}}_{\mathbf{M,1}}$$

M_2^{th} **Layer** In this layer, encoding function $f_{M,2}$ uses all the previous encoded frames and $\mathbf{X_{M,2}}$ as the side information to encode $\mathbf{X_{M,2}}$ to $S_{M,2}$,

$$f_{M,2}(\mathbf{X_1}, ..., \mathbf{X_M}, S_{1,1}, S_{1,2}, ..., S_{M,1}) = S_{M,2}$$

$S_{M,2}$ is decoded to $\hat{\mathbf{X}}_{\mathbf{M,2}}$ using decoding function $g_{M,2}$,

$$g_{M,2}(S_{1,1}, S_{1,2}, ..., S_{M,1}) = \hat{\mathbf{X}}_{\mathbf{M,2}}$$

In addition, it is easy to extend the achievable region in (6) as follows,

Theorem 10 *Given IID sources* $(X_1, ..., X_M)$*, the set of* $(R_1, ..., R_{M,1}, R_{M,2}, D_1, ..., D_{M,1}, D_{M,2})$ *is achievable if there exist auxiliary random variables* $(U_1, \ldots, U_M)$ *and* $(W_2, \ldots, W_M)$ *and functions* $(g_1, g_{2,1}, g_{2,2}, ..., g_{M,1}, g_{M,2})$ *such that*

$$\begin{aligned}
R_1 &\geq I(X_1; U_1) \\
R_{k,1} &\geq I(X_1, ..., X_k; W_k | U_1, ..., U_{k-1}, W_2, ..., W_{k-1}) + I(U_1; W_k | W_2, ..., W_{k-1}, U_2, ..., U_{k-1}) \\
R_{k,2} &\geq I(X_1, ..., X_k; U_k | U_1, ..., U_{k-1}, W_2, ..., W_k) \qquad k = 2, .., M-1 \\
R_{M,1} &\geq I(X_M; W_M | U_1, ..., U_{M-1}, W_2, ..., W_{M-1}) + I(U_1; W_M | U_2, ..., U_{M-1}, W_2, ..., W_{M-1}) \\
R_{M,2} &\geq I(X_M; U_M | U_1, ..., U_{M-1}, W_2, ..., W_M)
\end{aligned}$$

and

$$\begin{aligned}
D_1 &\geq E[d(X_1, g_1(U_1))] \\
D_{k,1} &\geq E[d(X_k, g_{k,1}(W_1, ..., W_k))] \\
D_{k,2} &\geq E[d(X_k, g_{k,2}(U_1, W_1, ..., W_k))]
\end{aligned}$$

and

$$\begin{aligned}
U_1 &\rightarrow X_1 \rightarrow (X_2, ..., X_M) \\
W_k &\rightarrow (X_1, ..., X_k, U_1, ..., U_{k-1}, W_2, ..., W_{k-1}) \rightarrow (X_{k+1}, ..., X_M) \\
U_k &\rightarrow (X_1, ..., X_k, U_1, ..., U_{k-1}, W_2, ..., W_k) \rightarrow (X_{k+1}, ..., X_M) \\
W_M &\rightarrow (X_M, U_1, ..., U_{M-1}, W_2, ..., W_{M-1}) \rightarrow (X_1, ..., X_{M-1}) \\
U_M &\rightarrow (X_M, U_1, ..., U_{M-1}, W_2, ..., W_M) \rightarrow (X_1, ..., X_{M-1})
\end{aligned}$$

The proof is straightforward, followed by the proof in Appendix C. Having modeled CVC-PL with M frames and having derived an achievable region for M-frame CVC-PL, in future work, we are interested in deriving the minimum total rate in the case of M jointly Gaussian distributed sources with covariance matrix shown in (4.2),

$$\sum_{X_1, ..., X_M} = \begin{pmatrix} 1 & a_{12} & \ldots & a_{1M} \\ a_{21} & 1 & \ldots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \ldots & 1 \end{pmatrix} \tag{4.2}$$

"What are the sufficient and necessary conditions to achieve the minimum total rate for different distortion profiles?" is the question that will be answered in the future.

APPENDICES

Appendix A

Achievable Region of a Multiple Description Coding with a Gaussian Distributed Source

In this appendix, we prove Theorem 4.

