Some aspects of Cantor sets

by

Ka Shing Ng

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

For every positive, decreasing, summable sequence $a = (a_i)$, we can construct a Cantor set C_a associated with a. These Cantor sets are not necessarily self-similar. Their dimensional properties and measures have been studied in terms of the sequence a.

In this thesis, we extend these results to a more general collection of Cantor sets. We study their Hausdorff and packing measures, and compare the size of Cantor sets with the more refined notion of dimension partitions. The properties of these Cantor sets in relation to the collection of cut-out sets are then considered. The multifractal spectrum of **p**-Cantor measures on these Cantor sets are also computed. We then focus on the special case of homogeneous Cantor sets and obtain a more accurate estimate of their exact measures. Finally, we prove the L^p-improving property of the **p**-Cantor measure on a homogeneous Cantor set as a convolution operator.

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Dedication

This work is dedicated to my family and my dear friends.

Table of Contents

Chapter 1

Preliminaries

1.1 Introduction

In fractal geometry, self-similar sets are well-known to many people. A self-similar set is the attractor of a family of contracting similarities. Their measures, dimensions, multifractal spectrum and many other properties have been studied (See [24, 22, 7, 11]). Among them the middle-third Cantor set is probably the most famous example.

A Cantor set is a perfect, totally disconnected, compact subset of the real line $\mathbb{R}$. There are different types of Cantor sets though, including the central Cantor sets and the Cantor sets C_a associated with a sequence $a = (a_i)$. They are not necessarily self-similar, so we do not have the same machinery to study them. In this thesis, we study a collection of Cantor sets which cover these examples and extend some results on them.

The Cantor set C_a associated with a decreasing summable sequence $a = (a_i)$ is a Cantor set having gaps with lengths a_i. We also call it a decreasing Cantor set in this thesis. Its Hausdorff measure and dimension have already been studied in [2]. The dimension can be calculated in terms of the tails of the sequence a.

However, even when α is the Hausdorff dimension of a set C, the Hausdorff measure, $H^\alpha(C)$, may still be 0 or ∞. We need a more general dimension function h and h-Hausdorff measure, $H^h(C)$, to measure the size of C. This gives us a more refined description of the dimension of a set. The packing measure and packing dimension introduced by Tricot ([32], see also [31]), as the dual concepts of the Hausdorff measure and Hausdorff dimension, together with the more general h-packing measure, provide an even more complete picture of the size of a set.

These h-measures of decreasing Cantor sets C_a are estimated in [3], [13] and [6]. The partition of dimension functions were used to classify these Cantor sets in [6].

We generalize these results in Chapter 2 to a more extensive collection of Cantor sets which we call balanced Cantor sets. They include both the decreasing Cantor sets and the central Cantor sets. In Chapter 3, we consider the relative size of the balanced Cantor sets within the collection of all cut-out sets.

The multifractal spectrum of **p**-Cantor measures defined on central Cantor sets [18] and decreasing Cantor sets [19] have been studied. We extend these results to the **p**-Cantor measures on the balanced Cantor sets with a fixed number of divisions in Chapter 4.

The homogeneous Cantor sets are generalizations of the central Cantor sets, and they are balanced Cantor sets, as well. The exact Hausdorff measures of homogeneous Cantor sets are calculated in [27, 28] and the exact packing measures of central Cantor sets are obtained in [14]. In Chapter 5, we improve an estimation in the former case and extend the latter packing measure result to the homogeneous Cantor sets.

In the last chapter, we will prove that the **p**-Cantor measures on homogeneous Cantor sets, acting as convolution operators, have the L^p-improving property.

In the remainder of this chapter, we establish notations and definitions. We begin by introducing certain collections of Cantor sets as our examples.

1.2 Cut-out Cantor sets

Any compact set E on the real line $\mathbb{R}$, with zero Lebesgue measure $m(E) = 0$, is of the form

$$E = I \setminus \bigcup_{i=1}^{\infty} A_i,$$

where I is a closed and bounded interval and $\{A_i\}$ is a sequence of disjoint open subintervals $A_i \subseteq I$ so that $|I| = \sum_{i=1}^{\infty} |A_i|$. The set E is called a **cut-out set** and A_i's are the gaps of E, following the terminology in [10].

Among these sets, which are totally disconnected, we are particularly interested in those perfect compact sets. It is well-known that these sets are homeomorphic to the middle-third Cantor set. We call them **cut-out Cantor sets**.

Every cut-out Cantor set can be associated with a binary tree structure, as is the middle-third Cantor set. Indeed, let C be a perfect, totally disconnected, compact set with zero Lebesgue measure. Let I be the smallest closed and bounded interval containing C, i.e. the convex hull of C. As above, the complement, $I \setminus C$, can be expressed as a disjoint union of open subintervals. Each open subinterval is a connected component of the complement of C.

Let G_1 be the largest of these open intervals. Since C is a perfect set, the endpoints of G_1 cannot coincide with either of the endpoints of I, otherwise $I \setminus G_1$ will contain an isolated point. The subset $I \setminus G_1$ will then be a union of two disjoint closed (non-trivial) intervals, I_0 and I_1, on the left and on the right respectively.

Since C is totally disconnected, there must be open subintervals in $I_0 \setminus C$ and $I_1 \setminus C$. Let G_2 and G_3 be the largest intervals in $I_0 \setminus C$ and $I_1 \setminus C$ respectively. By the same reasoning as above, the endpoints of G_2 and G_3 must stay away from the endpoints of I_0 and I_1 respectively. We then obtain closed subintervals I_{00}, I_{01} from $I_0 \setminus G_2$ and I_{10}, I_{11} from $I_1 \setminus G_3$.

We can continue this process and obtain a sequence of closed subintervals $I_w, w \in \{0,1\}^k$ for each $k \geq 1$. Every gap G_i in $I \setminus C$ is eventually removed. If $x \in C$, then for every $k \geq 1$, the point x must be in I_w for some $w \in \{0,1\}^k$. Therefore, C can be expressed in a way similar to the usual construction of the middle-third Cantor set:

$$C = \bigcap_{k=1}^{\infty} \bigcup_{w \in \{0,1\}^k} I_w.$$

We call this a **binary representation of** C. Every cut-out Cantor set has such a description.

For other ways to describe cut-out Cantor sets, we introduce the symbol space W. For each integer $k \geq 1$, let $n_k \geq 2$. Let $D_0 := \{e\}, D_k := \{w_1 \cdots w_k : 0 \leq w_l \leq n_l - 1 \text{ for } 1 \leq l \leq k\}$. Let

$$W := \bigcup_{k=0}^{\infty} D_k$$

be the set of all words with finite length. It is called a symbol space. If $w = w_1 \cdots w_k \in W$, its length is denoted as $|w| = k$. (In some cases, the symbol w_k may range over $\{1, \cdots, n_k\}$ instead.)

Let C and I be defined as above. If we fix a symbol space W, we can obtain a representation of C corresponding to W. Let $I_e := I$. For each $k \geq 1$ and $w \in W$ of length $|w| = k - 1$, we can find the $n_k - 1$ largest gaps $G_{w,i}$ in each $I_w \setminus C$ by the total disconnectedness and perfectness of C. Since C is perfect, the endpoints of the gaps will not coincide and $I_w \setminus \bigcup_i G_{w,i}$ gives n_k closed subintervals I_{wj} of I_w.

Inductively, we obtain a family of closed intervals $\mathcal{F} := \{I_w : w \in W\}$ in I such that

i $I_e := I$,

ii I_{wj} is a closed subinterval of I_w for any $k \geq 1, w \in D_{k-1}, 0 \leq j \leq n_k - 1$,

iii $I_{w(j-1)}$ is to the left of I_{wj} for any $1 \leq j \leq n_k - 1$,

iv I_{w0} shares the same left endpoint with I_w, and $I_{w(n_k-1)}$ shares the same right endpoint with I_w.

In this case
$$C = \bigcap_{k=1}^{\infty} \bigcup_{w \in D_k} I_w.$$
We call this a W**-representation of** C. If $D_k = \{0,1\}^k$, we obtain a binary representation.

If $|w| = k$, I_w is called a Cantor interval of level k. Denote the number of intervals at level k by
$$N_k = |D_k| = n_1 \cdots n_k$$
and the average length of Cantor intervals at level k by
$$s_k = \frac{1}{N_k} \sum_{w \in D_k} |I_w|.$$

Since $w \in D_k$ can be mapped bijectively to $1 \leq j \leq N_k$, we also label I_w as I_j^k, $j = 1, \cdots, N_k$. We will use both notations interchangeably.

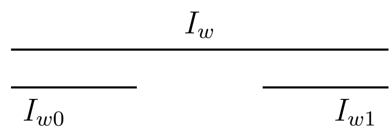
Note that in general there can be many different representations for the same set, but the flexibility here allows us to use a convenient one according to the situation.

1.3 Examples of Cantor sets

Now we define some classes of Cantor sets by specifying the symbol space W and the family of intervals $\mathcal{F} := \{I_w : w \in W\}$. Let I be a fixed closed and bounded interval. Since we can always normalize the interval length, we often assume $|I| = 1$.

1.3.1 Central Cantor sets and homogeneous Cantor sets

Suppose $n_k = 2$ for all k and $r = \{r_k\}$ is a sequence of numbers such that $0 < r_k \le \frac{1}{2}$. The r_k's are called the ratios of dissection at step k. For each interval I_w of level $k-1$, let I_{w0} and I_{w1} be the left and right intervals of level k obtained by removing an open interval G_w from I_w so that $|I_{w0}| = |I_{w1}| = |I_w| r_k$.



If $r_k < \frac{1}{2}$ for infinitely many k, then $K_r = \bigcap_{k=1}^{\infty} \bigcup_{w \in D_k} I_w$ is perfect and totally disconnected. It is called a **central Cantor set**. The middle third Cantor set is an example with $r_k = \frac{1}{3}$. The average length of the intervals of level k in this construction of K_r is

$$s_k = r_1 \cdots r_k = |I_w|$$

for any w with $|w| = k$.

More generally, let $n_k \ge 2$ be the number of divisions and r_k be the ratio of dissection with $n_k r_k \le 1$ for each level $k \ge 1$. Let $W = \bigcup_{k=0}^{\infty} D_k$ be the symbol space where $D_k = \{w_1 \cdots w_k : 0 \le w_l \le n_l - 1 \text{ for } 1 \le l \le k\}$ and $D_0 = \{e\}$.

Let $I_e = I$ be a closed and bounded interval. For each $k \ge 1$ and interval I_w of level $k-1$, let $I_{wj}, 0 \le j \le n_k - 1$, be n_k subintervals of equal length in I_w so that $|I_{w0}| = \cdots = |I_{w(n_k-1)}| = |I_w| r_k$. Moreover, we require the subintervals to be equally spaced, i.e. the gap lengths between adjacent subintervals I_{wj} and $I_{w(j+1)}$ are all the same. If $n_k r_k < 1$ for infinitely many k, then

$$C = C(\{n_k\}, \{r_k\}) = \bigcap_{k=1}^{\infty} \bigcup_{w \in D_k} I_w$$

is a **homogeneous Cantor set**. Any central Cantor set is a homogeneous Cantor set with $n_k = 2$ for all k.

The number of intervals at level k is $N_k = n_1 \cdots n_k$, and the length of each subinterval at level k is $s_k = r_1 \cdots r_k$.

Example. If K is the middle-fourth Cantor set (the central Cantor set where $r_k = \frac{1}{4}$ for all k), then $K + K$ is a homogeneous Cantor set with $I = [0, 2]$, $n_k = 3$ and $r_k = \frac{1}{4}$ for all k.

1.3.2 Cut-out sets and decreasing Cantor sets associated with a sequence

Let $a = (a_i)$ be a non-increasing summable sequence of positive numbers, i.e.

$$a_i \geq a_{i+1} > 0 \text{ and } \sum_{i=1}^{\infty} a_i < \infty.$$

Suppose $|I| = \sum_{i=1}^{\infty} a_i$ and $A_i \subseteq I$ is a sequence of disjoint open subintervals with $|A_i| = a_i$. Then $E := I \setminus \bigcup_{i=1}^{\infty} A_i$ is called a **cut-out set associated with the sequence** $a = (a_i)$. The collection of all such E is denoted by $\mathscr{C}_a$. Every compact subset $E \subseteq \mathbb{R}$ of measure 0 is of this form for a suitable sequence (a_i).

Let us single out one particular set in $\mathscr{C}_a$. Without loss of generality, assume $\sum_{i=1}^{\infty} a_i = 1$ and start with $I = [0, 1]$. Remove an open interval A_1 of length a_1 from I, leaving two closed non-trivial intervals I_1^1 and I_2^1 with lengths

$$|I_1^1| = \sum_{l=1}^{\infty} \sum_{p=0}^{2^{l-1}-1} a_{2^l+p} = a_2 + a_4 + a_5 + a_8 + \cdots$$

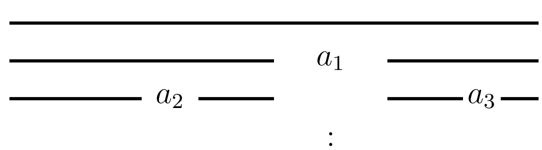
and

$$|I_2^1| = \sum_{l=1}^{\infty} \sum_{p=2^{l-1}}^{2^l-1} a_{2^l+p} = a_3 + a_6 + a_7 + a_{12} + \cdots .$$

Recursively, suppose we have constructed $\{I_j^k\}_{1 \leq j \leq 2^k}$ at step k, ordered from left to right. Remove from each interval I_j^k an open interval of length a_{2^k+j-1} and obtain two closed intervals $I_{2j-1}^{k+1}, I_{2j}^{k+1}$ of step $k+1$, where

$$|I_{2j-1}^{k+1}| = \sum_{l=0}^{\infty} \sum_{p=(2j-2)2^l}^{(2j-1)2^l-1} a_{2^{l+k+1}+p} \text{ and } |I_{2j}^{k+1}| = \sum_{l=0}^{\infty} \sum_{p=(2j-1)2^l}^{(2j)2^l-1} a_{2^{l+k+1}+p}.$$

The positions of the gaps A_i removed and the intervals I_j^k are uniquely determined.



Define

$$C_a := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{2^k} I_j^k. \tag{1.1}$$

Since the complement of C_a in I is exactly the union of gaps with lengths (a_i), the set C_a is in $\mathscr{C}_a$. It is totally disconnected because $m(I \setminus C_a) = \sum_i a_i = |I|$. Since C_a is also compact and perfect, it is a Cantor set. We call it the Cantor set associated with the sequence $a = (a_i)$, or simply a **decreasing Cantor set**. Note that (1.1) is also a binary representation of C_a.

At level k, the first $2^k - 1$ gaps of lengths $(a_i)_{1 \le i \le 2^k - 1}$ are removed. The average interval length is therefore

$$s_k = \frac{1}{2^k} \sum_{i \ge 2^k} a_i.$$

Example. Let K_r be a central Cantor set associated with ratios of dissection $r = (r_k)$. The gap length in a level k interval is $s_k(1 - 2r_{k+1}) = s_{k-1} r_k (1 - 2r_{k+1})$. One can check that if $r_k \le \frac{1}{3}$ for all $k \ge 1$, then $s_{k-1}(1 - 2r_k) \ge s_k(1 - 2r_{k+1})$, so the gap lengths are decreasing. More generally, whenever the ratios of dissection satisfy $1 \ge r_k(3 - 2r_{k+1})$, the gap lengths are decreasing. In these cases, the central Cantor set K_r is a decreasing Cantor set as well.

We can generalize the construction above. Let $W = \bigcup_{k=0}^{\infty} D_k$ be a general symbol space. At the first step, we remove $n_1 - 1$ open intervals $A_i, 1 \le i \le n_1 - 1$ with length a_i from I, and obtain n_1 closed intervals $I_j^1, 1 \le j \le n_1$. At step k, suppose we have already constructed N_k intervals $\{I_j^k\}_{1 \le j \le N_k}$ ordered from left to right in I. We then remove $n_{k+1} - 1$ open intervals A_i of length a_i from each interval I_j^k and obtain N_{k+1} closed intervals I_j^{k+1} of step $k+1$. Define

$$C_a^W := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{N_k} I_j^k$$

and we call it a **general decreasing Cantor set**. The set C_a^W exists and is unique; it is in $\mathscr{C}_a$ as well. The average interval length is

$$s_k = \frac{1}{N_k} \sum_{i \ge N_k} a_i.$$

In fact, we have some more flexibility here. Let $a = (a_i)_{i=1}^{\infty}$ and I be defined as above. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation of natural numbers such that for all $k \ge 1$, if

$N_k \le i \le N_{k+1} - 1$, then $N_k \le \sigma(i) \le N_{k+1} - 1$. Define a sequence $b = (b_i)$ by $b_i := a_{\sigma(i)}$. At the first step, we remove $n_1 - 1$ open intervals B_i with length b_i, $1 \le i \le n_1 - 1$, from I, and obtain n_1 closed intervals $I_j^1, 1 \le j \le n_1$. Repeat as above and define

$$C_b^W := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{N_k} I_j^k.$$

Note that even though b may not be a decreasing sequence any more, the average interval length remains the same:

$$s_k = \frac{1}{N_k} \sum_{i \ge N_k} a_i.$$

1.3.3 Self-similar sets

We include the definition of self-similar sets for comparison.

For each $1 \le i \le m$, let $f_i : \mathbb{R} \to \mathbb{R}$ be a similarity with ratio ρ_i, i.e.

$$|f_i(x) - f_i(y)| = \rho_i |x - y|$$

for $x, y \in \mathbb{R}$, where $0 < \rho_i < 1$. The non-empty compact set E such that

$$E = \bigcup_{i=1}^{m} f_i(E)$$

is called a self-similar set (or the attractor of the set $\{f_i\}_{i=1}^m$ of similarities). If the union is disjoint, we say that the strong separation condition (SSC) is satisfied. If there exists a non-empty bounded open set V such that $f_i(V) \subseteq V$ and $f_i(V) \cap f_j(V) = \emptyset$ for all $i \ne j$, we say that the open set condition (OSC) is satisfied.

Example. Let $I = [0, 1]$ and K_r be a central Cantor set associated with $r = (r_k)$. If there is a fixed $0 < r \le \frac{1}{2}$ such that $r_k = r$ for all k, then K_r is self-similar, where $\{f_1(x) = rx, f_2(x) = 1 - r + rx\}$ is the set of similarities. It satisfies the OSC taking $V = (0, 1)$.

The Cantor sets we study in this thesis need not be self-similar. An example is given in Section 1.5.