A.1 Outer-Bound

CASE 1:$D_1 + D_2 < D_0(1 + \exp(-2(R_1 + R_2)))$
If the set of $(R_0, R_1, R_2, R_3, D_0, D_1, D_2, D_3)$ is achievable then,

$$\begin{aligned}
R_0 &\geq \frac{1}{n}H(S_0) \geq \frac{1}{n}I(\mathbf{X};\hat{\mathbf{X}}_\mathbf{0}) \overset{(a)}{\geq} \frac{1}{2}\log(\frac{1}{D_0}) \\
R_0 + R_1 &\geq \frac{1}{n}(H(S_0) + H(S_1)) \geq \frac{1}{n}H(S_0, S_1) \geq \frac{1}{n}I(\mathbf{X};\hat{\mathbf{X}}_\mathbf{1}) \overset{(b)}{\geq} \frac{1}{2}\log(\frac{1}{D_1}) \\
R_0 + R_2 &\geq \frac{1}{n}(H(S_0) + H(S_2)) \geq \frac{1}{n}H(S_0, S_2) \geq \frac{1}{n}I(\mathbf{X};\hat{\mathbf{X}}_\mathbf{2}) \overset{(c)}{\geq} \frac{1}{2}\log(\frac{1}{D_2})
\end{aligned}$$

where (a,b,c) come from the rate-distortion function of zero-mean Gaussian dostributed random variable derived in Example 1. Hence,

$$\begin{aligned}
D_0 &\geq \exp(-2R_0) \\
D_1 &\geq \exp(-2(R_0 + R_1)) \\
D_2 &\geq \exp(-2(R_0 + R_2))
\end{aligned} \tag{A.1}$$

To derive an upper-bound on D_3, we proceed as follows

$$
\begin{aligned}
R_0 + R_1 + R_2 + R_3 &\geq \frac{1}{n}(H(S_0) + H(S_1) + H(S_2) + H(S_3)) \\
&\overset{(a)}{\geq} \frac{1}{n}[H(S_0) + H(S_3) + H(S_1|S_0) + H(S_2|S_0) \\
&+ H(S_1, S_2|S_0) - H(S_1, S_2|S_0)] \\
&\overset{(b)}{\geq} \frac{1}{n}I(\mathbf{X}; S_0, S_1, S_2, S_3) + \frac{1}{n}I(S_1; S_2|S_0) \\
&\overset{(c)}{\geq} \frac{1}{n}I(\mathbf{X}; \hat{\mathbf{X}}_\mathbf{3}) + \frac{1}{n}I(\hat{\mathbf{X}}_\mathbf{1}; \hat{\mathbf{X}}_\mathbf{2}|S_0) \\
&\overset{(d)}{\geq} \frac{1}{2}\log\frac{1}{D_3} + \frac{1}{n}I(\hat{\mathbf{X}}_\mathbf{1}; \hat{\mathbf{X}}_\mathbf{2}|S_0)
\end{aligned}
\tag{A.2}
$$

where

(a). Condition reduces entropy, i.e. $H(A|B) \leq H(A)$

(b). Entropy chain inequality, i.e. $H(A, B) \leq H(A) + H(B)$ and the entropy of a function of a random variable given that random variable is zero, i.e. $H(f(A)|A) = 0$

(c). Markov chain inequality, i.e. $I(A; B) \geq I(A; f(B))$

(d). Rate-distortion theory, i.e. Theorem(1)

Lets define random variable $Y = X + Z$, where Z is a zero-mean Gaussian distributed random variable with variance ϵ, independent of $S_0, \hat{\mathbf{X}}_\mathbf{1}, \hat{\mathbf{X}}_\mathbf{2}$, i.e., $Y \to X \to (S_0, \hat{\mathbf{X}}_\mathbf{1}, \hat{\mathbf{X}}_\mathbf{2})$. Hence,

$$
\begin{aligned}
I\; & (\hat{\mathbf{X}}_\mathbf{1}; \hat{\mathbf{X}}_\mathbf{2}|S_0) \\
\geq\; & I(\mathbf{Y}; \hat{\mathbf{X}}_\mathbf{1}|S_0) + I(\mathbf{Y}; \hat{\mathbf{X}}_\mathbf{2}|S_0) + I(\mathbf{Y}; S_0) + I(\mathbf{Y}; S_0) - 2I(\mathbf{Y}; S_0) - I(\mathbf{Y}; \hat{\mathbf{X}}_\mathbf{1}, \hat{\mathbf{X}}_\mathbf{2}|S_0) \\
=\; & I(\mathbf{Y}; \hat{\mathbf{X}}_\mathbf{1}, S_0) + I(\mathbf{Y}; \hat{\mathbf{X}}_\mathbf{2}, S_0) - I(\mathbf{Y}; S_0) - h(\mathbf{Y}) + h(\mathbf{Y}|\hat{\mathbf{X}}_\mathbf{1}, \hat{\mathbf{X}}_\mathbf{2}, S_0) \\
\overset{(a)}{\geq}\; & I(\mathbf{Y}; \hat{\mathbf{X}}_\mathbf{0}, \hat{\mathbf{X}}_\mathbf{1}) + I(\mathbf{Y}; \hat{\mathbf{X}}_\mathbf{0}, \hat{\mathbf{X}}_\mathbf{2}) - I(\mathbf{Y}; S_0) - \frac{n}{2}\log(2\pi e(1+\epsilon)) + h(\mathbf{Y}|\hat{\mathbf{X}}_\mathbf{1}, \hat{\mathbf{X}}_\mathbf{2}, S_0) \\
\overset{(b)}{\geq}\; & \frac{n}{2}\log\frac{1+\epsilon}{D_1+\epsilon} + \frac{n}{2}\log\frac{1+\epsilon}{D_2+\epsilon} - 2\frac{n}{2}\log(2\pi e(1+\epsilon)) + h(\mathbf{Y}|S_0) + h(\mathbf{Y}|\hat{\mathbf{X}}_\mathbf{1}, \hat{\mathbf{X}}_\mathbf{2}, S_0)
\end{aligned}
\tag{A.3}
$$

where

(a). Markov chain inequality, i.e. $I(A;B) \geq I(A;f(B))$

(b). Rate-distortion theory

Before going to the rest of the proof, we present a proposition called "Entropy Power Inequality" as follows,

Proposition[8]
Let $W \to X \to Y$ be a Markov chain and $Y = X + Z$ and Z is independent of W then

$$\exp(\frac{1}{n}h(\mathbf{Y}|\mathbf{W})) \geq \exp(\frac{1}{n}h(\mathbf{X}|\mathbf{W})) + \exp(\frac{1}{n}h(\mathbf{Z})) \tag{A.4}$$

Using entropy power inequality in (A.4), we have

$$\begin{aligned}\frac{2}{n}h(\mathbf{Y}|S_0) &\geq \log[\exp(\frac{2}{n}h(\mathbf{X}|S_0)) + 2\pi e\epsilon] \\ &\geq \log[2\pi e(\exp(-2R_0) + \epsilon)]\end{aligned} \tag{A.5}$$

and

$$\frac{2}{n}h(\mathbf{Y}|W) \geq \log(\exp(\frac{2}{n}h(\mathbf{X}|W)) + 2\pi e\epsilon) \tag{A.6}$$

where $W = (S_0, \hat{\mathbf{X}}_\mathbf{1}, \hat{\mathbf{X}}_\mathbf{2})$. Substituting (A.5) and (A.6) into (A.3), It can be easily shown that,

$$\begin{aligned}I(\hat{\mathbf{X}}_\mathbf{1};\hat{\mathbf{X}}_\mathbf{2}|S_0) &\geq \frac{n}{2}\log\frac{(\exp(-2R_0)+\epsilon)(1+\epsilon)}{(D_1+\epsilon)(D_2+\epsilon)} \\ &- \frac{n}{2}\log(2\pi e(1+\epsilon)) + \frac{n}{2}\log(\exp(-2(R_0+R_1+R_2)t+\epsilon)\end{aligned} \tag{A.7}$$

where

$$t = \frac{2}{n}I(\hat{\mathbf{X}}_\mathbf{1};\hat{\mathbf{X}}_\mathbf{2}|S_0)$$

Isolating t,

$$t \geq \frac{(\bar{\epsilon})(\bar{\epsilon}+1)}{\bar{\epsilon}^2 + \bar{\epsilon}(1+\Delta-\Pi)+\Delta} \tag{A.8}$$

Maximizing on $\bar{\epsilon}$,

$$t \geq \frac{1}{1-(\sqrt{\Pi}-\sqrt{\Delta})^2} \tag{A.9}$$

Substituting (A.9) into (A.2) plus the results in (A.1), we proved the converse of Theorem (4).
CASE 2:$D_1 + D_2 \geq D_0(1 + \exp(-2(R_1 + R_2)))$
It is easy to derive the outer-bound in case 2,