1.4 p-Cantor measures

We will make use of a class of measures on the Cantor sets when we study their dimensional properties.

Let $C = \bigcap_{k=1}^{\infty} \bigcup_{|w|=k} I_w$ be a Cantor set associated with the symbol space W. Let $\mathbf{p} = \{p_{kj} : k \geq 1, 0 \leq j \leq n_k - 1\}$ be probability weights with $\sum_{j=0}^{n_k-1} p_{kj} = 1$ for $k \geq 1$. Define $\mu_{\mathbf{p}} = \mu_{\mathbf{p}}^{C,W}$ on C by

$$\mu_{\mathbf{p}}(I_w) := \prod_{l=1}^{k} p_{lw_l}$$

for $w \in W$ of length $|w| = k$ and extend it to a measure. We call $\mu_{\mathbf{p}}$ a **p**-Cantor measure. It is a probability measure with C as support. The measure $\mu_{\mathbf{p}}$ is singular if $m(C) = 0$.

If $p_{k0} = \cdots = p_{k(n_k-1)}$ for all k, i.e. $p_{kj} = \frac{1}{n_k}$, then $\mu_{\mathbf{p}}$ is called the uniform Cantor measure. The classical Cantor measure is an example.

In this thesis we will impose some boundedness conditions on the probability weights when needed. For instance, we typically assume $\inf p_{kj} > 0$, which ensures $\mu_{\mathbf{p}}$ is a continuous measure.

1.5 Dimension functions and partitions

We are interested in the size of sets of Lebesgue measure 0 and we now define the dimensional concepts we will study.

The diameter of any set $A \subseteq \mathbb{R}$ is denoted by $|A|$.

Definition 1 (Hausdorff and packing measures [20, 32]). Let $\alpha \geq 0$. The α-Hausdorff measure of a set $E \subseteq \mathbb{R}$ is defined to be

$$H^{\alpha}(E) := \lim_{\delta \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} |E_i|^{\alpha} : E \subseteq \bigcup_{i=1}^{\infty} E_i, |E_i| \leq \delta \right\}.$$

A δ-packing of a set E is a countable, disjoint family of open balls $\{B_i\}_i$ centred at points in E with $|B_i| \leq \delta$. The α-packing pre-measure of E is

$$P_0^{\alpha}(E) := \lim_{\delta \to 0^+} \sup \left\{ \sum_{i=1}^{\infty} |B_i|^{\alpha} : \{B_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-packing of } E \right\}$$

and the α-packing measure of E is

$$P^\alpha(E) := \inf\left\{\sum_{i=1}^\infty P_0^\alpha(E_i) : E = \bigcup_{i=1}^\infty E_i\right\}.$$

The **Hausdorff dimension, pre-packing dimension** and **packing dimension** of E are

$$\begin{aligned}
\dim_H E &= \sup\{\alpha : H^\alpha(E) = \infty\} = \inf\{\alpha : H^\alpha(E) = 0\},\\
\dim_{P_0} E &= \sup\{\alpha : P_0^\alpha(E) = \infty\} = \inf\{\alpha : P_0^\alpha(E) = 0\},\\
\dim_P E &= \sup\{\alpha : P^\alpha(E) = \infty\} = \inf\{\alpha : P^\alpha(E) = 0\}
\end{aligned}$$

respectively. It is well known that $\dim_H E \le \dim_P E \le \dim_{P_0} E$.

Example (Self-similar sets [11])**.** Let E be a self-similar set with similarity ratios $0 < \rho_i < 1$ for $1 \le i \le m$. The root α of

$$\sum_{i=1}^m \rho_i^\alpha = 1$$

is called the similarity dimension of E. If the open set condition is satisfied, then $\dim_H E = \dim_P E = \alpha$ and $0 < H^\alpha(E) \le P^\alpha(E) < \infty$.

Related results are known for Moran sets with a weaker similarity property. See [24, 7, 21].

Example (p-Cantor sets [4])**.** Let $p > 1$, $a_i = \frac{1}{i^p}$ for $i \ge 1$. Then for $a = (a_i)$,

$$\dim_H C_a = \dim_P C_a = \frac{1}{p}$$

and

$$0 < H^{\frac{1}{p}}(C_a) \le P^{\frac{1}{p}}(C_a) < \infty.$$

Example (Central Cantor sets [5, 12, 14])**.** Let K_r be a central Cantor set with ratios of dissection $r = (r_k)$. Its Hausdorff and packing dimensions are known to be

$$\dim_H K_r = \liminf_{n\to\infty} \frac{-\log 2^k}{\log(r_1 \cdots r_k)}$$

and

$$\dim_P K_r = \limsup_{n\to\infty} \frac{-\log 2^k}{\log(r_1 \cdots r_k)}.$$

- Let $p > 1$ and $r_k = \frac{1}{2^p}$ for $k \geq 1$. The Hausdorff and packing dimensions of K_r are

$$\alpha = \lim_{k\to\infty} \frac{\log 2^k}{\log 2^{pk}} = \frac{1}{p}.$$

  Moreover,
$$0 < H^\alpha(K_r) \leq P^\alpha(K_r) < \infty.$$

- Let $p > 1$ and $r_k = \frac{1}{2^{p+\frac{1}{k}}}$ for $k \geq 1$. The dimensions of this central Cantor set K_r are still $\alpha = \frac{1}{p}$, but $H^\alpha(K_r) = P^\alpha(K_r) = 0$. In particular, this implies K_r is not self-similar.

- Let $p > 1$ and $k_0 \geq 1$ be such that $p - 1 > \frac{1}{k_0+1}$. Let $r_k = \frac{1}{2^{p-\frac{1}{k_0+k}}}$ for $k \geq 1$. The dimensions of this central Cantor set K_r are again $\alpha = \frac{1}{p}$, but $H^\alpha(K_r) = P^\alpha(K_r) = \infty$. This K_r is also not self-similar.

These results can also be deduced from Theorem 1 in Chapter 2.

As we can see from the last example, even when $\alpha = \dim_H E = \dim_P E$, the measures $H^\alpha(E)$ and $P^\alpha(E)$ may not be finite and positive. Thus it is helpful to have a more precise way to capture the size of a set. The more general concepts of dimension functions h and h-measures (introduced by Hausdorff himself) do this.

Definition 2 (Dimension function [20, 30, 31])**.** A function $h : [0, \delta) \to [0, \infty)$ is called a dimension function (or a gauge function) if h is continuous, increasing, doubling ($h(2x) \leq \tau h(x)$ for some $\tau > 0$) and $h(0) = 0$. Let $\mathbb{D}$ be the set of dimension functions.

The power functions $h(x) = x^\alpha$, $\alpha > 0$, are typical examples of dimension functions. The logarithmic perturbation of power functions, $h(x) = x^\alpha(\log \frac{1}{x})^\beta$, $\alpha, \beta > 0$, is another type of examples.

Definition 3 (Equivalence of dimension functions and sequences)**.** 1. Let $g, h \in \mathbb{D}$ be dimension functions. The function g is said to be equivalent to h if and only if there exist $\delta, A, B > 0$ such that

$$Ah(x) \leq g(x) \leq Bh(x)$$

for all $x \in [0, \delta)$. In this case, we write $g \equiv h$.

2. Let $x = \{x_k\}, y = \{y_k\}$ be two sequences. The sequence $\{x_k\}$ is said to be equivalent to $\{y_k\}$ if and only if there exist $A, B > 0$ such that

$$Ay_k \leq x_k \leq By_k$$

for all k. We also write $\{x_k\} \equiv \{y_k\}$.

The h-Hausdorff and h-packing measures are natural generalizations of the α-Hausdorff and α-packing measures.

Definition 4 (h-Hausdorff and h-packing measures [20, 30, 31])**.** Let $h \in \mathbb{D}$. The h-Hausdorff measure of a set $E \subseteq \mathbb{R}$ is defined to be

$$H^h(E) := \lim_{\delta \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} h(|E_i|) : E \subseteq \bigcup_{i=1}^{\infty} E_i, |E_i| \leq \delta \right\}.$$

The h-packing pre-measure of E is

$$P_0^h(E) := \lim_{\delta \to 0^+} \sup \left\{ \sum_{i=1}^{\infty} h(|B_i|) : \{B_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-packing of } E \right\}$$

and the h-packing measure of E is

$$P^h(E) := \inf \left\{ \sum_{i=1}^{\infty} P_0^h(E_i) : E = \bigcup_{i=1}^{\infty} E_i \right\}.$$

If $h(x) = h_\alpha(x) = x^\alpha$ for some $\alpha \geq 0$, we get back the usual Hausdorff measure $H^\alpha(E)$, packing pre-measure $P_0^\alpha(E)$ and packing measure $P^\alpha(E)$.

Definition 5 (Dimension partition [6])**.** The dimension partition of a set $E \subseteq \mathbb{R}$ is a partition of $\mathbb{D}$ into six sets, $\mathbb{H}_\beta^E \cap \mathbb{P}_\gamma^E$, for $\beta \leq \gamma \in \{0, 1, \infty\}$, where

$$\mathbb{H}_1^E = \{h \in \mathbb{D} : 0 < H^h(E) < \infty\},$$
$$\mathbb{P}_1^E = \{h \in \mathbb{D} : 0 < P^h(E) < \infty\},$$

and for $\eta = 0, \infty$,

$$\mathbb{H}_\eta^E = \{h \in \mathbb{D} : H^h(E) = \eta\},$$
$$\mathbb{P}_\eta^E = \{h \in \mathbb{D} : P^h(E) = \eta\}.$$

The numerical Hausdorff dimension and packing dimension can be written as

$$\dim_H E = \sup\{\alpha : h_\alpha \in \mathbb{H}_\infty^E\} = \inf\{\alpha : h_\alpha \in \mathbb{H}_0^E\},$$
$$\dim_P E = \sup\{\alpha : h_\alpha \in \mathbb{P}_\infty^E\} = \inf\{\alpha : h_\alpha \in \mathbb{P}_0^E\}$$

respectively.

When $h \in \mathbb{D}$, it is proved in [31] that

$$H^h(E) \leq P^h(E) \leq P_0^h(E)$$

for $E \subseteq \mathbb{R}$ since h is doubling. A set E is called h-regular if $0 < H^h(E) \leq P^h(E) < \infty$ and α-regular if $0 < H^\alpha(E) \leq P^\alpha(E) < \infty$. In such cases we also call E an h-set or an α-set respectively. We have seen that even when $\alpha = \dim_H E$ or $\dim_P E$, the measures $H^\alpha(E)$ or $P^\alpha(E)$ can be 0 or ∞. If there exists an $h \in \mathbb{D}$ such that E is h-regular, it will give a more precise description of the size of the set.

We will be studying a class of Cantor sets that are h-regular (for a suitable h). This will be the content of the next chapter.

Chapter 2

Balanced Cantor sets

In Chapter 1, we have seen that a cut-out Cantor set has the form

$$C = \bigcap_{k=1}^{\infty} \bigcup_{w \in D_k} I_w = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{N_k} I_j^k$$

corresponding to a symbol space W. In this thesis, we are interested in the dimensional properties of certain of these cut-out Cantor sets and the associated measures. In order to study these metric-related properties, we will start with a property of the intervals I_w in the representation above. The collection of Cantor sets under consideration will include all the central Cantor sets and the decreasing Cantor sets introduced in the preliminary chapter.

2.1 Cantor sets with a balanced property

Let us introduce the cut-out Cantor sets with a certain "balancing property" in its construction. Let $W = \bigcup_{k=0}^{\infty} D_k$ be the symbol space where $n_k \geq 2$ is the number of divisions at level k. Assume

$$M := \sup_k n_k < \infty.$$

Suppose a Cantor set C has the representation corresponding to W, as in Chapter 1, given by

$$C = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{N_k} I_j^k$$

where $I_w = I_j^k$ are the level k Cantor intervals. Recall that the number of intervals at level k is $N_k = n_1 \cdots n_k$ and the average length of the Cantor intervals at level k is

$$s_k = \frac{1}{N_k} \sum_{j=1}^{N_k} |I_j^k|.$$

Note that s_k is decreasing. In fact, $I^k_{n_k(j-1)+1} \cup \cdots \cup I^k_{n_k j} \subseteq I_j^{k-1}$ implies

$$n_k s_k = \frac{1}{N_{k-1}} \sum_{j=1}^{N_k} |I_j^k| \leq \frac{1}{N_{k-1}} \sum_{j=1}^{N_{k-1}} |I_j^{k-1}| = s_{k-1}.$$

Definition 6. A cut-out Cantor set C is said to be **balanced** (or W**-balanced**) if and only if C has a W-representation, with the associated Cantor intervals satisfying the property that there exist some K and $L_1, L_2 \in \mathbb{N}$ such that

$$s_{k+L_1} \leq |I_j^k| \leq s_{k-L_2}$$

for any $k \geq K$ and $1 \leq j \leq N_k$. Let $\mathscr{C}$ denote the collection of all balanced Cantor sets.

In particular, this condition implies that

$$|I_{j'}^{k+L_1+L_2}| \leq |I_j^k| \leq |I_{j''}^{k-L_1-L_2}|$$

for all $k > L_1 + L_2$ and any j, j', j''.

We show below that the balanced Cantor sets include both the central Cantor sets and the decreasing Cantor sets.

Example (Central and homogeneous Cantor sets)**.** The interval length of a central Cantor set or a homogeneous Cantor set at level k is

$$|I_j^k| = r_1 \cdots r_k = s_k,$$

so it satisfies the balanced property with $L_1 = L_2 = 0$.

Example (Binary decreasing Cantor sets)**.** The average interval length of a decreasing Cantor set C_a at level k is

$$s_k = \frac{1}{2^k} \sum_{i \geq 2^k} a_i.$$

For $0 \le j \le 2^{k-1}$,

$$|I_j^k| = \sum_{l=k}^{\infty} \sum_{p=(j-1)2^{l-k}}^{j2^{l-k}-1} a_{2^l+p} = \sum_{l=0}^{\infty} \sum_{p=(j-1)2^l}^{j2^l-1} a_{2^{l+k}+p}.$$

Since $a = (a_i)$ is decreasing, $\{|I_j^k|\}_{(k,j)}$ is lexicographically decreasing. In consequence,

$$s_{k+1} \le |I_1^{k+1}| \le |I_j^k| \le |I_{2^{k-1}}^{k-1}| \le s_{k-1}$$

for all j, thus the balanced property is satisfied with $L_1 = L_2 = 1$.

Example (General decreasing Cantor sets)**.** In the case of a general decreasing Cantor set C_a^W,

$$s_k = \frac{1}{N_k} \sum_{i \ge N_k} a_i.$$

For $0 \le j \le N_{k-1}$,

$$|I_j^k| = \sum_{l=k}^{\infty} \sum_{p=(j-1)(N_{l+1}-N_l)/N_k}^{j(N_{l+1}-N_l)/N_k-1} a_{N_l+p}.$$

(Note that $(N_{l+1} - N_l)/N_k = n_{k+1} \cdots n_l(n_{l+1} - 1)$ is the number of gaps to be removed from I_j^k at level l.) One can again see that

$$s_{k+1} \le |I_1^{k+1}| \le |I_j^k| \le |I_{N_{k-1}}^{k-1}| \le s_{k-1},$$

so the Cantor set C_a^W is also balanced.

Example. Let C_a^W be the general decreasing Cantor set as above. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation of natural numbers such that for all $k \ge 1$, if $N_k \le i \le N_{k+1} - 1$, then $N_k \le \sigma(i) \le N_{k+1} - 1$. Recall that the sequence $b = (b_i)$, where $b_i := a_{\sigma(i)}$, defines a Cantor set C_b^W as in Section 1.3.2. For each level k, let $\{I_j^k\}_{1 \le j \le N_k}$ be the Cantor intervals of C_b^W, and $\{\widetilde{I}_j^k\}_{1 \le j \le N_k}$ be the Cantor intervals of the general decreasing Cantor set C_a^W. Note that the average interval length

$$s_k = \frac{1}{N_k} \sum_{i \ge N_k} a_i$$

of C_b^W is the same as that of C_a^W. We can check that

$$s_{k+1} \le |\widetilde{I}_1^{k+1}| \le |I_j^k| \le |\widetilde{I}_{N_{k-1}}^{k-1}| \le s_{k-1}$$

for all j, so C_b^W is balanced as well.

2.2 Hausdorff and packing measure of a balanced Cantor set

First, we estimate the Hausdorff measure and packing pre-measure of a balanced Cantor set. This generalizes the results in [2] and [13]. Recall that $M = \sup_k n_k < \infty$.

Theorem 1. *Let C be a balanced Cantor set. For any $h \in \mathbb{D}$, we have:*

1. $\frac{1}{M^{L_1+2}} \liminf_{k\to\infty} N_k h(s_k) \le H^h(C) \le M^{L_2} \liminf_{k\to\infty} N_k h(s_k)$,
2. $\frac{1}{M^{L_1+1}} \limsup_{k\to\infty} N_k h(s_k) \le P_0^h(C) \le M^{L_2+2} \limsup_{k\to\infty} N_k h(s_k)$.

Proof. 1. (a) With the balanced property, we have

$$\sum_{j=1}^{N_k} h(|I_j^k|) \le N_k h(s_{k-L_2}) \le M^{L_2} N_{k-L_2} h(s_{k-L_2})$$

when k is large enough. For any $\delta > 0$ we can take k_δ such that $|I_j^k| < \delta$ for any $k \ge k_\delta$. Then the intervals of level k form a δ-covering of C and hence

$$H^h(C) \le M^{L_2} \liminf_{k\to\infty} N_k h(s_k).$$

(b) Let $\lambda = \liminf_{k\to\infty} N_k h(s_k)$. If $\lambda = 0$, we trivially have

$$H^h(C) \ge \frac{1}{M^{L_1+2}} \liminf_{k\to\infty} N_k h(s_k),$$

so assume $\lambda > 0$. Then, for any $\varepsilon > 0$, there exists K_0 such that for any $k \ge K_0$ we have

$$(1-\varepsilon)\lambda < N_k h(s_k)$$

and also

$$s_{k+L_1} \le |I_j^k|$$

by the balanced property.

Let $0 < \delta < \min_j |I_j^{K_0}|$. Let $\{B_i\}_i$ be a δ-covering of C by open intervals and let $R = \bigcup_i B_i$. As C is compact we can assume the covering consists of finitely many intervals, say $\{B_i\}_{i=1}^M$. There exists $K > 1$ such that

$$\bigcup_{j=1}^{N_K} I_j^K \subseteq R,$$

for otherwise $\{\bigcup_{j=1}^{N_k} I_j^k \backslash R\}_k$ would be a decreasing sequence of non-empty closed set with empty intersection and this contradicts the finite intersection property of a compact set.

We claim we can also assume the intervals B_i in the covering are disjoint. This is because if $B_i \cap B_j \neq \emptyset$ for some $i \neq j$, the intersection being open must contain some gap of the Cantor set. We can then shrink down the intervals to make them disjoint and get a lower estimate of $H^h(C)$.

In order to obtain a further lower bound, let us replace each B_i by the smallest possible single closed interval V_i containing $B_i \cap \bigcup_{j=1}^{N_K} I_j^K$. Then $\sum h(|V_i|) \leq \sum h(|B_i|)$. If $V_i = \emptyset$, we simply discard it.

Let τ_i be the number of intervals of level K contained in V_i. Then $\tau_i \geq 1$,

$$V_i \supset \bigcup_{l=1}^{\tau_i} I_{j_l}^K$$

and

$$\sum_i \tau_i = N_K = n_1 \cdots n_K.$$

For each i, let p_i be the non-negative integer such that

$$\frac{N_K}{N_{K-p_i}} \leq \tau_i < \frac{N_K}{N_{K-p_i-1}}.$$

Let

$$Q(x) = \frac{N_K}{N_x}.$$

If $p_i = 0$, then $1 \leq \tau_i < n_k$ and V_i contains some I_j^K. In this case $|I_j^K| \leq |V_i| < \delta$ and hence $K > K_0$. Then

$$\frac{1}{N_{K-p_i+L_1+1}}(1-\varepsilon)\lambda < \frac{1}{N_{K+L_1}}(1-\varepsilon)\lambda < h(s_{K+L_1}) \leq h(|I_j^K|) \leq h(|V_i|).$$

If $p_i \geq 1$, then $2 \leq Q(K-p_i) \leq \tau_i < Q(K-p_i-1)$. Note that V_i contains at least $Q(K-p_i)$ consecutive intervals of level K and

$$Q(K-p_i) \geq 2Q(K-p_i+1). \tag{2.1}$$

Consider the level $K-p_i+1$. Each interval $I_j^{K-p_i+1}$ contains $Q(K-p_i+1)$ subintervals of level K. Let $I_{j,l}^K$, $1 \leq l \leq Q(K-p_i+1)$, be these subintervals ordered from left to right. For some j, V_i contains at least one interval $I_{j,l}^K$.

- If V_i contains both $I_{j,1}^K$ and $I_{j,Q(K-p_i+1)}^K$, then it must also contain all the other intervals $I_{j,l}^K$ in between and hence the whole interval $I_j^{K-p_i+1}$. This implies $|V_i| \geq |I_j^{K-p_i+1}|$.
- If V_i does not contain $I_{j,1}^K$, then the number of intervals $I_{j,l}^K$ contained in V_i is less than $Q(K-p_i+1)$, and V_i must contain more than $Q(K-p_i+1)$ intervals $I_{j+1,l}^K$ on the right because of (2.1). In this case, V_i will contain the whole interval $I_{j+1}^{K-p_i+1}$ and $|V_i| \geq |I_{j+1}^{K-p_i+1}|$. Similarly if V_i does not contain $I_{j,Q(K-p_i+1)}^K$, V_i will contain the whole interval $I_{j-1}^{K-p_i+1}$ and $|V_i| \geq |I_{j+1}^{K-p_i+1}|$.

In either case, the length of V_i must be at least $|I_j^{K-p_i+1}|$ for some j, so we have

$$|I_j^{K-p_i+1}| \leq |V_i| < \delta.$$

In particular this forces $K - p_i + 1 > K_0$ and hence

$$\frac{1}{N_{K-p_i+L_1+1}}(1-\varepsilon)\lambda < h(s_{K-p_i+L_1+1}) \leq h(|I_j^{K-p_i+1}|) \leq h(|V_i|).$$

by the balanced property.

Finally, we have

$$\begin{aligned}
\frac{1}{M^{L_1+2}}(1-\varepsilon)\lambda &= \frac{(1-\varepsilon)\lambda}{M^{L_1+2}} \sum_i \frac{\tau_i}{N_k} \\
&< (1-\varepsilon)\lambda \sum_i \frac{1}{N_{K-p_i-1} M^{L_1+2}} \\
&\leq (1-\varepsilon)\lambda \sum_i \frac{1}{N_{K-p_i+L_1+1}} \\
&< \sum_i h(|V_i|) \leq \sum_i h(|B_i|).
\end{aligned}$$

Since $\{B_i\}_i$ is any δ-covering of C and $\varepsilon > 0$ is arbitrary, we get

$$\frac{1}{M^{L_1+2}} \liminf_{k\to\infty} N_k h(s_k) \leq H^h(C).$$

2. (a) Let $d < \limsup_{k\to\infty} N_k h(s_k)$. Then, there exists a subsequence $\{k_p\}_{p\geq 1}$ such that $d < N_{k_p} h(s_{k_p}) \leq N_{k_p} h(|I_j^{k_p - L_1}|)$ where the second inequality follows from the balanced property.

For any $\delta > 0$ take k_p large enough such that $|I_j^{k_p-L_1}| < \delta$ for all j. Let us take the family of intervals $\{B_i := B(x_i, r_i)\}_{i=1}^{N_{k_p-L_1-1}}$, where $r_i = s_{k_p}/2$ and x_i is the left endpoint of $I_{in}^{k_p-L_1}$, $n = n_{k_p-L_1}$. The point x_i is the left endpoint of the right most level $k_p - L_1$ interval contained in the i-th interval of level $k_p - L_1 - 1$, $I_i^{k_p-L_1-1}$. The balls $\{B_\delta(x_i)\}_i$ are centred in C. Since $|I_j^{k_p-L_1}| \geq s_{k_p}$, the right endpoint, $x_i + r_i$, of each ball B_i cannot exceed the right end of the interval $I_i^{k_p-L_1-1}$. On the other hand, the left endpoint $x_i - r_i$ cannot go below the left end of the interval $I_i^{k_p-L_1-1}$. Hence, the balls $B_i \subseteq I_i^{k_p-L_1-1}$ and they are pairwise disjoint.

Since $|B_i| = s_{k_p} < \delta$, this is a δ-packing and

$$\sum_i h(|B_i|) \geq \sum_{i=1}^{N_{k_p-L_1-1}} h(s_{k_p}) = N_{k_p-L_1-1} h(s_{k_p}) > \frac{d}{M^{L_1+1}}.$$

That means for any $\delta > 0$ we can find a δ-packing satisfying the above inequality. Therefore, $\frac{d}{M^{L_1+1}} \leq P_0^h(C)$ for any $d < \limsup_{k\to\infty} N_k h(s_k)$, which shows

$$\frac{1}{M^{L_1+1}} \limsup_{k\to\infty} N_k h(s_k) \leq P_0^h(C).$$

(b) Let $\varepsilon > 0$. There exists k_0 such that

$$\sup_{k \geq k_0} N_k h(s_k) \leq \limsup_{k\to\infty} N_k h(s_k) + \varepsilon.$$

Choose δ small enough so that $2\delta < |I_j^{k_0+L_2+2}|$ for all j.

Let $\{B_i\}_i$ be a δ-packing of C_a and take $k_i := \min\{k : I_j^k \subseteq B_i$ for some $1 \leq j \leq N_k\}$. Then $k_i \geq k_0 + L_2 + 2$ and B_i is centred at a point of an interval of level $k_i - 1$ but does not contain the interval. Therefore, $|B_i|/2 < |I_{j_i}^{k_i-1}|$ where $I_{j_i}^{k_i-1}$ is the interval of level $k_i - 1$ containing the center of B_i. As $n_k \geq 2$,

$$|B_i| < 2|I_{j_i}^{k_i-1}| \leq n_{k_i-L_2-1} s_{k_i-L_2-1} \leq s_{k_i-L_2-2}$$

from the balanced property and

$$\sum_i h(|B_i|) \leq \sum_i h(s_{k_i-L_2-2}).$$

Let $l_1 < \cdots < l_m$ be the distinct k_i's and let θ_p be the number of repetitions of l_p, i.e. θ_p is the number of B_i's containing an interval of level l_p but none of those at level $l_p - 1$. Each ball, B_i, of the packing associated to l_p, contains at least $\frac{N_{l_m}}{N_{l_p}}$ intervals of step l_m. Since $\{B_i\}_i$ is a disjoint family, $\sum_{p=1}^{m-1} \theta_p \frac{N_{l_m}}{N_{l_p}}$ intervals of level l_m are already covered by the B_i's corresponding to $l_1, \cdots, l_{m-1}$. θ_m can only be at most the number of the remaining intervals at level l_m:

$$\theta_m \leq N_{l_m} - \sum_{p=1}^{m-1} \theta_p \frac{N_{l_m}}{N_{l_p}} = N_{l_m}(1 - \sum_{p=1}^{m-1} \frac{\theta_p}{N_{l_p}}).$$

This implies $\frac{\theta_m}{N_{l_m}} \leq 1 - \sum_{p=1}^{m-1} \frac{\theta_p}{N_{l_p}}$, i.e.

$$\sum_{p=1}^{m} \frac{\theta_p}{N_{l_p}} \leq 1.$$

As a result,

$$\begin{aligned}
\sum_i h(|B_i|) &\leq \sum_i h(s_{k_i - L_2 - 2}) \\
&= \sum_{p=1}^{m} \theta_p h(s_{l_p - L_2 - 2}) = \sum_{p=1}^{m} \frac{\theta_p}{N_{l_p}} N_{l_p} h(s_{l_p - L_2 - 2}) \\
&\leq M^{L_2+2} \sum_{p=1}^{m} \frac{\theta_p}{N_{l_p}} N_{l_p - L_2 - 2} h(s_{l_p - L_2 - 2}) \\
&\leq M^{L_2+2} (\limsup_{k\to\infty} N_k h(s_k) + \varepsilon)
\end{aligned}$$

since $l_p - L_2 - 2 \geq k_0$. Hence,

$$P_0^h(C) \leq M^{L_2+2} \limsup_{k\to\infty} N_k h(s_k).$$

$\blacksquare$

Only the packing pre-measure is estimated above. In general we only know that $P^h(E) \leq P_0^h(E)$ and $\dim_P E \leq \dim_{P_0} E$ for $E \subseteq \mathbb{R}$, and the strict inequality can happen. However, the packing measure $P^h(C)$ and the packing pre-measure $P_0^h(C)$ are finite and positive simultaneously for a balanced Cantor set C, and its packing dimension is equal to its pre-packing dimension. To prove this, we will make use of the following version of the mass distribution principle. The proof given in [6] is included here for completeness.

Lemma 2 ([6]). *Let $E \subseteq \mathbb{R}$. Let μ be a finite regular Borel measure and $h \in \mathbb{D}$ be a dimension function. If*

$$\liminf_{r\to 0} \frac{\mu(B(x_0, r))}{h(r)} < c \tag{2.2}$$

for all $x_0 \in E$, then

$$P^h(E) \geq \frac{\mu(E)}{c}.$$

Proof. For each $\delta > 0$, let

$$P^h_{0,\delta}(E) = \sup\{\sum h(|B_i|) : \{B_i\}_i \text{ is a } \delta\text{-packing of } E\}.$$

Let $\mathcal{B}_\delta = \{B(x, r) : x \in E, r \leq \delta \text{ and } \mu(B(x, r)) < ch(r)\}$. The hypothesis (2.2) tells us that, for any $x \in E$, there are balls $B(x, r) \in \mathcal{B}_\delta$ with r arbitrarily small. By the Vitali covering theorem, there is a sequence of disjoint balls $\{B_i\}_{i=1}^\infty$ from $\mathcal{B}_\delta$ such that $\mu(E \setminus \bigcup B_i) = 0$. Thus

$$P^h_{0,\delta}(E) \geq \sum h(|B_i|) \geq \frac{1}{c} \sum \mu(B_i) = \frac{1}{c}\mu(\bigcup B_i) = \frac{1}{c}\mu(E)$$

and hence $P^h_0(E) \geq \frac{1}{c}\mu(E)$.

For any partition $E = \bigcup_{i=1}^\infty E_i$,

$$\sum P^h_0(E_i) \geq \sum \frac{1}{c}\mu(E_i) \geq \frac{\mu(E)}{c}$$

and therefore we have

$$P^h(E) \geq \frac{\mu(E)}{c}.$$

■

Theorem 3. *Let C be a balanced Cantor set and $h \in \mathbb{D}$. If $P^h_0(C) = \infty$, then $P^h(C) = \infty$. If $P^h_0(C) > 0$, then $P^h(C) > 0$.*

Proof. Let μ be the uniform Cantor measure defined by $\mu(I^k_j) = \frac{1}{N_k}$. Let $x_0 \in C$ and $r > 0$. The balanced property tells us that there exist L_1, L_2 such that $s_{k+L_1} \leq |I^k_j| \leq s_{k-L_2}$ for large enough k.

Suppose k is the minimal integer such that $B(x_0, r)$ contains an interval of level k. The minimality of k ensures that $B(x_0, r)$ can only intersect at most $2n_k$ intervals of level

k, which implies $\mu(B(x_0,r)) \le 2n_k \frac{1}{N_k} = \frac{2}{N_{k-1}}$. Let I_j^k be a level k interval contained in $B(x_0,r)$, so $|I_j^k| \le 2r$. Since h is doubling, there exists some $\tau > 0$ such that

$$h(s_{k+L_1}) \le h(|I_j^k|) \le h(2r) \le \tau h(r).$$

Then

$$\frac{\mu(B(x_0,r))}{h(r)} \le \frac{2\tau}{N_{k-1}h(s_{k+L_1})} = \frac{2\tau M^{L_1+1}}{N_{k+L_1}h(s_{k+L_1})}$$

and

$$\liminf_{r\to 0} \frac{\mu(B(x_0,r))}{h(r)} \le \frac{2\tau M^{L_1+1}}{\limsup_{k\to\infty} N_k h(s_k)}.$$

By the inequalities in Theorem 1, $P_0^h(C) > 0$ implies $\limsup N_k h(s_k) > 0$, while $P_0^h(C) = \infty$ implies $\limsup N_k h(s_k) = \infty$. Let $c_0 := \liminf_{r\to 0} \frac{\mu(B(x_0,r))}{h(r)}$. If $P_0^h(C) > 0$, then $c_0 < \infty$ and $P^h(C) \ge \frac{\mu(C)}{c_0} > 0$ by the lemma. Correspondingly, if $P_0^h(C) = \infty$, then $c_0 = 0$. Hence $P^h(C) \ge \frac{\mu(C)}{c} > 0$ for every $c > 0$ and $P^h(C) = \infty$. ■

Corollary 4. *Let C be a balanced Cantor set and $h \in \mathbb{D}$. Then*

1. $P_0^h(C) = 0$ *if and only if* $P^h(C) = 0$,
2. $P_0^h(C) = \infty$ *if and only if* $P^h(C) = \infty$, *and*
3. $0 < P_0^h(C) < \infty$ *if and only if* $0 < P^h(C) < \infty$.

In other words,

$$\mathbb{P}_1^C = \{h \in \mathbb{D} : 0 < P^h(C) < \infty\} = \{h \in \mathbb{D} : 0 < P_0^h(C) < \infty\}$$

and

$$\mathbb{P}_\beta^C = \{h \in \mathbb{D} : P^h(C) = \beta\} = \{h \in \mathbb{D} : P_0^h(C) = \beta\}$$

for $\beta = 0, \infty$.

In particular, if we take $h(x) = x^\alpha$, we get the following corollary.

Corollary 5. *If C is a balanced Cantor set, then* $\dim_P C = \dim_{P_0} C$.

2.3 Dimension partition of a balanced Cantor set

For any balanced Cantor set $C \in \mathscr{C}$ and any dimension function $h \in \mathbb{D}$, by Theorem 1 we have

$$\frac{1}{M^{L_1+2}} \liminf_{k\to\infty} N_k h(s_k) \le H^h(C) \le M^{L_2} \liminf_{k\to\infty} N_k h(s_k)$$
$$\text{and } \frac{1}{M^{L_1+1}} \limsup_{k\to\infty} N_k h(s_k) \le P_0^h(C) \le M^{L_2+2} \limsup_{k\to\infty} N_k h(s_k).$$

Define $h_C(s_k) = \frac{1}{N_k}$ and extend h_C to a piecewise linear function on $[0, s_1]$ with $h_C(0) := \lim_{x\to 0^+} h_C(x) = 0$. Then h_C is continuous and increasing, and we can see that $h_C \in \mathbb{D}$ by the following proposition. From the above inequalities and Corollary 4, C is h_C-regular. Let us call h_C the **associated dimension function** of C. More generally, any continuous increasing function h such that $\{h(s_k)\} \equiv \{\frac{1}{N_k}\}$ will also make C h-regular. Note that all such h are doubling by the next proposition.

Proposition 6. *Let C be a balanced Cantor set with $M = \sup_k n_k < \infty$. Suppose h is an increasing function on $[0, \delta)$. If $\{h(s_k)\} \equiv \{\frac{1}{N_k}\}$, then h is doubling. In particular, $h_C \in \mathbb{D}$.*

Proof. As $\{h(s_k)\} \equiv \{\frac{1}{N_k}\}$, there exist $c_1, c_2 > 0$ such that

$$c_1 \frac{1}{N_k} \le h(s_k) \le c_2 \frac{1}{N_k}$$

for all k. If $s_{k+1} \le x \le s_k$, then

$$\begin{aligned} h(2x) &\le h(2s_k) \le h(s_{k-1}) \\ &\le c_2 \frac{1}{N_{k-1}} \le c_2 M^2 \frac{1}{N_{k+1}} \le \frac{c_2 M^2}{c_1} h(s_{k+1}) \\ &\le \frac{c_2 M^2}{c_1} h(x), \end{aligned}$$

so h is doubling.

■

That means $\mathbb{H}_1^C \cap \mathbb{P}_1^C$ is always non-empty for the balanced Cantor sets. Indeed we get immediately from Theorem 1 and Corollary 4 the following description of the dimension partition for the balanced Cantor sets.