$$\begin{aligned}
R_0 &\geq \frac{1}{n}H(S_0) \geq \frac{1}{n}I(\mathbf{X};\hat{\mathbf{X}}_{\mathbf{0}}) \geq \frac{1}{2}\log(\frac{1}{D_0}) \\
R_0 + R_1 &\geq \frac{1}{n}(H(S_0) + H(S_1)) \geq \frac{1}{n}H(S_0, S_1) \geq \frac{1}{n}I(\mathbf{X};\hat{\mathbf{X}}_{\mathbf{1}}) \geq \frac{1}{2}\log(\frac{1}{D_1}) \\
R_0 + R_2 &\geq \frac{1}{n}(H(S_0) + H(S_2)) \geq \frac{1}{n}H(S_0, S_2) \geq \frac{1}{n}I(\mathbf{X};\hat{\mathbf{X}}_{\mathbf{2}}) \geq \frac{1}{2}\log(\frac{1}{D_2}) \\
R_0 + R_1 + R_2 + R_3 &\geq \frac{1}{n}(H(S_0) + H(S_1) + H(S_2) + H(S_3)) \\
&\geq \frac{1}{n}H(S_0, S_1, S_2, S_3) \\
&\geq \frac{1}{n}I(\mathbf{X};\hat{\mathbf{X}}_{\mathbf{3}}) \\
&\geq \frac{1}{2}\log(\frac{1}{D_3})
\end{aligned}$$

A.2 Inner-bound

Before going through the direct part of the proof, we present the following achievable region which is an extension to the achievable region in [16]. According to the rate-distortion achievable region in [16], the set $(R_0, R_1, R_2, R_3, D_0, D_1, D_2, D_3)$ is achievable if there exist auxiliary random variables (U_0, U_1, U_2, U_3) jointly distributed with X and if there exist functions (g_0, g_1, g_2, g_3) such that

$$\begin{aligned}
R_0 &\geq I(X; U_0) \\
R_1 &\geq I(X; U_1|U_0) \\
R_2 &\geq I(X; U_2|U_0) \\
R_1 + R_2 &\geq I(X; U_1, U_2|U_0) + I(U_1; U_2|U_0) \\
R_3 &\geq I(X; U_3|U_0, U_1, U_2)
\end{aligned}$$

and

$$
\begin{aligned}
E[d(X, g_0(U_0))] &\leq D_0 \\
E[d(X, g_1(U_0, U_1))] &\leq D_1 \\
E[d(X, g_2(U_0, U_2))] &\leq D_2 \\
E[d(X, g_3(U_0, U_1, U_2, U_3))] &\leq D_3
\end{aligned}
\tag{A.10}
$$

CASE 1:$D_1 + D_2 < D_0(1 + \exp(-2(R_1 + R_2)))$
We introduce a coding scheme and show that the concluded inner-bound coincides with the outer-bound derived in the *Convers* part. In the first step, we quantize X such that $X = \hat{X}_0 + W_0$ and $W_0 \sim \mathbf{N}(0, D_0)$. Afterwards, we send U_1 and U_2 such that

$$
\begin{aligned}
U_1 &= W_0 + N_1 \\
U_2 &= W_0 + N_2
\end{aligned}
$$

where the covariance matrix of N_1 and N_2, is

$$
\sum_{N_1, N_2} = \begin{pmatrix} \sigma_{N_1}^2 & \rho\sigma_{N_1}\sigma_{N_2} \\ \rho\sigma_{N_1}\sigma_{N_2} & \sigma_{N_2}^2 \end{pmatrix}
$$

and for $i = 1, 2$

$$
D_i = \frac{D_0 \sigma_{N_i}^2}{D_0 + \sigma_{N_i}^2}
$$

The MMSE of W_0 given U_i, i.e. $MMSE(W_0|U_i)$ for i=1,2 is

$$
MMSE(W_0|U_i) = \frac{D_0}{D_0 + \sigma_{N_i}^2} U_i
$$

Therefor, $\hat{X}_1$ and $\hat{X}_2$ would be,

$$
\begin{aligned}
\hat{X}_1 &= \hat{X}_0 + MMSE(W_0|U_1) \\
\hat{X}_2 &= \hat{X}_0 + MMSE(W_0|U_2)
\end{aligned}
$$

We define U_3 as the residue of W_0 and $MMSE(W_0|U_1, U_2)$, i.e. $U_3 = W_0 - MMSE(W_0|U_1, U_2)$, which is a zero-mean Gaussian distributed random variable with variance d, where

$$
MMSE(W_0|U_1, U_2) = c_1 U_1 + c_2 U_2 \tag{A.11}
$$

and for $i, j \in \{1, 2\}$ and $i \neq j$,

$$c_i = \frac{D_0(\sigma_{N_j}^2 - \rho\sigma_{N_1}\sigma_{N_2})}{\sigma_{N_1}^2\sigma_{N_2}^2(1-\rho^2) + D_0(\sigma_{N_1}^2 + \sigma_{N_2}^2) - 2D_0\rho\sigma_{N_1}\sigma_{N_2}}$$

The variance d is derived as follows,

$$d = \frac{D_0\sigma_{N_1}^2\sigma_{N_2}^2(1-\rho^2)}{\sigma_{N_1}^2\sigma_{N_2}^2(1-\rho^2) + D_0(\sigma_{N_1}^2 + \sigma_{N_2}^2) - 2D_0\rho\sigma_{N_1}\sigma_{N_2}} \tag{A.12}$$

Finally U_3 is quantized such that $U_3 = \hat{X}_3 + W_3$ and $W_3 \sim \mathbf{N}(0, D_3)$.