Theorem 7. *Let C be a balanced Cantor set with the number of Cantor intervals $N_k = n_1 \cdots n_k$ and the average length s_k at level k. Then*

$$\begin{aligned}
\mathbb{H}_1^C &= \{h \in \mathbb{D} : 0 < \liminf_{k\to\infty} N_k h(s_k) < \infty\}, \\
\mathbb{H}_\beta^C &= \{h \in \mathbb{D} : \liminf_{k\to\infty} N_k h(s_k) = \beta\}, \\
\mathbb{P}_1^C &= \{h \in \mathbb{D} : 0 < \limsup_{k\to\infty} N_k h(s_k) < \infty\}, \\
\mathbb{P}_\beta^C &= \{h \in \mathbb{D} : \limsup_{k\to\infty} N_k h(s_k) = \beta\}
\end{aligned}$$

where $\beta = 0, \infty$.

Corollary 8. *Let C be a balanced Cantor set with the number of Cantor intervals $N_k = n_1 \cdots n_k$ and the average length s_k at level k. Then*

$$\dim_H C = \liminf_{k\to\infty} \frac{-\log N_k}{\log s_k}$$

and

$$\dim_P C = \limsup_{k\to\infty} \frac{-\log N_k}{\log s_k}.$$

Let $N_\delta(C)$ be the smallest number of sets of diameter at most δ which can cover C. Recall that the lower and upper box dimensions are given by

$$\underline{\dim}_B C = \liminf_{\delta\to 0} \frac{\log N_\delta(C)}{-\log \delta} \text{ and } \overline{\dim}_B C = \limsup_{\delta\to 0} \frac{\log N_\delta(C)}{-\log \delta}.$$

Corollary 9. *Let C be a balanced Cantor set with the number of Cantor intervals $N_k = n_1 \cdots n_k$ and the average length s_k at level k. Then*

$$\underline{\dim}_B C = \liminf_{k\to\infty} \frac{\log N_k}{-\log s_k}$$

and

$$\overline{\dim}_B C = \limsup_{k\to\infty} \frac{\log N_k}{-\log s_k}.$$

Proof. If $s_{k+1} \le \delta < s_k$, then $|I_j^{k+L_2+1}| \le s_{k+1} \le \delta$ for all $1 \le j \le N_{k+L_2+1}$. It follows that

$$\frac{\log N_\delta(C)}{-\log \delta} \le \frac{\log N_{k+L_2+1}}{-\log s_k} \le \frac{\log M^{L_2+1} + \log N_k}{-\log s_k}.$$

Therefore,

$$\underline{\dim}_B C = \liminf_{\delta\to 0} \frac{\log N_\delta(C)}{-\log\delta} \le \liminf_{k\to\infty} \frac{\log N_k}{-\log s_k}.$$

The same is true for the upper box dimension.

Since $\dim_H C \le \underline{\dim}_B C \le \liminf_{k\to\infty} \frac{\log N_k}{-\log s_k}$ and $\dim_P C \le \overline{\dim}_B C \le \limsup_{k\to\infty} \frac{\log N_k}{-\log s_k}$, by Corollary 8, we get

$$\underline{\dim}_B C = \liminf_{k\to\infty} \frac{\log N_k}{-\log s_k} \text{ and } \overline{\dim}_B C = \limsup_{k\to\infty} \frac{\log N_k}{-\log s_k}.$$

■

2.4 Dimension partition in terms of h_C

In fact, we can further describe the dimension partition of a balanced Cantor set in terms of its associated dimension function h_C.

Proposition 10. *Let C be a balanced Cantor set with an associated dimension function h_C and $g \in \mathbb{D}$.*

1. *If $\liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} > 0$, then $H^g(C) > 0$. In particular, if $\liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} = \infty$, then $H^g(C) = \infty$.*
2. *If $\limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} < \infty$, then $P_0^g(C) < \infty$. In particular, if $\limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} = 0$, then $P_0^g(C) = 0$.*
3. *If $\liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} < \infty$, then $H^g(C) < \infty$. In particular, if $\liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} = 0$, then $H^g(C) = 0$.*
4. *If $\limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} > 0$, then $P_0^g(C) > 0$. In particular, if $\limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} = \infty$, then $P_0^g(C) = \infty$.*

Proof. 1. Let

$$\lambda_* := \liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} > 0.$$

For any $0 < \alpha < \lambda_*$ there is a $\delta > 0$ such that $g(x) \geq \alpha h_C(x)$ for all $0 < x < \delta$. Then

$$H^g(C) \geq \alpha H^{h_C}(C) > 0$$

by the definition of Hausdorff measure.

If $\lambda_* = \infty$, then α can be arbitrarily large and hence $H^g(C) = \infty$.

2. Let
$$\lambda^* := \limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} < \infty.$$
For any $\lambda^* < \alpha < \infty$ there is a $\delta > 0$ such that $g(x) \leq \alpha h_C(x)$ for all $0 < x < \delta$. Then
$$P_0^g(C) \leq \alpha P_0^{h_C}(C) < \infty$$
by the definition of packing premeasure.

If $\lambda^* = 0$, then $P_0^g(C) \leq \alpha P_0^{h_C}(C)$ for any $\alpha > 0$ and hence $P_0^g(C) = 0$.

3. Let
$$\lambda_* := \liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} < \infty.$$
For any $\alpha > \lambda_*$, there exists a positive decreasing sequence $\{\delta_m\}_m$ such that $\lim_{m\to\infty} \delta_m = 0$ and $g(\delta_m) \leq \alpha h_C(\delta_m)$. Let k be the integer such that $s_k \leq \delta_m < s_{k-1}$. Then
$$|I_j^{k+L_2}| \leq s_k \leq \delta_m$$
and $\{I_j^{k+L_2}\}_j$ is a δ_m-covering of C. Therefore
$$\begin{aligned} H^g_{\delta_m}(C) &\leq \sum_{j=1}^{N_{k+L_2}} g(|I_j^{k+L_2}|) \leq \sum_{j=1}^{N_{k+L_2}} g(\delta_m) \\ &< N_{k+L_2} \alpha h_C(\delta_m) \leq \alpha M^{L_2+1} N_{k-1} h_C(s_{k-1}). \end{aligned}$$
Taking limits implies
$$H^g(C) \leq \lambda_* M^{L_2+1} \liminf_{k\to\infty} N_k h_C(s_k) < \infty.$$
If $\lambda_* = 0$, then $H^g(C) = 0$.

4. Let
$$\lambda^* := \limsup_{x \to 0^+} \frac{g(x)}{h_C(x)} > 0.$$
For any $0 < \alpha < \lambda^*$, there exists a positive decreasing sequence $\{\delta_m\}_m$ such that $\lim_{m\to\infty} \delta_m = 0$ and $g(\delta_m) \geq \alpha h_C(\delta_m)$. Let k be the integer such that $s_k \leq \delta_m < s_{k-1}$. Take the left endpoint of the interval $I_{jn}^{k-L_1-1}$, $n = n_{k-L_1-1}$, at level $k - L_1 - 1$ as x_j for $1 \leq j \leq N_{k-L_1-2}$ and put $r := \delta_m/2$. The collection $\{B(x_j, r)\}_j$ will then be disjoint as $\delta_m < s_{k-1} \leq |I_j^{k-L_1-1}|$ and hence a δ_m-packing. Therefore
$$\begin{aligned} P^g_{\delta_m}(C) &\geq \sum_{j=1}^{N_{k-L_1-2}} g(\delta_m) \geq \sum_{j=1}^{N_{k-L_1-2}} \alpha h_C(\delta_m) \\ &\geq \alpha N_{k-L_1-2} h_C(s_k) \geq \frac{\alpha}{M^{L_1+2}} N_k h_C(s_k) \end{aligned}$$
and
$$P^g_0(C) \geq \frac{\lambda^*}{M^{L_1+2}} \limsup_{k\to\infty} N_k h_C(s_k).$$
If $\lambda^* = \infty$, then $P^g_0(C) = \infty$.

■

Corollary 11. *Let C be a balanced Cantor set with an associated dimension function h_C. Then*
$$\begin{aligned} \mathbb{H}^C_1 &= \{g \in \mathbb{D} : 0 < \liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} < \infty\}, \\ \mathbb{H}^C_\beta &= \{g \in \mathbb{D} : \liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} = \beta\}, \\ \mathbb{P}^C_1 &= \{g \in \mathbb{D} : 0 < \limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} < \infty\}, \\ \mathbb{P}^C_\beta &= \{g \in \mathbb{D} : \limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} = \beta\} \end{aligned}$$
where $\beta = 0, \infty$.

Corollary 12. *Let C_1 and C_2 be balanced Cantor sets with h_{C_1} and h_{C_2} as their respective associated dimension functions. Then the following are equivalent.*

(a) $h_{C_1} \equiv h_{C_2}$.

(b) $\mathbb{H}_\beta^{C_1} \cap \mathbb{P}_\gamma^{C_1} = \mathbb{H}_\beta^{C_2} \cap \mathbb{P}_\gamma^{C_2}$ *for all* $\beta \le \gamma \in \{0, 1, \infty\}$.

(c) $\mathbb{H}_1^{C_1} \cap \mathbb{P}_1^{C_1} = \mathbb{H}_1^{C_2} \cap \mathbb{P}_1^{C_2}$.

Note that C_1 and C_2 can be balanced with respect to different symbol spaces.

Definition 7. Let C_1 and C_2 be balanced Cantor sets. C_1 and C_2 are said to be equivalent if and only if $h_{C_1} \equiv h_{C_2}$, and then we write $C_1 \sim C_2$.

2.5 Equivalence of balanced Cantor sets

If two Cantor sets C_1 and C_2 are balanced with respect to the same symbol space W, we can also characterize their equivalence in terms of the sequence s_k.

Let us recall the definition of a $\mathbf{p}$-Cantor measure $\mu_{\mathbf{p}}^{C,W}$ on the Cantor set C associated with the symbol space W. For each $k \ge 1$ and $0 \le j \le n_k - 1$, let $p_{kj} \ge 0$ be probability weights such that $\sum_j p_{kj} = 1$ and $\mathbf{p} = (p_{kj})$. The $\mathbf{p}$-Cantor measure $\mu_{\mathbf{p}} = \mu_{\mathbf{p}}^{C,W}$ on C is defined by

$$\mu_{\mathbf{p}}(I_w) := \prod_{l=1}^{k} p_{l w_l}$$

for $w \in W$ of length $|w| = k$.

In the remaining part of this chapter, we assume the probability weights are uniformly bounded away from 0, i.e. there exists $b > 0$ such that $p_{kj} \ge b$ for all k, j. Then, also, $p_{kj} \le 1 - b$ for all k and j.

When $\mu_{\mathbf{p}} = \mu_{\mathbf{p}}^{C,W}$ is a $\mathbf{p}$-Cantor measure on C, define

$$\Lambda_{\mathbf{p}}^{C}(w) = \Lambda_{\mathbf{p}}^{C,W}(w) := \left\{ h \in \mathbb{D} : 0 < \liminf_{k \to \infty} \frac{\mu_{\mathbf{p}}(I_{w|k})}{h(s_k)} \le \limsup_{k \to \infty} \frac{\mu_{\mathbf{p}}(I_{w|k})}{h(s_k)} < \infty \right\}$$

for any infinite word $w \in W^\infty := \prod_{k=1}^{\infty} \{0, \cdots, n_k - 1\}$, where $w|k = w_1 \cdots w_k$.

Theorem 13. *Let C_1 and C_2 be two Cantor sets which are balanced with respect to the same symbol space W. Then the following are equivalent.*

(a) $h_{C_1} \equiv h_{C_2}$.

(b) *There exists an integer L such that $s_{k+L}^{C_2} \le s_k^{C_1} \le s_{k-L}^{C_2}$ for all $k > L$. (Here $\{s_k^{C_i}\}$ are the average interval lengths of C_i.)*

(c) $\Lambda_{\mathbf{p}}^{C_1}(w) = \Lambda_{\mathbf{p}}^{C_2}(w)$ *for all* $\mu_{\mathbf{p}}$ *and all* $w \in W^\infty$.

(d) $\Lambda_{\mathbf{p}}^{C_1}(w) = \Lambda_{\mathbf{p}}^{C_2}(w)$ *for some* $\mu_{\mathbf{p}}$ *and some* $w \in W^\infty$.

Proof. We will prove in the following order: (a) $\Rightarrow$ (b) $\Rightarrow$ (a), and then (d) $\Rightarrow$ (b) $\Rightarrow$ (c). The implication (c) $\Rightarrow$ (d) is immediate.

(a) $\Rightarrow$ **(b)** Let $A, B > 0$ be constants such that

$$A\frac{1}{N_k} \leq h_{C_1}(s_k^{C_1}) \text{ and } h_{C_2}(s_k^{C_2}) \leq B\frac{1}{N_k}$$

for all k. Suppose $h_{C_1}(x) \leq K h_{C_2}(x)$ for some $K > 0$. Then

$$h_{C_2}(s_k^{C_1}) \geq \frac{1}{K}h_{C_1}(s_k^{C_1}) \geq \frac{A}{K}\frac{1}{N_k} = B(\frac{A}{KB})\frac{1}{N_k}.$$

Choose an integer L (independent of k) such that

$$\frac{KB}{A} \leq 2^L \leq \frac{N_{k+L}}{N_k}.$$

Then for all K,

$$h_{C_2}(s_k^{C_1}) \geq B\frac{1}{N_{k+L}} \geq h_{C_2}(s_{k+L}^{C_2}).$$

Since h_{C_2} is non-decreasing, we have $s_k^{C_1} \geq s_{k+L}^{C_2}$.
By similar reasoning it follows that, when $h_{C_1} \equiv h_{C_2}$, there is an integer L such that

$$s_{k+L}^{C_2} \leq s_k^{C_1} \leq s_{k-L}^{C_2}$$

for all $k > L$.

(b) $\Rightarrow$ **(a)** It follows from $s_{k+L}^{C_2} \leq s_k^{C_1} \leq s_{k-L}^{C_2}$ that for $\beta = 0, \infty$,

- $\liminf_{k\to\infty} N_k h(s_k^{C_1}) = \beta$ if and only if $\liminf_{k\to\infty} N_k h(s_k^{C_2}) = \beta$,
- $\limsup_{k\to\infty} N_k h(s_k^{C_1}) = \beta$ if and only if $\limsup_{k\to\infty} N_k h(s_k^{C_2}) = \beta$.

Consequently $\mathbb{H}_\beta^{C_1} = \mathbb{H}_\beta^{C_2}$ and $\mathbb{P}_\beta^{C_1} = \mathbb{P}_\beta^{C_2}$ for $\beta = 0$ and ∞. In turn this forces $\mathbb{H}_1^{C_1} = \mathbb{H}_1^{C_2}$ and $\mathbb{P}_1^{C_1} = \mathbb{P}_1^{C_2}$. It follows from Corollary 12 that $h_{C_1} \equiv h_{C_2}$.

(d) $\Rightarrow$ (b) Define $h(s_k^{C_1}) := \mu_{\mathbf{p}}^{C_1}(I_{w|k}^{C_1})$ and extend h to a piecewise linear function with $h(0) = 0$. We first check that h is a dimension function. Since $s_{k+1}^{C_1} \leq s_k^{C_1}$ and $h(s_{k+1}^{C_1}) = \prod_{l=1}^{k+1} p_{lw_l} \leq \prod_{l=1}^{k} p_{lw_l} = h(s_k^{C_1})$, h is increasing. If $s_{k+1}^{C_1} \leq x \leq s_k^{C_1}$,

$$\begin{aligned} h(2x) &\leq h(2s_k^{C_1}) \\ &\leq h(s_{k-1}^{C_1}) = \prod_{l=1}^{k-1} p_{lw_l} \leq \frac{1}{b^2} \prod_{l=1}^{k+1} p_{lw_l} \\ &= \frac{1}{b^2} h(s_{k+1}^{C_1}) \leq \frac{1}{b^2} h(x), \end{aligned}$$

so h is doubling and hence $h \in \mathbb{D}$. Since $\mu_{\mathbf{p}}^{C_1}(I_{w|k}^{C_1}) = \prod_{l=1}^{k} p_{lw_l} = \mu_{\mathbf{p}}^{C_2}(I_{w|k}^{C_2})$, we also have $h(s_k^{C_1}) = \mu_{\mathbf{p}}^{C_2}(I_{w|k}^{C_2})$.

We can see from the definition that $h \in \Lambda_{\mathbf{p}}^{C_1}(w)$. Since $\Lambda_{\mathbf{p}}^{C_1}(w) = \Lambda_{\mathbf{p}}^{C_2}(w)$, $h \in \Lambda_{\mathbf{p}}^{C_2}(w)$ as well and therefore there exist $A, B > 0$ such that

$$A \leq \frac{\mu_{\mathbf{p}}^{C_2}(I_{w|k}^{C_2})}{h(s_k^{C_2})} \leq B$$

for all k.

Let L be an integer such that $(1-b)^L \leq \frac{1}{B}$ and $(1-b)^L \leq A$. When $k > L$,

$$h(s_k^{C_2}) \leq \frac{\mu_{\mathbf{p}}^{C_2}(I_{w|k}^{C_2})}{A} \leq \mu_{\mathbf{p}}^{C_2}(I_{w|k-L}^{C_2}) \frac{(1-b)^L}{A} \leq \mu_{\mathbf{p}}^{C_2}(I_{w|k-L}^{C_2}) = h(s_{k-L}^{C_1})$$

and

$$h(s_k^{C_2}) \geq \frac{\mu_{\mathbf{p}}^{C_2}(I_{w|k}^{C_2})}{B} \geq \mu_{\mathbf{p}}^{C_2}(I_{w|k+L}^{C_2}) \frac{1}{(1-b)^L B} \geq \mu_{\mathbf{p}}^{C_2}(I_{w|k+L}^{C_2}) = h(s_{k+L}^{C_1}).$$

Since h is increasing,

$$s_{k+L}^{C_1} \leq s_k^{C_2} \leq s_{k-L}^{C_1}.$$

(b) $\Rightarrow$ (c) Fix $\mu_{\mathbf{p}}$ and $w \in W^\infty$. Recall $\mu_{\mathbf{p}}^{C_1}(I_{w|k}^{C_1}) = \mu_{\mathbf{p}}^{C_2}(I_{w|k}^{C_2})$. By assumption $s_{k+L}^{C_1} \leq s_k^{C_2} \leq s_{k-L}^{C_1}$, thus

$$b^L \frac{\mu_{\mathbf{p}}^{C_1}(I_{w|k-L}^{C_1})}{h(s_{k-L}^{C_1})} \leq \frac{\mu_{\mathbf{p}}^{C_1}(I_{w|k}^{C_1})}{h(s_{k-L}^{C_1})} \leq \frac{\mu_{\mathbf{p}}^{C_2}(I_{w|k}^{C_2})}{h(s_k^{C_2})} \leq \frac{\mu_{\mathbf{p}}^{C_1}(I_{w|k}^{C_1})}{h(s_{k+L}^{C_1})} \leq \frac{1}{b^L} \frac{\mu_{\mathbf{p}}^{C_1}(I_{w|k+L}^{C_1})}{h(s_{k+L}^{C_1})}$$

If $h \in \Lambda_{\mathbf{p}}^{C_1}(w)$, then

$$0 < \liminf_{k\to\infty} \frac{\mu_{\mathbf{p}}^{C_1}(I_{w|k-L}^{C_1})}{h(s_{k-L}^{C_1})} \text{ and } \limsup_{k\to\infty} \frac{\mu_{\mathbf{p}}^{C_1}(I_{w|k+L}^{C_1})}{h(s_{k+L}^{C_1})} < \infty$$

and hence $h \in \Lambda_{\mathbf{p}}^{C_2}(w)$. By a symmetric argument we get

$$\Lambda_{\mathbf{p}}^{C_1}(w) = \Lambda_{\mathbf{p}}^{C_2}(w).$$

■

Chapter 3

Cut-out sets and balanced Cantor sets

Let $a = (a_i)$ be a decreasing summable positive sequence and $I = [0, \sum_{i=1}^{\infty} a_i]$. Let $\mathscr{C}_a$ be the collection of cut-out sets associated with a contained in I. It is known that among all the sets in $\mathscr{C}_a$, the set C_a (as defined in Section 1.3.2) has the maximal Hausdorff dimension and maximal Hausdorff measure up to a constant [2, 17]. On the other hand, the prepacking dimensions of all the sets in $\mathscr{C}_a$ are the same and equal to the upper box dimension [10]. Since the packing and prepacking dimensions of C_a coincide [6], it follows that $\dim_P E \le \dim_{P_0} E = \dim_{P_0} C_a = \dim_P C_a$ for any $E \in \mathscr{C}_a$. However, it is shown in [17] that C_a has the least packing premeasure up to a constant among the sets in $\mathscr{C}_a$.

In the following we will prove similar results for the balanced Cantor sets in $\mathscr{C}_a$.

3.1 Balanced Cantor sets within $\mathscr{C}_a$

For any $E \subseteq \mathbb{R}$ and $r > 0$, let

$$N(E, r) = \min\{k : E \subseteq \bigcup_{i=1}^{k} B(x_i, r)\},$$
$$P(E, r) = \max\{k : \{B(x_i, r)\}_{1 \le i \le k} \text{ is an } r\text{-packing of } E\}$$

and

$$E(r) = \{x \in \mathbb{R} : |x - y| < r \text{ for some } y \in E\}.$$

Geometric reasoning shows that

$$N(E,2r) \leq P(E,r) \leq N(E,r/2)$$

and

$$P(E,r)2r \leq m(E(r)) \leq N(E,r)4r$$

where m denotes the Lebesgue measure.

If $E_1, E_2 \in \mathscr{C}_a$ are cut-out sets associated with $a = (a_i)$, then $m(E_1(r)) = m(E_2(r))$ [10], so

$$P(E_1,r)2r \leq m(E_1(r)) = m(E_2(r)) \leq N(E_2,r)4r.$$

From these inequalities we have the following lemma.

Lemma 14. *For any $E_1, E_2 \in \mathscr{C}_a$ and $r > 0$,*

$$P(E_2,r) \leq 2N(E_1,r) \leq 2P(E_1,r/2) \leq 4N(E_2,r/2).$$

Theorem 15. *Let C be a balanced Cantor set in $\mathscr{C}_a$ for some $a = (a_i)$. If $h \in \mathbb{D}$ and E is any set in $\mathscr{C}_a$, then $H^h(E) \leq A\, H^h(C)$ and $P_0^h(C) \leq B\, P_0^h(E)$ for some constants A and B, which depend only on h and C.*

Proof. Recall that $h \in \mathbb{D}$ is doubling, say, $h(2x) \leq \tau h(x)$ for some τ. For any cut-out set $E \in \mathscr{C}_a$ we have

$$N(E,r)h(2r) \leq 2\tau^2 N(C,r/2)h(r/2)$$

and

$$H^h(E) \leq \liminf_{r\to 0} N(E,r)h(2r) \leq 2\tau^2 \liminf_{r\to 0} N(C,r)h(r).$$

On the other hand, for a balanced Cantor set C,

$$H^h(C) \geq \frac{1}{M^{L_1+2}} \liminf_{k\to\infty} N_k h(s_k)$$

by Theorem 1 from Chapter 2. If $r > 0$ is small, there exists $k \in \mathbb{N}$ such that $s_k \leq r \leq s_{k-1}$. Consider the Cantor intervals $\{I_j^{k+L_2}\}_{j=1}^{N_{k+L_2}}$ at level $k + L_2$. Take their left endpoints as centres and form N_{k+L_2} balls with radius r (which is at least the length of any Cantor interval of level $k + L_2$). This is a r-covering of C. So $N(C,r) \leq N_{k+L_2}$ and

$$N(C,r)h(r) \leq N_{k+L_2} h(s_{k-1}) \leq M^{L_2+1} \cdot N_{k-1} h(s_{k-1}).$$

Combining all these,

$$\begin{aligned} H^h(E) &\le 2\tau^2 \liminf_{r\to 0} N(C,r)h(r) \\ &\le 2\tau^2 M^{L_2+1} \liminf_{k\to\infty} N_k h(s_k) \\ &\le 2\tau^2 M^{L_1+L_2+3} H^h(C). \end{aligned}$$

In other words, there exists a constant $A := 2\tau^2 M^{L_1+L_2+3}$ such that

$$H^h(E) \le A H^h(C)$$

for any $E \in \mathscr{C}_a$.

Similarly, for the pre-packing measure, for any $E \in \mathscr{C}_a$,

$$\frac{1}{2} P(C,r)h(r) \le P(E, r/2)h(r) \le P_0^h(E)$$

by the lemma above. After taking the limit we get

$$\limsup_{r\to 0} P(C,r)h(r) \le 2P_0^h(E).$$

As above, let $r > 0$ be small and take $k \in \mathbb{N}$ such that $s_{k+1} \le r \le s_k$. Take the subset of Cantor intervals $\{I_{jn}^{k-L_1}\}_{j=1}^{N_{k-L_1-1}}$ at level $k - L_1$, where $n = n_{k-L_1}$. Take their left endpoints as centres and form N_{k-L_1-1} balls with radii r (which is at most the length of any level $k - L_1$ Cantor interval). This forms a r-packing of C. So $N_{k-L_1-1} \le P(C,r)$ and

$$\frac{1}{M^{L_1+2}} \cdot N_{k+1} h(s_{k+1}) \le N_{k-L_1-1} h(s_{k+1}) \le P(C,r)h(r).$$

Again, applying Theorem 1 from Chapter 2, we deduce that

$$\begin{aligned} P_0^h(C) &\le M^{L_2+2} \limsup_{k\to\infty} N_k h(s_k) \\ &\le M^{L_1+L_2+4} \limsup_{r\to 0} P(C,r)h(r) \\ &\le 2M^{L_1+L_2+4} P_0^h(E), \end{aligned}$$

i.e. for any $E \in \mathscr{C}_a$,

$$P_0^h(C) \le B P_0^h(E)$$

with $B := 2M^{L_1+L_2+4}$. ■

Corollary 16. *Let C be a balanced Cantor set in $\mathscr{C}_a$ for some $a = (a_i)$. If $h \in \mathbb{D}$ and E is any set in $\mathscr{C}_a$, then $\dim_H E \leq \dim_H C$.*

Proof. This is because $H^\alpha(E) \leq A\, H^\alpha(C)$ for any $\alpha \geq 0$. ■

Corollary 17. *If $C_1, C_2 \in \mathscr{C}_a$ are both balanced Cantor sets, then*

$$A_1 H^h(C_2) \leq H^h(C_1) \leq A_2 H^h(C_2)$$

and

$$B_1 P_0^h(C_2) \leq P_0^h(C_1) \leq B_2 P_0^h(C_2)$$

for some positive constants A_1, A_2, B_1, B_2. Hence $\mathbb{H}_\beta^{C_1} = \mathbb{H}_\beta^{C_2}$ and $\mathbb{P}_\beta^{C_1} = \mathbb{P}_\beta^{C_2}$ for $\beta = 0, 1$ and ∞, i.e. $C_1 \sim C_2$. In particular $\dim_H C_1 = \dim_H C_2$ and $\dim_P C_1 = \dim_P C_2$.

It follows that $C \sim C_a$ for any balanced Cantor set C in $\mathscr{C}_a$. Note that once the sequence $a = (a_i)$ is fixed, we can construct many general decreasing Cantor sets C_a^W, with respect to different symbol spaces W. As long as they are balanced Cantor sets, they all have the maximal Hausdorff dimension within the collection $\mathscr{C}_a$, namely $\dim_H C_a$.

3.2 Central Cantor set and decreasing Cantor set in each equivalence class

The central Cantor sets and the decreasing Cantor sets have served as our prototype of the balanced Cantor sets. It turns out that for each balanced Cantor set C there exists at least one central Cantor set and one decreasing Cantor set having the same dimension partition as C.

Proposition 18. *For any balanced Cantor set C, there exists a central Cantor set K_r, with decreasing gap lengths, such that $C \sim K_r$.*

Proof. Each balanced Cantor set C is a cut-out set, i.e. $C \in \mathscr{C}_a$ for some positive, summable and decreasing sequence $a = (a_i)$. We know from Corollary 17 that $C \sim C_a$. Next, we will show that $C_a \sim K_r$ for some central Cantor set K_r.

As usual, let

$$s_k = \frac{1}{2^k} \sum_{i \geq 2^k} a_i$$

be the average interval lengths of C_a. We can construct a central Cantor set K_r as follows. Let $r_1 = s_1$, $r_2 = \frac{s_2}{s_1}$ and inductively define $r_{k+1} = \frac{s_{k+1}}{s_k}$. Note that $2s_{k+1} < s_k$ ensures that $r_{k+1} = \frac{s_{k+1}}{s_k} < \frac{1}{2}$. As $r_1 r_2 \cdots r_k = s_k$, C is h-regular if and only if $h(s_k) \approx \frac{1}{2^k}$ if and only if K_r is h-regular. Hence $K_r \sim C_a \sim C$.

Let $g = (g_n)$ be the sequence of gap lengths of K_r:

$$\begin{aligned} g_1 &= a_1, \\ g_2 = g_3 = s_1 - 2s_2 &= \frac{1}{2}(\sum_{i\geq 2} a_i - \sum_{i \geq 4} a_i) \\ &= \frac{1}{2}(a_2 + a_3) \leq a_1 = g_1. \end{aligned}$$

In general, for $2^k \leq n < 2^{k+1}$,

$$\begin{aligned} g_n = s_k - 2s_{k+1} &= \frac{1}{2^k}(\sum_{i \geq 2^k} a_i - \sum_{i \geq 2^{k+1}} a_i) \\ &= \frac{1}{2^k}(a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1}) \leq g_{n-1}. \end{aligned}$$

Hence, K_r obtained here has decreasing gap lengths.

∎

In particular,

$$\begin{aligned} \dim_H K_r &= \liminf_{k\to\infty} \frac{\ln 2^k}{|\ln r_1 \cdots r_k|} = \dim_H C_a \\ \text{and } \dim_P K_r &= \limsup_{k\to\infty} \frac{\ln 2^k}{|\ln r_1 \cdots r_k|} = \dim_P C_a. \end{aligned}$$

However, note that K_r is not necessarily in $\mathscr{C}_a$ for the same sequence $a = (a_i)$.

3.3 Size of balanced Cantor sets in $\mathscr{C}_a$

Below, we try to explore how many balanced Cantor sets there are within the collection of cut-out sets $\mathscr{C}_a$.

3.3.1 Cardinality

In this section, we use $|\mathbb{R}|$ to denote the cardinality of the real line and $|\mathscr{C} \cap \mathscr{C}_a|$ to denote the cardinality of the collection of balanced Cantor sets within the collection of cut-out sets associated with $a = (a_i)$.

Theorem 19. *Let $\mathscr{C}$ be the collection of balanced Cantor sets and $\mathscr{C}_a$ be the collection of cut-out sets associated with $a = (a_i)$. Then $|\mathscr{C} \cap \mathscr{C}_a| \geq |\mathbb{R}|$.*

Proof. To estimate the cardinality $|\mathscr{C} \cap \mathscr{C}_a|$, let us start with $C_a \in \mathscr{C}_a$. We want to generate $|\mathbb{R}|$ many balanced Cantor sets in $\mathscr{C}_a$ from it. For each permutation ρ defined below, we will construct a Cantor set $C_{\tilde{a}} \in \mathscr{C}_a$ with a permuted sequence of gaps $\tilde{a} := (a_{\rho(i)})$, in the same manner as we construct a decreasing Cantor set.

Consider the gap lengths of C_a at each level k: $a_{2^k}, a_{2^k+1}, \cdots, a_{2^{k+1}-1}$. There are two cases.

1. Suppose there are infinitely many levels k_n such that $a_{2^{k_n}} > a_{2^{k_n+1}-1}$. Then we take a subsequence (l_n) such that $l_n \geq 2$, $2^{k_n} \leq l_n < 2^{k_n+1} - 1$ and $a_{l_n} > a_{l_n+1}$ for all n.

   For each symbol $w = w_1 w_2 \cdots w_n \cdots \in \{0,1\}^{\mathbb{N}}$, define a permutation ρ_w on the natural numbers by

$$\rho_w(l) := \begin{cases} l_n + 1, & \text{if } w_n = 1 \text{ and } l = l_n, \\ l_n, & \text{if } w_n = 1 \text{ and } l = l_n + 1, \\ l, & \text{otherwise.} \end{cases}$$

   The permutation works within each level, and there are $|\mathbb{R}|$ many of them in $\{\rho_w : w \in \{0,1\}^{\mathbb{N}}\}$.

   If $\rho_w \neq \rho_{\tilde{w}}$, let $n = \min\{i : w_i \neq \tilde{w}_i\}$. One of them will permute l_n and $l_n + 1$, while the other will keep them fixed. Since the gaps a_{l_n} and a_{l_n+1} are at level $k_n + 1$ in the binary tree structure, there must be some gap a_l from a level $k \leq k_n$ between them. The set of gap lengths to the left and right of a_l are different after the permutation; the resulting Cantor sets, $C_{(a_{\rho_w(i)})}$ and $C_{(a_{\rho_{\tilde{w}}(i)})}$, are therefore distinct.

   Let $|\tilde{I}_j^k|$ be the interval lengths of the new Cantor set $C_{\tilde{a}}$, and $\tilde{s}_k$ be the new average interval length of level k. For all k,

$$\begin{aligned}\tilde{s}_k &= \frac{1}{2^k}\sum_{l=2^k}^{\infty} a_{\rho(l)} \\ &= \frac{1}{2^k}\sum_{l=2^k}^{\infty} a_l = s_k.\end{aligned}$$

Next, we want to prove $|\tilde{I}_j^k| \geq |I_{j+1}^k| \geq s_{k+1}$ for $1 \leq j < 2^k$. Let $A_j^k := \{a_i : a_i \subseteq I_j^k\}$ and $\tilde{A}_j^k := \{a_i : a_i \subseteq \tilde{I}_j^k\}$ be the set of gaps contained in I_j^k and $\tilde{I}_j^k$ respectively. Note that the gaps in A_j^k have greater lengths than the gaps in A_{j+1}^k. Since the gaps a_{l_n} are exchanged with the ones next to them and of the same level, after permutation, $\tilde{A}_j^k$ may have some common gaps as A_{j+1}^k. The gaps in $\tilde{A}_j^k \setminus A_{j+1}^k$ are in one-one correspondence with and have greater lengths than those in $A_{j+1}^k \setminus \tilde{A}_j^k$. Therefore,

$$|\tilde{I}_j^k| = \sum_{a_i \in \tilde{A}_j^k} a_i \geq \sum_{a_i \in A_{j+1}^k} a_i = |I_{j+1}^k| \geq s_{k+1}.$$

By similar reasoning we also have

$$s_{k-1} \geq |I_{j-1}^k| \geq |\tilde{I}_j^k|$$

for $1 < j \leq 2^k$. If $j = 1$ or 2^k, we can see that

$$|I_1^k| \geq |\tilde{I}_1^k| \text{ and } |\tilde{I}_{2^k}^k| \geq |I_{2^k}^k|$$

because the permutations always occur within a level. Hence, we still have

$$\tilde{s}_{k+1} = s_{k+1} \leq |\tilde{I}_j^k| \leq s_{k-1} = \tilde{s}_{k-1}$$

and the new Cantor set $C_{\tilde{a}}$ is balanced.

2. The second possibility is that $a_{2^k} > a_{2^{k+1}-1}$ only occurs at finitely many levels. That is, we have

$$a_{2^k} = a_{2^k+1} = \cdots = a_{2^{k+1}-1}$$

except on finitely many levels. The decrease in gap lengths happen at $a_{2^{k_n}-1} > a_{2^{k_n}}$ where $k_n \to \infty$. Take $l_n = 2^{k_n} - 1$ a subsequence such that $a_{l_n} > a_{l_n+1}$ for all n.

Define ρ_w in a similar fashion to the first case. If $\rho_w \neq \rho_{\tilde{w}}$ and $n = \min\{i : w_i \neq \tilde{w}_i\}$, one of them will permute l_n and $l_n + 1$. To the right of a_1, the number of gap lengths equal to a_{l_n} will be different after the permutation. Again, the resulting Cantor sets will be different.

For each k, either $\rho_w(2^k) = 2^k$, in which case (using the same notation again)

$$\tilde{s}_k = \sum_{l=2^k}^{\infty} a_{\rho(l)} = \frac{1}{2^k} \sum_{l=2^k}^{\infty} a_l = s_k,$$

or $\rho_w(2^k) = 2^k - 1$, in which case

$$\tilde{s}_k = \frac{1}{2^k}\Big(\sum_{l=2^k+1}^{\infty} a_l + a_{2^k-1}\Big) < \frac{1}{2^k} \sum_{l=2^{k-1}}^{\infty} a_l \le \frac{1}{2} s_{k-1}$$

and

$$s_k = \frac{1}{2^k} \sum_{l=2^k}^{\infty} a_l \le \frac{1}{2^k} \sum_{l=2^k}^{\infty} a_{\rho(l)} = \tilde{s}_k.$$

As only the rightmost and leftmost gaps are swapped, only the rightmost and leftmost interval lengths are affected. Hence for $1 < j < 2^k$, we still have $|\tilde{I}_j^k| = |I_j^k|$ and so

$$\tilde{s}_{k+2} \le s_{k+1} \le |\tilde{I}_j^k| \le s_{k-1} \le \tilde{s}_{k-1}.$$

For the remaining intervals, observe that a comparison of gap lengths contained in the old and new intervals shows

$$|I_1^k| \le |\tilde{I}_1^k| \le |I_{2^{k-1}}^{k-1}|$$

and

$$|I_1^{k+1}| \le |\tilde{I}_{2^k}^k| \le |I_{2^k}^k|.$$

Thus we have

$$\tilde{s}_{k+1} \le s_k \le |\tilde{I}_1^k| \le s_{k-1} \le \tilde{s}_{k-1}$$

and

$$\tilde{s}_{k+2} < s_{k+1} \le |\tilde{I}_{2^k}^k| \le s_k \le \tilde{s}_k.$$

The balanced property is satisfied.