Substituting all the random variables in (A.10), we have,

$$\begin{aligned} D_0 &\geq \exp(-2R_0) && \text{(A.13)} \\ D_i &\geq D_0\exp(-2R_i) && \text{(A.14)} \\ R_1 + R_2 &\geq \frac{1}{2}\log(\frac{D_0^2}{D_1D_2(1-\rho^2)}) && \text{(A.15)} \\ D_3 &\geq d\exp(-2R_3) && \text{(A.16)} \end{aligned}$$

From (A.15), we have

$$\rho^2 \leq \frac{\bar{D}_1\bar{D}_2 - \exp(-2(R_1+R_2))}{\bar{D}_1\bar{D}_2} \tag{A.17}$$

where $\bar{Y} = Y/D_0$.

Using inequality in (A.17), we arbitrarily choose ρ as,

$$\rho = -\frac{\sqrt{\bar{D}_1\bar{D}_2 - \exp(-2(R_1+R_2))}}{\sqrt{\bar{D}_1\bar{D}_2}} \tag{A.18}$$

Substitute (A.18) into (A.12) and (A.16) we have,

$$D_3 \geq \frac{D_0\exp\left(-2(R_1+R_2+R_3)\right)}{1-(\sqrt{\Pi}-\sqrt{\Delta})^2}$$

Now, we got an achievable region as follows,

$$\begin{aligned} D_0 &\geq \exp(-2R_0) \\ D_1 &\geq \exp(-2(R_0+R_1)) \\ D_2 &\geq \exp(-2(R_0+R_2)) \\ D_3 &\geq D_0\exp(-2(R_1+R_2+R_3))\frac{1}{1-(\sqrt{\Pi}-\sqrt{\Delta})^2} && \text{(A.19)} \end{aligned}$$

$$\Delta = \bar{D}_1\bar{D}_2 - \exp(-2(R_1+R_2))$$
$$\Pi = (1-\bar{D}_1)(1-\bar{D}_2)$$

where $\bar{Y} = Y/D_0$. Therefor, one can say that the following region is achievable which is a subspace of the region in (A.19)

$$\begin{aligned}
D_0 &= \exp(-2R_0)\\
D_1 &\geq \exp(-2(R_0+R_1))\\
D_2 &\geq \exp(-2(R_0+R_2))\\
D_3 &\geq D_0\exp(-2(R_1+R_2+R_3))\frac{1}{1-\left(\sqrt{\Pi}-\sqrt{\Delta}\right)^2}
\end{aligned} \tag{A.20}$$

Since in (A.20), $D_0 = \exp(-2R_0)$, we could substitute all of D_0 in (A.20) with $\exp(-2R_0)$,

$$\begin{aligned}
D_0 &= \exp(-2R_0)\\
D_1 &\geq \exp(-2(R_0+R_1))\\
D_2 &\geq \exp(-2(R_0+R_2))\\
D_3 &\geq \exp(-2(R_0+R_1+R_2+R_3))\frac{1}{1-\left(\sqrt{\Pi}-\sqrt{\Delta}\right)^2}
\end{aligned} \tag{A.21}$$

where $\bar{Y} = Y/\exp(-2R_0)$. We conclude that the following region is achievable,

$$\begin{aligned}
D_0 &\geq \exp(-2R_0)\\
D_1 &\geq \exp(-2(R_0+R_1))\\
D_2 &\geq \exp(-2(R_0+R_2))\\
D_3 &\geq \exp(-2(R_0+R_1+R_2+R_3))\frac{1}{1-\left(\sqrt{\Pi}-\sqrt{\Delta}\right)^2}
\end{aligned} \tag{A.22}$$