Therefore, in both cases, we have generated $|\mathbb{R}|$ many distinct balanced Cantor sets in $\mathscr{C}_a$. ■

3.3.2 Denseness

Let I be a closed interval of length $|I| = \sum_i a_i$. Let X be the collection of non-empty compact subsets of I. Recall that the Hausdorff metric is defined by

$$d(E, F) := \max\{\sup_{x\in E}\inf_{y\in F}|x-y|, \sup_{y\in F}\inf_{x\in E}|x-y|\}$$

where E, F are two sets in X. It is known that X with the Hausdorff metric is a compact metric space. As a subset of X, the collection of cut-out sets $\mathscr{C}_a$ is also a compact metric space [34]. We will see that the collection of balanced Cantor sets is dense within $\mathscr{C}_a$.

Let $E \in \mathscr{C}_a$ for $a = (a_i)$. We label its gaps as follows. Let $E = I \setminus \bigcup_{i=1}^{\infty} A_i^E$ where $|A_i^E| = a_i$. If $|A_i^E| = |A_{i+1}^E|$ for some i, then A_i^E lies to the left of A_{i+1}^E, i.e., $x < y$ for any $x \in A_i^E$ and $y \in A_{i+1}^E$.

For each fixed $n \geq 1$, consider the gaps $A_1^E, \cdots, A_n^E$. There is a permutation $\sigma_n^E \in \mathrm{Sym}(\{1, \cdots, n\})$ such that if $x_i \in A_{\sigma_n^E(i)}^E$ for $1 \leq i \leq n$, then $x_1 < \cdots < x_n$ (i.e. $A_{\sigma_n^E(i)}^E$ is to the left of $A_{\sigma_n^E(i+1)}^E$). Define

$$\mathscr{C}_a^n(E) := \{F \in \mathscr{C}_a : \sigma_n^F = \sigma_n^E\}.$$

In these sets, the relative positions of the first n gaps are the same. We can see that the sets in $\mathscr{C}_a^n(E)$ are close to E from the following lemma. Recall that the diameter of a subset Y in a metric space X is defined as $\mathrm{diam} Y := \sup\{d(E, F) : E, F \in Y\}$.

Lemma 20 ([34]). $\mathrm{diam}\mathscr{C}_a^n(E) \leq 3r_{n+1}$, *where* $r_n := \sum_{j\geq n} a_j$.

Proof. Let $A_i^F = (L_i(F), R_i(F))$ be the gaps for each $F \in \mathscr{C}_a$, and let

$$F_n = \bigcup_{1\leq i\leq n} \{L_i(F), R_i(F)\}.$$

For any $F, F' \in \mathscr{C}_a^n(E)$, the relative position of A_i^F among the first n gaps is the same as that of $A_i^{F'}$. For each i, the leftmost possible position of the endpoint $L_i(F)$ and the rightmost possible position of the endpoint $L_i(F')$ can only differ by at most $\sum_{j=n+1}^{\infty} a_j = r_{n+1}$. This is also true for the right endpoints $R_i(F)$ and $R_i(F')$. In consequence, we have $|L_i(F) - L_i(F')| \leq r_{n+1}$ and $|R_i(F) - R_i(F')| \leq r_{n+1}$ for $1 \leq i \leq n$, so

$$d(F_n, F_n') \leq r_{n+1}.$$

On the other hand, $d(F, F_n) \leq r_{n+1}$ and $d(F', F_n') \leq r_{n+1}$, since the distance between any consecutive two of the first n gaps is at most $\sum_{j=n+1}^{\infty} a_j = r_{n+1}$. Hence,

$$d(F, F') \leq 3r_{n+1}.$$

■

Theorem 21. *The collection of balanced Cantor sets $\mathscr{C} \cap \mathscr{C}_a$ is dense in $\mathscr{C}_a$.*

Proof. Let $E \in \mathscr{C}_a$ and $\varepsilon > 0$. Assume $E = I \setminus \bigcup_{i=1}^{\infty} A_i^E$. We show that we can find a balanced Cantor set $C \in \mathscr{C} \cap \mathscr{C}_a$ such that $d(E, C) < \varepsilon$.

Let $n = 2^l - 1$ be sufficiently large that diam $\mathscr{C}_a^n(E) \leq 3r_{n+1} < \varepsilon$. Remove the gaps $A_i^E, 1 \leq i \leq n$, from I in the same left to right order as they lie in $I \setminus E$. There will be 2^l intervals left behind. We then remove gaps of lengths $\{|A_i^E| = a_i : i \geq 2^l\}$ from the remaining intervals, from left to right in each level, as we do in constructing C_a. In this way we obtain a Cantor set $C \in \mathscr{C}_a^n(E)$ and thus $d(E, C) < \varepsilon$.

Since the Cantor intervals I_j^k and the average interval lengths s_k are determined by the remaining gap lengths $\{a_i : i \geq 2^l\}$, C satisfies the same balanced property as the decreasing Cantor set C_a. ■

Chapter 4

Multifractal box dimensions and Multifractal analysis

Let μ be a finite Borel regular measure on $\mathbb{R}$. The local dimension of μ at a point $x \in \mathbb{R}$, given by

$$\dim_{loc} \mu(x) := \lim_{r \to 0^+} \frac{\log \mu(B(x, r))}{\log r},$$

describes the power law behaviour of $\mu(B(x, r))$ for small r. For $\alpha \geq 0$ we consider the level sets

$$\mathcal{E}(\alpha) = \mathcal{E}(\mu, \alpha) := \{x \in \operatorname{supp}(\mu) : \dim_{loc} \mu(x) = \alpha\}$$

of the support of the measure. If $\mathcal{E}(\alpha) \neq \emptyset$ for a range of α, these sets can be viewed as a decomposition of the support into a family of fractals.

More generally, if $h \in \mathbb{D}$ is a dimension function, define

$$\mathcal{E}_h = \mathcal{E}_h(\mu) := \left\{ x \in \operatorname{supp}(\mu) : \liminf_{r \to 0^+} \frac{\log \mu(B_r(x))}{\log h(r)} = 1 \right\},$$

$$\mathcal{E}^h = \mathcal{E}^h(\mu) := \left\{ x \in \operatorname{supp}(\mu) : \limsup_{r \to 0^+} \frac{\log \mu(B_r(x))}{\log h(r)} = 1 \right\}$$

and

$$\mathcal{E}(h) = \mathcal{E}(\mu, h) = \mathcal{E}_h \cap \mathcal{E}^h.$$

If $h(t) = t^\alpha$, these sets will be correspondingly denoted as $\mathcal{E}_\alpha = \mathcal{E}_\alpha(\mu) := \mathcal{E}_h(\mu)$ and $\mathcal{E}^\alpha = \mathcal{E}^\alpha(\mu) := \mathcal{E}^h(\mu)$. In this notation, $\mathcal{E}(\alpha) := \mathcal{E}_\alpha \cap \mathcal{E}^\alpha$.

We are interested in the dimensions of these level sets $\mathcal{E}(\alpha)$. The dimensions $f_H(\alpha) := \dim_H \mathcal{E}(\alpha)$ and $f_P(\alpha) := \dim_P \mathcal{E}(\alpha)$ are called the Hausdorff multifractal spectrum and the packing multifractal spectrum respectively.

In the physics literature ([16]) it is suggested that these dimensions are often equal to the Legendre transform of some other functions. This sometimes provides an alternate way to calculate the dimensions.

The Legendre transform of a function τ is defined as

$$\tau^*(\alpha) = \inf_q (q\alpha - \tau(q)) .$$

The multifractal formalism is the following relationship: the multifractal spectrum $f(\alpha) = \dim_H(\mathcal{E}_\alpha)$ or $\dim_P(\mathcal{E}_\alpha)$ is the Legendre transform

$$f(\alpha) = \tau^*(\alpha)$$

of some suitable auxiliary function τ. This statement is justified and the multifractal spectrum is calculated in different cases, including self-similar measures ([10, 7, 26]), p-Cantor measures on the central Cantor sets ([18]), decreasing Cantor sets ([19]), and many other examples.

In this chapter we want to extend the calculations to the balanced Cantor sets. As the most general case is complicated in notation and not substantially different, we will restrict ourselves to the following assumptions.

Assumption 1. (a) We assume C is a balanced Cantor set where $M \geq 2$ is a fixed integer and $n_k = M$ for all k. Then $N_k = M^k$. This includes the homogeneous Cantor sets with $n_k = M$.

(b) We study the $\mathbf{p}$-Cantor measure on C where $p_{kj} = p_j$ for each $j = 1, \cdots, M$ and for all k. We assume $p = p_{\min} := \min_{1\leq j\leq M} p_j > 0$ throughout this chapter.

4.1 Multifractal box dimensions

In this section let us introduce the multifractal box dimension as our auxiliary function. For $q \in \mathbb{R}$ let

$$S_\delta(q) = \sup\left\{\sum_i \mu(B(x_i,\delta))^q : \{B(x_i,\delta)\}_i \text{ are disjoint closed balls with } x_i \in \operatorname{supp}(\mu)\right\}.$$

The lower and upper multifractal q-box dimensions of μ are defined in [26] as

$$C_q = \liminf_{\delta\to 0^+} \frac{\log S_\delta(q)}{|\log\delta|} \text{ and } C^q = \limsup_{\delta\to 0^+} \frac{\log S_\delta(q)}{|\log\delta|}.$$

It is denoted as $C(q)$ if $\lim_{\delta\to 0^+} \frac{\log S_\delta(q)}{|\log\delta|}$ exists. The quantity $\tau(q) := -C^q$ is also known as the L^q-spectrum.

Lemma 22. *Let μ be a $\mathbf{p}$-Cantor measure on a balanced Cantor set C satisfying Assumption 1. Suppose $q \in \mathbb{R}$ and $s_k \le \delta < s_{k-1}$. Then there exist $A, B > 0$ such that*

$$A\left(\sum_{j=1}^{M} p_j^q\right)^k \le S_\delta(q) \le B\left(\sum_{j=1}^{M} p_j^q\right)^k. \tag{4.1}$$

Proof. Let $\{B(x_i,\delta)\}_i$ be a collection of disjoint closed balls with $x_i \in \operatorname{supp}(\mu)$. The balanced property of the Cantor set tells us that $|I_j^{k+L_1}| \le s_k \le \delta < s_{k-1} \le |I_j^{k-1-L_2}|$, hence if $I_j^{k-1-L_2} = I^{k-1-L_2}(x_i)$, then

$$I^{k+L_1}(x_i)\cap C \subseteq B_\delta(x_i)\cap C \subseteq (I_{j-1}^{k-1-L_2}\cup I_j^{k-1-L_2}\cup I_{j+1}^{k-1-L_2})\cap C.$$

1. Let us first consider $q \ge 0$. For the upper bound,

$$\begin{aligned}\mu(B_\delta(x_i))^q &\le \left(\mu(I_{j-1}^{k-1-L_2}\cap C)+\mu(I_j^{k-1-L_2}\cap C)+\mu(I_{j+1}^{k-1-L_2}\cap C)\right)^q\\ &\le K_q\left(\mu(I_{j-1}^{k-1-L_2}\cap C)^q+\mu(I_j^{k-1-L_2}\cap C)^q+\mu(I_{j+1}^{k-1-L_2}\cap C)^q\right)\end{aligned}$$

   for some constant K_q. This is due to Holder's inequality when $q \ge 1$ and concavity of x^q when $0 < q < 1$. Note that each $I_j^{k-1-L_2}$ can intersect at most $M^{L_1+L_2+1}$ of the balls $B_\delta(x_i)$ since the balls are disjoint and each of them contains an interval $I^{k+L_1}(x_i)$. Thus

$$\begin{aligned}\sum_i \mu(B_\delta(x_i))^q &\le M^{L_1+L_2+1}K_q \sum_{j=1}^{M^{k-L_2-1}} \mu(I_j^{k-L_2-1})^q\\ &= M^{L_1+L_2+1}K_q \sum_{|w|=k-L_2-1}\left(\prod_{l=1}^{k-L_2-1} p_{w_l}\right)^q\\ &= M^{L_1+L_2+1}K_q\left(\sum_{j=1}^{M} p_j^q\right)^{k-L_2-1},\end{aligned}$$

so we can take $B = M^{L_1+L_2+1} K_q \left(\sum_{j=1}^M p_j^q\right)^{-L_2-1}$.

Now we consider the lower bound. Take x_i to be the left endpoints of the intervals $I_{iM}^{k-L_2-1}$ for $1 \leq i \leq M^{k-L_2-2}$. The balls $\{B_\delta(x_i)\}_i$ are centred in C and pairwise disjoint.

The interval $I^{k+L_1}(x_i)$ is the leftmost subinterval of $I^{k-L_2-1}(x_i)$ at level $k+L_1$, so

$$\mu(I^{k+L_1}(x_i)) = \left(\prod_{l=1}^{k-L_2-2} p_{w_l}\right) p_M\, p_1^{L_1+L_2+1} \geq p^{L_1+L_2+2} \prod_{l=1}^{k-L_2-2} p_{w_l}$$

where $I^{k-L_2-1}(x_i) = I_{w_1 \cdots w_{k-L_2-2} M}$. Since $I^{k+L_1}(x_i) \cap C \subseteq B_\delta(x_i) \cap C$,

$$\begin{aligned} S_\delta(q) \geq \sum_i \mu(B_\delta(x_i))^q &\geq \sum_{i=1}^{M^{k-L_2-2}} \mu(I^{k+L_1}(x_i))^q \\ &\geq (p^{L_1+L_2+2})^q \sum_{|w|=k-L_2-2} \left(\prod_{l=1}^{k-L_2-2} p_{w_l}\right)^q \\ &= (p^{L_1+L_2+2})^q \left(\sum_{j=1}^M p_j^q\right)^{k-L_2-2}. \end{aligned}$$

The last equality holds because the sum is over all the Cantor intervals of level $k - L_2 - 2$.

2. Consider $q < 0$. For any δ-packing $\{B_\delta(x_i)\}_i$, the Cantor intervals $I^{k+L_1}(x_i)$ are contained in $B_\delta(x_i)$ and hence are disjoint. As $q < 0$, we have $\mu(B_\delta(x_i))^q \leq \mu(I^{k+L_1}(x_i))^q$. Thus

$$\begin{aligned} \sum_i \mu(B_\delta(x_i))^q \leq \sum_i \mu(I^{k+L_1}(x_i))^q &\leq \sum_{j=1}^{M^{k+L_1}} \mu(I_j^{k+L_1})^q \\ &= \left(\sum_{j=1}^M p_j^q\right)^{k+L_1}. \end{aligned}$$

For the lower bound, take x_i to be the the left endpoints of $I_{iM}^{k-L_2-1}$ for $1 \leq i \leq M^{k-L_2-2}$. The choice of the left endpoint for x_i gives the inclusion $B_\delta(x_i) \subseteq I_i^{k-L_2-2}$

and that implies $\mu(B_\delta(x_i))^q \geq \mu(I_i^{k-L_2-2})^q$ when $q < 0$. Thus

$$\sum_i \mu(B_\delta(x_i))^q \geq \sum_{i=1}^{M^{k-L_2-2}} \mu(I_i^{k-L_2-2})^q = \left(\sum_{j=1}^{M} p_j^q\right)^{k-L_2-2}.$$

■

From the above inequalities, we obtain the following theorem.

Theorem 23. *Let μ be a $\mathbf{p}$-Cantor measure on a balanced Cantor set satisfying Assumption 1. The multifractal q-box dimensions of μ are given by*

$$C_q = \liminf_{k\to\infty} \frac{\log \sum_{j=1}^{M} p_j^q}{\frac{1}{k}|\log s_k|} \text{ and } C^q = \limsup_{k\to\infty} \frac{\log \sum_{j=1}^{M} p_j^q}{\frac{1}{k}|\log s_k|}.$$

Hence, if the lower and upper limits exist and are equal, then

$$C(q) = \lim_{k\to\infty} \frac{\log \sum_{j=1}^{M} p_j^q}{\frac{1}{k}|\log s_k|}.$$

Proof. Let $q \in \mathbb{R}$ and $s_k \leq \delta < s_{k-1}$. Taking the logarithm of (4.1), we get

$$\log A + k \log \sum_{j=1}^{M} p_j^q \leq \log S_\delta(q) \leq \log B + k \log \sum_{j=1}^{M} p_j^q. \tag{4.2}$$

On the other hand,

$$|\log s_k| \geq |\log \delta| \geq |\log s_{k-1}|. \tag{4.3}$$

Note that $\lim_{k\to\infty} \frac{\log A}{|\log s_k|} = 0$ and $\lim_{k\to\infty} \frac{\log B}{|\log s_k|} = 0$. If we consider the division of (4.2) by (4.3) and take the liminfs, then

$$\liminf_{\delta\to 0} \frac{\log S_\delta(q)}{|\log \delta|} = \liminf_{k\to\infty} \frac{\log \sum_{j=1}^{M} p_j^q}{\frac{1}{k}|\log s_k|}$$

The same is true for the upper limits. ■

4.2 Separation conditions

In the multifractal analysis on self-similar sets or other Cantor-like sets, some separation conditions, like the strong separation condition and the open set condition (see Section 1.3.3), are usually required. In our setting, the balanced Cantor sets may not be self-similar. We need to formulate another version of the separation condition.

Let $G_j^k, 1 \leq j \leq M^{k-1}(M-1)$, be the gaps between the k-th level intervals I_j^k of the Cantor set C. In this section, we suppose that C satisfies the following separation condition.

Assumption 2. There is an $\epsilon > 0$ such that for any k and $1 \leq j \leq M^{k-1}(M-1)$,

$$|G_j^k| \geq \epsilon s_k.$$

For $x \in C$ and $k \geq 1$, let $I^k(x)$ be the Cantor interval containing x at level k. The following lemma shows that the measures of balls and Cantor intervals are comparable under the separation condition above.

Lemma 24. *Let C be a balanced Cantor set satisfying Assumptions 1 and 2. There is an $N \in \mathbb{N}$ such that*

$$I^k(x) \cap C \subseteq B_{|I^k(x)|}(x) \cap C \subseteq I^{k-N}(x) \cap C$$

for any $x \in C$ and $k > N$.

Proof. We only need to prove the second inclusion. Choose $N \geq L_2$ such that $2^{N-L_2}\epsilon > 1$. Fix $k > N$ and let $1 \leq i \leq k-N$, say $i = k - N'$ with $N' \geq N$. Then

$$|I^k(x)| \leq s_{k-L_2} \leq \frac{1}{2^{N'-L_2}} s_{k-N'} \leq \frac{1}{2^{N'-L_2}\epsilon}|G_j^{k-N'}| < |G_j^i|.$$

Thus the radius of $B_{|I^k(x)|}(x)$ is smaller than the gap lengths up to level $k-N$, and so $B_{|I^k(x)|}(x)$ is contained in the union of $I^{k-N}(x)$ and its adjacent gaps. Therefore, we have the second inclusion. ■

Lemma 25. *Let μ be a $\mathbf{p}$-Cantor measure on C satisfying Assumptions 1 and 2. Then for any $h \in \mathbb{D}$ and $x \in C$,*

$$\liminf_{\delta \to 0^+} \frac{\log \mu(B_\delta(x))}{\log h(\delta)} = \liminf_{k \to \infty} \frac{\log \mu(I^k(x))}{\log h(s_k)}$$

and

$$\limsup_{\delta \to 0^+} \frac{\log \mu(B_\delta(x))}{\log h(\delta)} = \limsup_{k \to \infty} \frac{\log \mu(I^k(x))}{\log h(s_k)}.$$

Proof. Let $|I^{k+L_2}| \le s_k \le \delta < s_{k-1} \le |I^{k-L_1-1}|$. Lemma 24 implies that for suitable N and sufficiently large k,

$$I^{k+L_2}(x) \cap C \subseteq B_\delta(x) \cap C \subseteq B_{|I^{k-L_1-1}(x)|}(x) \cap C \subseteq I^{k-L_1-1-N}(x) \cap C.$$

Since there exists $p > 0$ such that $p_j \ge p > 0$ for all $1 \le j \le M$,

$$c_1\mu(I^{k-1}(x)) \le \mu(I^{k+L_2}(x)) \le \mu(B_\delta(x)) \le \mu(I^{k-L_1-1-N}(x)) \le c_2\mu(I^k(x))$$

where $c_1 = p^{L_2+1}$ and $c_2 = (\frac{1}{p})^{L_1+1+N}$ are independent of x and k. Thus

$$\frac{\log(c_2\mu(I^k(x)))}{\log h(s_k)} \le \frac{\log \mu(B_\delta(x))}{\log h(\delta)} \le \frac{\log(c_1\mu(I^{k-1}(x)))}{\log h(s_{k-1})}.$$

Taking limits we get the conclusion. ∎

4.3 Multifractal formalism

In this section we will calculate the multifractal spectrum of a **p**-Cantor measure μ on a balanced Cantor set C satisfying Assumptions 1 and 2. Our goal is to obtain a dimensional description of the level sets $\mathcal{E}(\alpha)$. We will first work with the more general $\mathcal{E}(h)$ for dimension functions h as in [19] and obtain the dimensions of $\mathcal{E}(\alpha)$ in the end.

Let

$$b_q = \left(\sum_{j=1}^{M} p_j^q\right)^{-1} \left(\sum_{j=1}^{M} p_j^q \log p_j\right)$$

and

$$\theta_q = q - \frac{\log\left(\sum_{j=1}^{M} p_j^q\right)}{b_q}.$$

Define an auxiliary measure $\nu = \nu_q$ as follows. For $|w| = k$, let

$$\nu(I_w) := \mu(I_w)^q \left(\sum_{j=1}^{M} p_j^q\right)^{-k}. \tag{4.4}$$

One can check that for each k,

$$\sum_{|w|=k} \nu(I_w) = \left(\sum_{j=1}^{M} p_j^q\right)^{-k} \sum_{|w|=k} \mu(I_w)^q = 1.$$

It follows that ν is a probability measure and we will see that it is concentrated on $\mathcal{E}_h$ or $\mathcal{E}^h$ with appropriate q and h. Moreover, ν is a $\mathbf{p}$-Cantor measure where $\mathbf{p} = \{\frac{p_1^q}{\sum_{j=1}^M p_j^q}, \cdots, \frac{p_M^q}{\sum_{j=1}^M p_j^q}\}$.

Lemma 26. *1. If* $\liminf_{k\to\infty} \frac{1}{k} \log h(s_k) = b_q$, *then* $\nu_q(\mathcal{E}_h) = 1$.

2. If $\limsup_{k\to\infty} \frac{1}{k} \log h(s_k) = b_q$, *then* $\nu_q(\mathcal{E}^h) = 1$.

Proof. 1. Let $\delta > 0$ and

$$\begin{aligned} E_k :=& \{x \in \mathrm{supp}(\mu) : \log \mu(I^k(x)) \geq (1-\epsilon) \log h(s_k)\} \\ =& \{x \in \mathrm{supp}(\mu) : \mu(I^k(x)) \geq h(s_k)^{(1-\epsilon)}\}. \end{aligned}$$

Then

$$\begin{aligned} \nu(E_k) &\leq \int \mu(I^k(x))^\delta h(s_k)^{\delta(\epsilon-1)} d\nu(x) \\ &= \sum_{|w|=k} h(s_k)^{\delta(\epsilon-1)} \mu(I_w)^\delta \nu(I_w) \\ &= h(s_k)^{\delta(\epsilon-1)} \sum_{|w|=k} \mu(I_w)^{q+\delta} \left(\sum_{j=1}^M p_j^q\right)^{-k} \\ &= h(s_k)^{\delta(\epsilon-1)} \left(\frac{\sum_{j=1}^M p_j^{q+\delta}}{\sum_{j=1}^M p_j^q}\right)^k =: \Phi^+(k) \end{aligned}$$

Consider the Taylor expansion of $g(t) := \log \sum_{j=1}^M p_j^t$, centred at q,

$$g(q+\delta) = \log \sum_{j=1}^M p_j^q + \left(\sum_{j=1}^M p_j^q\right)^{-1} \left(\sum_{j=1}^M p_j^q \log p_j\right) \delta + O(\delta^2).$$

Thus

$$\log \Phi^+(k) = \delta(\epsilon - 1)\log h(s_k) + k\left(\log \sum_{j=1}^{M} p_j^{q+\delta} - \log \sum_{j=1}^{M} p_j^q\right)$$
$$= k\delta\left((\epsilon-1)\frac{1}{k}\log h(s_k) + \left(\sum_{j=1}^{M} p_j^q\right)^{-1}\left(\sum_{j=1}^{M} p_j^q \log p_j\right) + O(\delta)\right).$$

When $\liminf_{k\to\infty} \frac{1}{k}\log h(s_k) = b_q = \left(\sum_{j=1}^{M} p_j^q\right)^{-1}\left(\sum_{j=1}^{M} p_j^q \log p_j\right)$, for any $\epsilon_0 > 0$ there exists a k_0 such that for all $k \geq k_0$ we have

$$\frac{1}{k}\log h(s_k) \geq b_q - \epsilon_0.$$

Thus

$$\begin{aligned}\log \Phi^+(k) &\leq k\delta\left((\epsilon-1)(b_q - \epsilon_0) + b_q + O(\delta)\right)\\ &= k\delta(\epsilon b_q + (1-\epsilon)\epsilon_0 + O(\delta)).\end{aligned}$$

Since $b_q < 0$, we can take ϵ_0 and δ small so that

$$\Phi^+(k) \leq \exp(k\delta\epsilon b_q/2)$$

for all $k \geq k_0$. It follows that $\{\nu(E_k)\}$ is summable. By the Borel-Cantelli lemma, $\nu(\bigcap_{k_1=1}^{\infty}\bigcup_{k\geq k_1} E_k) = 0$. In other words, for ν-almost $x \in \mathrm{supp}(\mu)$, there is a k_1 such that

$$\log \mu(I^k(x)) < (1-\epsilon)\log h(s_k)$$

for all $k \geq k_1$.

Analogously, when

$$\begin{aligned}E^k :=&\{x \in \mathrm{supp}(\mu) : \log \mu(I^k(x)) \leq (1+\epsilon)\log h(s_k)\}\\ =&\{x \in \mathrm{supp}(\mu) : \mu(I^k(x)) \leq h(s_k)^{(1+\epsilon)}\}\end{aligned}$$

we have

$$\begin{aligned}\nu(E^k) &\leq \int \mu(I^k(x))^{-\delta} h(s_k)^{\delta(1+\epsilon)} d\nu(x)\\ &= h(s_k)^{\delta(1+\epsilon)}\left(\frac{\sum_{j=1}^{M} p_j^{q-\delta}}{\sum_{j=1}^{M} p_j^q}\right)^k =: \Phi^-(k)\end{aligned}$$

and

$$\log \Phi^-(k) \le k\delta \left((1+\epsilon)\frac{1}{k}\log h(s_k) - \left(\sum_{j=1}^{M} p_j^q\right)^{-1} \left(\sum_{j=1}^{M} p_j^q \log p_j\right) + O(\delta) \right).$$

This time $\liminf_{k\to\infty} \frac{1}{k}\log h(s_k) = b_q$ implies

$$\begin{aligned}\log \Phi^-(k) &\le k\delta\left((1+\epsilon)(b_q+\epsilon_0) - b_q + O(\delta)\right)\\ &= k\delta\left(\epsilon b_q + (1+\epsilon)\epsilon_0 + O(\delta)\right)\end{aligned}$$

along a subsequence. Again we can take ϵ_0 and δ to be small so that

$$\Phi^-(k) \le \exp(k\delta\epsilon b_q/2)$$

and hence $\nu(E^k)$ is summable in that subsequence. Then for ν-almost $x \in \mathrm{supp}(\mu)$

$$\log \mu(I^k(x)) > (1+\epsilon)\log h(s_k)$$

for infinitely many k.

Therefore

$$1-\epsilon \le \liminf_{k\to\infty} \frac{\log \mu(I^k(x))}{\log h(s_k)} \le 1+\epsilon$$

for ν-almost $x \in \mathrm{supp}(\mu)$. Consequently $\nu(E_h) = 1$.

2. The proof is similar.

■

We also need a version of the mass distribution principle in terms of dimension functions.

Lemma 27. *Let ν be a measure and $h \in \mathbb{D}$ be a dimension function.*

1. *If $\liminf_{r\to 0^+} \frac{\log \nu(B_r(x))}{\log h(r)} \ge \theta > 0$ for all $x \in E$, then $H^{h^{\lambda\theta}}(E) \ge \nu(E)$ for all $\lambda < 1$.*
2. *If $\limsup_{r\to 0^+} \frac{\log \nu(B_r(x))}{\log h(r)} \le \theta < \infty$ for all $x \in E$, then $P^{h^{\lambda\theta}}(E) \le \nu(E)$ for all $\lambda > 1$.*

Proof. 1. Let $\lambda < 1$ and

$$F_k := \{x \in E : \nu(B_r(x)) \leq h(r)^{\lambda\theta} \text{ for all } r < \frac{1}{k}\}.$$

Then $F_k \subseteq F_{k+1}$ and $E = \bigcup F_k$ by the assumption.

Let $\{U_i\}$ be a δ-covering of E with $\delta < \frac{1}{k}$. It is also a δ-covering of F_k. If $U_i \cap F_k \neq \emptyset$, let $x_i \in U_i \cap F_k$ and $r_i = |U_i|$, so that $U_i \subseteq B_{r_i}(x)$. By definition of F_k,

$$\nu(U_i) \leq \nu(B_{r_i}(x_i)) \leq h(|U_i|)^{\lambda\theta}$$

and hence

$$\nu(F_k) \leq \sum_{i:U_i \cap F_k \neq \emptyset} \nu(U_i) \leq \sum_i h(|U_i|)^{\lambda\theta}.$$

This is true for any δ-covering of F_k, so

$$\nu(F_k) \leq H^{h^{\lambda\theta}}(E).$$

Letting $k \to \infty$,

$$\nu(E) \leq H^{h^{\lambda\theta}}(E).$$

2. Let $\lambda > 1$ and $\epsilon > 0$. There exists an open set $V \supset E$ such that $\nu(V) \leq \nu(E) + \epsilon$. Let

$$V_k := \{x \in V : h(r)^{\lambda\theta} \leq \nu(B_r(x)) \text{ and } B_r(x) \subseteq V \text{ for all } r < \frac{1}{k}\}.$$

Then $V_k \subseteq V_{k+1}$ and $E \subseteq \bigcup V_k$ by the assumption.

Fix k and let $\{B_j = B_{r_j}(x_j)\}$ be a δ-packing of V_k with $\delta < \frac{1}{k}$. Then

$$\sum_j h(r_j)^{\lambda\theta} \leq \sum_j \nu(B_j) = \sum_j \nu(V \cap B_j) = \nu(V \cap (\bigcup_j B_j)) \leq \nu(V).$$

This is true for any packing of V_k, so

$$P^{h^{\lambda\theta}}(V_k) \leq \nu(V) \leq \nu(E) + \epsilon.$$

Letting $k \to \infty$,

$$P^{h^{\lambda\theta}}(E) \leq P^{h^{\lambda\theta}}(\bigcup_k V_k) \leq \nu(E) + \epsilon$$

for any ϵ.

■

Recall the definition of b_q and θ_q on p.49.

Theorem 28. *Let $h \in \mathbb{D}$ and μ be a $\mathbf{p}$-Cantor measure on a balanced Cantor set C satisfying Assumptions 1 and 2.*

1. *If $\liminf_{k\to\infty} \frac{1}{k} \log h(s_k) = b_q$, then $H^{h^\lambda}(\mathcal{E}_h) \geq 1$ for all $\lambda < \theta_q$.*
2. *If $\limsup_{k\to\infty} \frac{1}{k} \log h(s_k) = b_q$, then $P^{h^\lambda}(\mathcal{E}^h) \leq 1$ for all $\lambda > \theta_q$.*
3. *If $\lim_{k\to\infty} \frac{1}{k} \log h(s_k) = b_q$, then $H^{h^\lambda}(\mathcal{E}(h)) \geq 1$ for all $\lambda < \theta_q$ and $P^{h^{\lambda'}}(\mathcal{E}(h)) \leq 1$ for all $\lambda' > \theta_q$.*

Proof. 1. Let $\nu = \nu_q$ be defined as in (4.4). Then

$$\frac{\log \nu(I^k(x))}{\log h(s_k)} = q\frac{\log \mu(I^k(x))}{\log h(s_k)} - \frac{\log \left(\sum_{j=1}^M p_j^q\right)^k}{\log h(s_k)}.$$

As suitable balls and Cantor intervals have comparable measures by Lemma 25, if $x \in \mathcal{E}_h$, then

$$\liminf_{k\to\infty} \frac{\log \nu(I^k(x))}{\log h(s_k)} \geq q - \frac{\log \left(\sum_{j=1}^M p_j^q\right)}{b_q} = \theta_q.$$

We note that $\theta_q > 0$. Indeed, if $q \geq 0$, then

$$\begin{aligned} qb_q &= q\left(\left(\sum_{j=1}^M p_j^q\right)^{-1}\left(\sum_{j=1}^M p_j^q \log p_j\right)\right) \\ &\leq q \max_j \log p_j < \log \sum_{j=1}^M p_j^q. \end{aligned}$$

If $q < 0$, then

$$qb_q \leq q \min_j \log p_j < \log \sum_{j=1}^M p_j^q.$$

Another application of Lemma 25 gives

$$\liminf_{\delta\to 0} \frac{\log \nu(B_\delta(x))}{\log h(\delta)} = \liminf_{k\to\infty} \frac{\log \nu(I^k(x))}{\log h(s_k)} \geq \theta_q > 0.$$

Hence

$$H^{h^\lambda}(\mathcal{E}_h) \geq \nu(\mathcal{E}_h) = 1$$

for all $\lambda < \theta_q$ by Lemma 26 and 27 .

2. The proof is similar to (1).

3. By Lemma 26, $\nu(\mathcal{E}_h \cap \mathcal{E}^h) = 1$. For any $x \in \mathcal{E}_h \cap \mathcal{E}^h$,

$$\lim_{k\to\infty} \frac{\log \nu(I^k(x))}{\log h(s_k)} = q - \frac{\log\left(\sum_{j=1}^M p_j^q\right)}{b_q} = \theta_q,$$

which is finite and positive. The conclusion then follows from Lemma 27.

■

Corollary 29. *Let* $G^- := \liminf_{k\to\infty} \frac{1}{k} \log s_k$ *and* $G^+ := \limsup_{k\to\infty} \frac{1}{k} \log s_k$.

1. *If* $\alpha = \frac{b_q}{G^-}$, *then*

$$\dim_H \mathcal{E}_\alpha \geq q\alpha - \frac{\log\left(\sum_{j=1}^M p_j^q\right)}{G^-}.$$

2. *If* $\alpha = \frac{b_q}{G^+}$, *then*

$$\dim_P \mathcal{E}^\alpha \leq q\alpha - \frac{\log\left(\sum_{j=1}^M p_j^q\right)}{G^+}.$$

Proof. 1. If $h(x) = t^\alpha$, then $\liminf_{k\to\infty} \frac{1}{k} \log h(s_k) = \alpha G^- = b_q$. By Theorem 28, $H^{\alpha\lambda}(\mathcal{E}_\alpha) \geq 1$ and $\dim_H \mathcal{E}_\alpha \geq \alpha\lambda$ for all $\lambda < \theta_q$. Hence

$$\dim_H \mathcal{E}_\alpha \geq \alpha\theta_q = q\alpha - \frac{\log\left(\sum_{j=1}^M p_j^q\right)}{G^-}.$$

2. The proof is similar.

■

Note that

$$\lim_{q\to\infty} b_q = \log p_{\max} \text{ and } \lim_{q\to-\infty} b_q = \log p_{\min}$$

where $p_{\max} = \max_{1\leq j\leq M} p_j$ and $p_{\min} = \min_{1\leq j\leq M} p_j$. The range of b_q is $(\log p_{\min}, \log p_{\max})$.