CASE 2:$D_1 + D_2 \geq D_0(1+\exp(-2(R_1+R_2)))$
We introduce $\hat{X}_0, U_1, U_2$ and U_3 such that they are independent of each other and independent of N_3,

$$X = \hat{X}_0 + U_1 + U_2 + U_3 + N_3$$

where

$$\begin{aligned}
\hat{U}_0 &\sim \mathbf{N}(0, 1-D_0)\\
\hat{U}_1 &\sim \mathbf{N}(0, D_0-D_1)\\
\hat{U}_2 &\sim \mathbf{N}(0, D_0-D_2)\\
U_3 &\sim \mathbf{N}(0, D_2+D_1-D_0-D_3)\\
N_3 &\sim \mathbf{N}(0, D_3)
\end{aligned} \tag{A.23}$$

and

$$\begin{aligned}
\hat{X}_1 &= \hat{X}_0 + U_1 \\
\hat{X}_2 &= \hat{X}_0 + U_2 \\
\hat{X}_3 &= \hat{X}_0 + U_1 + U_2 + U_3
\end{aligned}$$

Substituting $(A.23)$ into $(A.10)$, we have

$$\begin{aligned}
R_0 &\geq \frac{1}{2}\log\frac{1}{D_0} && \text{(A.24)} \\
R_1 &\geq \frac{1}{2}\log\frac{D_0}{D_1} && \text{(A.25)} \\
R_2 &\geq \frac{1}{2}\log\frac{D_0}{D_2} && \text{(A.26)} \\
R_1 + R_2 &\geq I(X;U_1,U_2|U_0) + I(U_1;U_2|U_0) && \text{(A.27)} \\
R_3 &\geq I(X;U_3|U_0,U_1,U_2) && \text{(A.28)}
\end{aligned}$$

Since U_0, U_1 and U_2 are independent, $I(U_1;U_2|U_0) = 0$. Hence,

$$I(X;U_1,U_2|U_0) + I(U_1;U_2|U_0) = \frac{1}{2}\log\frac{D_0}{D_1 + D_2 - D_0} \qquad \text{(A.29)}$$

On the other hand, we can show that

$$I(X;U_3|U_0,U_1,U_2) = \frac{1}{2}\log\frac{D_1 + D_2 - D_0}{D_3} \qquad \text{(A.30)}$$

Substituting (A.29) and (A.30) into (A.27) and substituting U_3 introduced in (A.23) into (A.28), we have

$$\begin{aligned}
R_1 + R_2 &\geq \frac{1}{2}\log\frac{D_0}{D_1 + D_2 - D_0} && \text{(A.31)} \\
R_3 &\geq \frac{1}{2}\log\frac{D_1 + D_2 - D_0}{D_3} && \text{(A.32)}
\end{aligned}$$

From (A.31), we can get an outer-bound on $D_1 + D_2 - D_0$ as follows

$$D_1 + D_2 - D_0 \geq D_0 \exp(-2(R_1 + R_2)) \qquad \text{(A.33)}$$

Using (A.32) to find an outer-bound on D_3 and then substituting (A.24) and (A.33) into the derived outer-bound, we have

$$\begin{aligned} D_3 &\geq (D_1 + D_2 - D_0)\exp(-2(R_3)) \\ &\geq D_0 \exp(-2(R_1 + R_2 + R_3)) \\ &\geq \exp(-2(R_0 + R_1 + R_2 + R_3)) \end{aligned} \tag{A.34}$$

From (A.24), (A.25),(A.26) and (A.34), the distortion-rate region in case 2 is complete,

$$\begin{aligned} D_0 &\geq \exp(-2R_0) \\ D_1 &\geq \exp(-2(R_0 + R_1)) \\ D_2 &\geq \exp(-2(R_0 + R_2)) \\ D_3 &\geq \exp(-2(R_0 + R_1 + R_2 + R_3)) \end{aligned}$$

Appendix B

Proof of Lemma 1

It is easy to see that

$$det(Cov(U_1, W|Y)) = \sigma^2_{U_1|Y}\sigma^2_{W|Y} - [r + \rho\sigma_{N_1'}\sigma_{N_W} - \frac{r}{1+\epsilon}]$$

So to satisfy $h(U_1|Y) + h(W|Y) - h(U_1, W|Y) = 0$, we need to have $r + \rho\sigma_{N_1'}\sigma_{N_W} - \frac{r}{1+\alpha} = 0$, or

$$\alpha = \frac{-\rho\sigma_{N_1'}\sigma_{N_W}}{r + \rho\sigma_{N_1'}\sigma_{N_W}}$$

To make α non-negative, we have $-r \leq \rho\sigma_{N_1'}\sigma_{N_W} \leq 0$. So, sufficient conditions to have $-r \leq \rho\sigma_{N_1'}\sigma_{N_W} \leq 0$ for any $0 \leq D_2 \leq 1$ are

$$\frac{1}{D_3} > -1 + \frac{1}{D_2} + \frac{1}{D_r}$$

and

$$D_3 > -1 + D_2 + D_r.$$

Appendix C

Generating Codebooks, Encoding and Decoding of an Achievable Rate-Distortion Region for CVC-PL with Two Frames

C.1 Generating Codebooks and Encoding

Generate a rate-distortion codebook C_1 consisting of 2^{nR_1} sequences $\mathbf{U}_1$ drawn IID$\sim \prod_i p(u_{1i})$. Denote the sequences $\mathbf{U}_1(1), \ldots, \mathbf{U}_1(2^{nR_1})$. Given a sequence $\mathbf{x}_1$, index it by w_1 if there exists a w_1 such that $(\mathbf{x}_1, \mathbf{U}_1(w_1)) \in A_\epsilon^{*(n)}$, the strongly jointly typical set. If there is more than one such w_1, send the first in lexicographic order. If there is no such w_1, let $w_1 = 1$. Generate a rate-distortion codebook C_2 consisting of 2^{nR_2} sequences $\mathbf{U}_2$ drawn IID$\sim \prod_i p(u_{2i})$. Denote the sequences $\mathbf{U}_2(1), \ldots, \mathbf{U}_2(2^{nR_2})$. Given a sequence $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}_1(w_1))$, index it by w_2 if there exists a w_2 such that $(\mathbf{x}_1, \mathbf{x}_2, , \mathbf{u}_1(w_1), \mathbf{U}_2(w_2)) \in A_\epsilon^{*(n)}$, the strongly jointly typical set. If there is more than one such w_2, send the first in lexicographic order. If there is no such w_2, let $w_2 = 1$. Generate a rate-distortion codebook C_3 consisting of 2^{nR_3} sequences $\mathbf{U_3}$ drawn IID$\sim \prod_i p(u_{3i}|u_{1i}, u_{2i})$. Denote the sequences $\mathbf{U}_3(1), \ldots, \mathbf{U}_3(2^{nR_3})$. Given a sequence $\mathbf{x_2}$, index it by w_3 if there exists a w_3 such that $(\mathbf{x}_2, \mathbf{U}_1(w_1), \mathbf{U}_2(w_2), \mathbf{U}_3(w_3)) \in A_\epsilon^{*(n)}$, the strongly jointly typical set. If there is more than one such w_3, send the first in lexicographic order. If there is no such w_3, let $w_3 = 1$.

C.2 Decoding

Decoder i receives $\mathbf{U}_i(w_i)$ and produces $\hat{\mathbf{X}}_i = g_i(\mathbf{U}_i(w_i))$ where $i = 1, 2$. The third decoder receives $(\mathbf{U}_1(w_1), \mathbf{U}_2(w_2), \mathbf{U}_3(w_3))$ and produces $\hat{\mathbf{X}}_3 = g_3(\mathbf{U}_1(w_1), \mathbf{U}_2(w_2), \mathbf{U}_3(w_3))$.

Appendix D

Generating Codebooks, Encoding and Decoding of the Rate-Distortion Achievable Region in Theorem (3.1)

D.1 Generating Codebooks and Encodings

Generate a rate-distortion codebook C_1 consisting of 2^{nR_1} sequences $\mathbf{U}_1$ drawn IID$\sim \prod_i p(u_{1i})$. Denote the sequences $\mathbf{U}_1(1), \ldots, \mathbf{U}_1(2^{nR_1})$. Generate a rate-distortion codebook C_{21} consisting of $2^{nR_{21}}$ sequences $\mathbf{U}_{21}$ drawn IID$\sim \prod_i p(u_{21i})$. Denote the sequences $\mathbf{U}_{21}(1), \ldots, \mathbf{U}_{21}(2^{nR_2})$. Given a sequence $(\mathbf{x}_1)$, index them by (w_1, w_{21}) if there exists a (w_1, w_{21}) such that $(\mathbf{x}_1, \mathbf{U}_1(w_1), \mathbf{U}_{21}(w_{21})) \in A_\epsilon^{*(n)}$, the strongly jointly typical set. If there is more than one such (w_1, w_{21}), send the first in lexicographic order. If there is no such (w_1, w_{21}), let $(w_1, w_{21}) = (1, 1)$. Generate a rate-distortion codebook C_{22} consisting of $2^{nR_{22}}$ sequences $\mathbf{U}_{22}$ drawn IID$\sim \prod_i p(u_{22i}|u_{21i})$. Denote the sequences $\mathbf{U}_{22}(1), \ldots, \mathbf{U}_{22}(2^{nR_3})$. Given a sequence $\mathbf{x}_2$, index it by w_{22} if there exists a w_{22} such that $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{U}_{21}(w_{21}), \mathbf{U}_{22}(w_{22})) \in A_\epsilon^{*(n)}$, the strongly jointly typical set. If there is more than one such w_{22}, send the first in lexicographic order. If there is no such w_{22}, let $w_{22} = 1$. Generate a rate-distortion codebook C_3 consisting of 2^{nR_3} sequences $\mathbf{U}_3$ drawn IID$\sim \prod_i p(u_{3i}|u_{1i}, u_{21i}, u_{22i})$. Denote the sequences $\mathbf{U}_3(1), \ldots, \mathbf{U}_3(2^{nR_3})$. Given a sequence $\mathbf{X}_2$, index it by w_3 if there exists a w_3 such that $(\mathbf{x}_2, \mathbf{U}_1(w_1), \mathbf{U}_2(w_2), \mathbf{U}_3(w_3)) \in A_\epsilon^{*(n)}$, the strongly jointly typical set. If there is more than one such w_3, send the first in lexicographic order. If there is no such w_3, let $w_3 = 1$.

D.2 Decoding

The first decoder receives $\mathbf{U}_1(w_2)$ and produces $\hat{\mathbf{X}}_1 = g_1(\mathbf{U}_1(w_1))$ where $i = 1, 2$. The second decoder receives $(\mathbf{U}_{21}(w_{21}), \mathbf{U}_{22}(w_{22}))$ and produces $\hat{\mathbf{X}}_2 = g_2(\mathbf{U}_{21}(w_{21}), \mathbf{U}_{22}(w_{22}))$. The third decoder receives $(\mathbf{U}_1(w_1), \mathbf{U}_{21}(w_{21}), \mathbf{U}_{22}(w_{22}), \mathbf{U}_3(w_3))$ and produces $\hat{\mathbf{X}}_3 = g_3(\mathbf{U}_1(w_1), \mathbf{U}_{21}(w_{21}), \mathbf{U}_{22}(w_{22}))$.

Appendix E

Generating Codebooks, Encoding and Decoding of an Achievable Rate-Distortion Region when Both Sources are Available at the Encoders

E.1 Generating Codebooks and Encoding

Generate a rate-distortion codebook C_1 consisting of 2^{nR_1} sequences $\mathbf{U}_1$ drawn IID$\sim \prod_i p(u_{1i})$. Denote the sequences $\mathbf{U}_1(1), \dots, \mathbf{U}_1(2^{nR_1})$. Generate a rate-distortion codebook C_2 consisting of 2^{nR_2} sequences $\mathbf{U_2}$ drawn IID$\sim \prod_i p(u_{2i})$. Denote the sequences $\mathbf{U}_2(1), \dots, \mathbf{U}_2(2^{nR_2})$. Given a sequence $(\mathbf{x_1}, \mathbf{x_2})$, index them by (w_1, w_2) if there exists a (w_1, w_2) such that $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{U}_1(w_1), \mathbf{U}_2(w_2)) \in A_\epsilon^{*(n)}$, the strongly jointly typical set. If there is more than one such (w_1, w_2), send the first in lexicographic order. If there is no such (w_1, w_2), let $(w_1, w_2) = (1, 1)$. Generate a rate-distortion codebook C_3 consisting of 2^{nR_3} sequences $\mathbf{U}_3$ drawn IID$\sim \prod_i p(u_{3i}|u_{1i}, u_{2i})$. Denote the sequences $\mathbf{U}_3(1), \dots, \mathbf{U}_3(2^{nR_3})$. Given a sequence $\mathbf{x}_2$, index it by w_3 if there exists a w_3 such that $(\mathbf{x}_2, \mathbf{U}_1(w_1), \mathbf{U}_2(w_2), \mathbf{U}_3(w_3)) \in A_\epsilon^{*(n)}$, the strongly jointly typical set. If there is more than one such w_3, send the first in lexicographic order. If there is no such w_3, let $w_3 = 1$.

E.2 Decoding

Decoder i receives $\mathbf{U}_i(w_i)$ and produces $\hat{\mathbf{X}}_\mathbf{i} = g_i(\mathbf{U}_\mathbf{i}(w_i))$ where $i = 1, 2$. The third decoder receives $(\mathbf{U}_1(w_1), \mathbf{U}_2(w_2), \mathbf{U}_3(w_3))$ and produces $\hat{\mathbf{X}}_\mathbf{3} = g_3(\mathbf{U}_1(w_1), \mathbf{U}_2(w_2), \mathbf{U}_3(w_3))$.

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