Corollary 30. *Suppose*

$$G := \lim_{k\to\infty} \frac{1}{k} \log s_k$$

exists. Let $\alpha \in (\frac{\log p_{\max}}{G}, \frac{\log p_{\min}}{G})$ *and* q *be such that* $\alpha = \frac{b_q}{G}$*. Then*

$$\dim_H \mathcal{E}(\alpha) = \dim_P \mathcal{E}(\alpha) = q\alpha - \frac{\log\left(\sum_{j=1}^M p_j^q\right)}{\lim_{k\to\infty} \frac{1}{k} \log s_k}.$$

Proof. When $h(t) = t^\alpha$, by part (3) of Theorem 28, $H^{\alpha\lambda}(\mathcal{E}(\alpha)) \geq 1$ for all $\lambda < \theta_q$ and $P^{\alpha\lambda'}(\mathcal{E}(\alpha)) \leq 1$ for all $\lambda' > \theta_q$. Then

$$\alpha\lambda \leq \dim_H \mathcal{E}(\alpha) \leq \dim_P \mathcal{E}(\alpha) \leq \alpha\lambda'$$

for all $\lambda < \theta_q$ and $\lambda' > \theta_q$. Therefore

$$\dim_H \mathcal{E}(\alpha) = \dim_P \mathcal{E}(\alpha) = \alpha\theta_q = q\alpha - \frac{\log\left(\sum_{j=1}^M p_j^q\right)}{\lim_{k\to\infty} \frac{1}{k} \log s_k}.$$

■

Remark. The Hausdorff and packing multifractal spectra $f(\alpha) := q\alpha - \frac{\log\left(\sum_{j=1}^M p_j^q\right)}{\lim_{k\to\infty} \frac{1}{k} \log s_k} = q\alpha + C(q)$ obtained above is indeed the Legendre transform of $-C(q)$, provided that $\lim_{k\to\infty} \frac{1}{k} \log s_k$ exists, since $q \mapsto q\alpha + C(q)$ is minimized at $\alpha = -C'(q) = \frac{b_q}{G}$.

Chapter 5

Exact measures of homogeneous Cantor sets

In this chapter we focus on the special case of homogeneous Cantor sets $C = C(\{n_k\}, \{r_k\})$, in which the interval and gap lengths at each level are all the same. The definition of the homogeneous Cantor sets can be found in Section 1.3.1.

Recall that for each level $k \geq 1$, $n_k \geq 2$ is the number of divisions and r_k is the ratio of dissection with

$$n_k r_k \leq 1.$$

The number of intervals at level k is $N_k = n_1 \cdots n_k$, and the length of each subinterval at level k is $s_k = r_1 \cdots r_k$. Let y_k be the length of a gap between two subintervals I^k at level k within the same parent interval I^{k-1}.

When $M := \sup_k n_k < \infty$, we already have bounds on the Hausdorff and pre-packing measures from Theorem 1 :

$$\frac{1}{M^2} \liminf_{k\to\infty} N_k h(s_k) \leq H^h(C) \leq \liminf_{k\to\infty} N_k h(s_k)$$

and

$$\frac{1}{M} \limsup_{k\to\infty} N_k h(s_k) \leq P_0^h(C) \leq M^2 \limsup_{k\to\infty} N_k h(s_k).$$

In this chapter we try to obtain more precise estimates of the measures.

5.1 Hausdorff measures

The Hausdorff dimension of a homogeneous Cantor set $C = C(\{n_k\}, \{r_k\})$ is given by [12]

$$\alpha = \liminf_{k\to\infty} \frac{\log N_k}{-\log s_k},$$

which can also be obtained from Corollary 8 in Chapter 2. It is known from [12] and [28] that we have the following bound on its Hausdorff measure:

$$\frac{1}{2} \liminf_{k\to\infty} N_k s_k^\alpha \le H^\alpha(C) \le \liminf_{k\to\infty} N_k s_k^\alpha. \tag{5.1}$$

Moreover, if the gap lengths are decreasing, i.e. $y_{k+1} \le y_k$ for all $k \ge 1$, then the exact Hausdorff measure is [27]

$$H^\alpha(C) = \liminf_{k\to\infty} N_k s_k^\alpha.$$

In this section we will improve (5.1) in the case where the gap lengths are not necessarily decreasing. One question we ask is whether the lower bound of the ratio $\frac{H^\alpha(C)}{\liminf_{k\to\infty} N_k s_k^\alpha}$ is always $\frac{1}{2}$. It turns out that the lower bound can be improved for different values of α.

From now on, let C be a fixed homogeneous Cantor set. Let

$$\mathcal{F}_k = \{I_\sigma : \sigma \in D_k\}$$

be the collection of Cantor intervals of level k defined as in Section 1.3.1. For $k \ge 1$ and $\sigma \in D_{k-1}$, let

$$\mathcal{G}_{k,\sigma} = \{I = \bigcup_{i=1}^{m} I_{\sigma w_i} : w_i \in \{0, \cdots, n_k - 1\}, 1 \le m \le n_k\}$$

be the collection of all possible unions of level k Cantor intervals in I_σ. Finally, let

$$\mathcal{G}_k := \bigcup_{\sigma\in D_{k-1}} \mathcal{G}_{k,\sigma},\ \mathcal{G} := \bigcup_{k=1}^{\infty} \mathcal{G}_k$$

and

$$H^\alpha_{\mathcal{G}}(C) := \liminf_{\delta\to 0} \left\{ \sum_i |U_i|^\alpha : C \subseteq \bigcup_i U_i, |U_i| \le \delta, U_i \in \mathcal{G} \right\}$$

for $\alpha \ge 0$. The special collection $\mathcal{G}$ of sets is used as coverings here. We will estimate $H^\alpha(C)$ through this intermediate quantity $H^\alpha_{\mathcal{G}}(C)$.

Let $B := \liminf_{k\to\infty} N_k s_k^\alpha$. For each $\sigma \in W_k$, let $I_\sigma = [a(\sigma), b(\sigma)]$. Let μ be the uniform Cantor measure on C so that $\mu(I_\sigma) = \frac{1}{N_k}$.

Lemma 31 ([28]). *Let $h(x) = x^\alpha$. Suppose $0 < B < \infty$ for some α. For any $\epsilon > 0$, there exists k_0 such that if $k \geq k_0$, $\sigma, \tau \in W_k$, $\sigma|_{k-1} = \tau|_{k-1}$ and $a(\sigma) < b(\tau)$, then*

$$\mu([a(\sigma), b(\tau)]) \leq \frac{1}{B-\epsilon} h(b(\tau) - a(\sigma)).$$

Remark. This lemma is also true for concave function $h \in \mathbb{D}$, with B modified accordingly.

Proof. Let $\sigma, \tau \in W_k$ be as in the statement. The assumption $\sigma|_{k-1} = \tau|_{k-1}$ and $a(\sigma) < b(\tau)$ implies that I_σ and I_τ lie in the same interval of level $k-1$ and are separated by, say, $i \geq 1$ intervals and $i-1$ gaps of level k. As $n_k s_k + (n_k - 1) y_k = s_{k-1}$,

$$b(\tau) - a(\sigma) = i s_k + (i-1) y_k = \frac{n_k - i}{n_k - 1} s_k + \frac{i-1}{n_k - 1} s_{k-1}.$$

By the concavity of h,

$$\begin{aligned} h(b(\tau) - a(\sigma)) &= h\left(\frac{n_k - i}{n_k - 1} s_k + \frac{i-1}{n_k - 1} s_{k-1}\right) \\ &\geq \frac{n_k - i}{n_k - 1} h(s_k) + \frac{i-1}{n_k - 1} h(s_{k-1}). \end{aligned}$$

For any $\epsilon > 0$, there exists k_0 such that for any $j \geq k_0 - 1$ we have

$$B - \epsilon \leq N_j h(s_j),$$

i.e.

$$\frac{B-\epsilon}{N_j} \leq h(s_j).$$

Therefore, for $k \geq k_0$,

$$\begin{aligned} h(b(\tau) - a(\sigma)) &\geq \frac{n_k - i}{n_k - 1} \frac{1}{N_k} (B - \epsilon) + \frac{i-1}{n_k - 1} \frac{1}{N_{k-1}} (B - \epsilon) \\ &= \frac{B-\epsilon}{N_k} \left(\frac{n_k - i}{n_k - 1} + \frac{(i-1) n_k}{n_k - 1}\right) \\ &= \frac{(B-\epsilon) i}{N_k} = (B - \epsilon) \mu([a(\sigma), b(\tau)]). \end{aligned}$$

■

Proposition 32 ([28])**.** *For $\alpha \geq 0$,*

$$H^{\alpha}_{\mathcal{G}}(C) = \liminf_{k\to\infty} N_k s_k^{\alpha}.$$

Proof. Since $\mathcal{F}_k = \{I_\sigma : \sigma \in W_k\}$ is a $\mathcal{G}$-covering of C, the upper bound

$$H^{\alpha}_{\mathcal{G}}(C) \leq \liminf_{k\to\infty} N_k s_k^{\alpha}$$

is easy to obtain. We now prove the lower bound.

If $B = 0$, then $H^{\alpha}_{\mathcal{G}}(C) = 0$ by the inequality above. If $B = \infty$, then $H^{\alpha}(C) = \infty$ by (5.1). In the definition of $H^{\alpha}(C)$, the infimum is taken over a larger collection of coverings than for $H^{\alpha}_{\mathcal{G}}(C)$, so $H^{\alpha}(C) \leq H^{\alpha}_{\mathcal{G}}(C)$ and hence $H^{\alpha}_{\mathcal{G}}(C) = \infty$. The conclusion holds in both cases. We can suppose $0 < B < \infty$.

For $\epsilon > 0$, there exists k_0 satisfying the previous lemma. Let $\delta \leq s_{k_0}$ and take any δ-covering $\{U_i\} \subseteq \mathcal{G}$ of C in $\mathcal{G}$.

Each U_i is in $G_{k(i)}$ for some $k(i) \geq k_0$, so

$$U_i = I_{\sigma^{(i)}} \cup \cdots \cup I_{\tau^{(i)}} \subseteq J$$

for some $J \in \mathcal{F}_{k(i)-1}$. We may assume U_i and $I_{\sigma^{(i)}}$ have the same left endpoints, and U_i and $I_{\tau^{(i)}}$ have the same right endpoints, so $|U_i| = b(\tau^{(i)}) - a(\sigma^{(i)})$. Since $\sigma^{(i)}|_{k(i)-1} = \tau^{(i)}_{k(i)-1}$,

$$\mu([a(\sigma^{(i)}), b(\tau^{(i)})]) \leq \frac{1}{B-\epsilon}(b(\tau^{(i)}) - a(\sigma^{(i)}))^{\alpha}$$

by the lemma. Taking summation over i, we get

$$1 = \mu(C) \leq \sum_i \mu(U_i) \leq \sum_i \mu([a(\sigma^{(i)}), b(\tau^{(i)})]) \leq \frac{1}{B-\epsilon}\sum_i |U_i|^{\alpha}.$$

Hence, $B - \epsilon \leq H^{\alpha}_{\mathcal{G}}(C)$ for any $\epsilon > 0$, and therefore

$$B \leq H^{\alpha}_{\mathcal{G}}(C).$$

■

Next, our aim is to compare $H^{\alpha}(C)$ with $H^{\alpha}_{\mathcal{G}}(C)$. We will need the following lemmas to replace a general open covering by a $\mathcal{G}$-covering.

Lemma 33. *Let U be a half-open interval. Suppose $U \cap C \neq \emptyset$ and U is in one of the two forms:*

- *$U = [a, b)$ and a is the left endpoint of some Cantor interval, or*
- *$U = (a, b]$ and b is the right endpoint of some Cantor interval.*

Fix an integer $N \geq 2$. Then there exists $G_1, \cdots, G_N \in \mathcal{G}$ such that

$$U \cap C \subseteq G_1 \cap \cdots \cap G_N$$

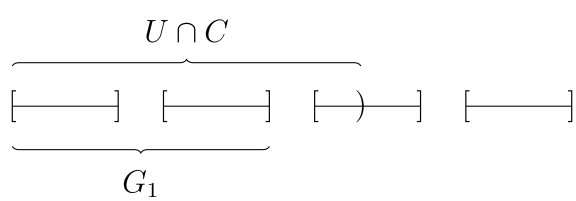
and

$$|G_1| + \cdots + |G_N| \leq \left(1 + \frac{1}{2^N - 1}\right)|U|.$$

Proof. Without loss of generality, suppose $U = [a, b)$ and a is the left endpoint of some Cantor interval.

- Since $U \cap C \neq \emptyset$, there is a level k interval, I_0, such that $a \in I_0 \subseteq U$, i.e. the left endpoints of U and I_0 coincide.

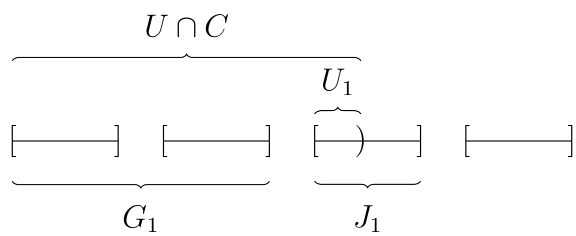
  Let $G_1 := \bigcup\{I_\sigma \in \mathcal{F}_k : I_\sigma \subseteq U\}$.



  If $U \cap C \subseteq G_1$, then we can stop and let $G_2, \cdots, G_N = \emptyset$ obtaining

$$|G_1| \leq |U| \leq (1 + \frac{1}{2^N - 1})|U|.$$

  Otherwise, $U \setminus G_1$ will have non-empty intersection with precisely one level k interval $J_1 \in \mathcal{F}_k$. Let $U_1 := U \cap J_1$ so $U \cap C \subseteq G_1 \cup U_1$ and the left endpoints of U_1 and J_1 coincide. Then there exists $k_1 > k$ such that U_1 contains an interval of level k_1 and the left endpoint $l(U_1)$ of U_1 is also the left endpoint of this interval.



- Now take
$$G_2 := \bigcup \{I_\sigma \in \mathcal{F}_{k_1} : I_\sigma \subseteq U_1\}.$$
If $U \cap C \subseteq G_1 \cup G_2$, then we stop and
$$|G_1| + |G_2| \leq |U| \leq (1 + \frac{1}{2^N - 1})|U|.$$
Otherwise, $U \setminus G_1 \cup G_2$ will have non-empty intersection with a level k_1 interval $J_2 \in \mathcal{F}_{k_1}$. Let $U_2 := U \cap J_2$. Then there exists $k_2 > k_1$ such that U_2 contains an interval of level k_2 and the left endpoint $l(U_2)$ of U_2 is also the left endpoint of J_2 and this interval. We then take $G_3 := \bigcup \{I_\sigma \in \mathcal{F}_{k_2} : I_\sigma \subseteq U_2\}$.

- In general, suppose we have obtained $G_1, \cdots G_n \in \mathcal{G}$ where $G_n \in \mathcal{G}_{k_{n-1}}$, $n < N$, and intervals $U_1, \cdots, U_{n-1}$ as above. If $U \cap C \subseteq G_1 \cup G_2 \cup \cdots \cup G_n$, then we can stop and
$$|G_1| + \cdots + |G_n| \leq |U| \leq (1 + \frac{1}{2^N - 1})|U|.$$
Otherwise, $U \setminus (G_1 \cup \cdots \cup G_n)$ will have non-empty intersection with a level k_{n-1} interval $J_n \in \mathcal{F}_{k_{n-1}}$. Let $U_n := U_{n-1} \cap J_n$. Then there exists $k_n > k_{n-1}$ such that U_n contains an interval of level k_n and $l(U_n)$ is the left endpoint of J_n and this interval.
If $n + 1 < N$, let $G_{n+1} := \bigcup \{I_\sigma \in \mathcal{F}_{k_n} : I_\sigma \subseteq U_n\}$ and continue this process.
If $n + 1 = N$, let
$$G_N := \bigcup \{I_\sigma \in \mathcal{F}_{k_{N-1}} : I_\sigma \cap U_{N-1} \neq \emptyset\}.$$

In this case
$$U \cap C \subseteq G_1 \cup \cdots \cup G_N$$
and
$$G_1 \cup \cdots \cup G_{N-1} \cup U_{N-1} \subseteq U.$$

At most one interval of level k_{N-1} intersects U_{N-1} without being fully contained in it, so

$$|G_N| \leq |U_{N-1}| + s_{k_{N-1}}.$$

Since U_{N-1} contains at least one interval of level k_{N-1},

$$s_{k_{N-1}} \leq |U_{N-1}|.$$

On the other hand, for each $1 \leq j \leq N-1$, G_j contains at least one interval of level k_{j-1}. Thus the number of interval at level k_{N-1} contained in G_j is at least $N_{k_{N-1}}/N_{k_{j-1}} \geq 2^{N-j}$ and hence

$$2^{N-j} s_{k_{N-1}} \leq |G_j|.$$

Therefore,

$$(2^N - 1)s_{k_{N-1}} = (2^{N-1} + 2^{N-2} + \cdots + 2 + 1)s_{k_{N-1}} \leq |G_1| + \cdots + |G_{N-1}| + |U_{N-1}|$$

and

$$\begin{aligned}
|G_1| + \cdots + |G_N| &\leq |G_1| + \cdots + |G_{N-1}| + |U_{N-1}| + s_{k_{N-1}} \\
&\leq (1 + \frac{1}{2^N - 1})(|G_1| + \cdots + |G_{N-1}| + |U_{N-1}|) \\
&\leq (1 + \frac{1}{2^N - 1})|U|
\end{aligned}$$

as the sets G_j, U_{N-1} are disjoint. ■

Corollary 34. *Let $N \geq 2$. If U is an open interval and $U \cap C \neq \emptyset$, then there exist $G_1, \cdots, G_{2N} \in \mathcal{G}$ such that*

$$U \cap C \subseteq G_1 \cap \cdots \cap G_{2N}$$

and

$$(1 - \frac{1}{2^N}) \sum_{j=1}^{2N} |G_j| \leq |U|$$

or, equivalently,

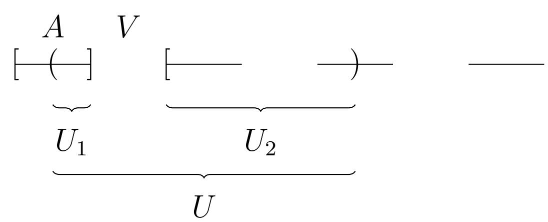
$$\sum_{j=1}^{2N} |G_j| \leq (1 + \frac{1}{2^N - 1})|U|.$$

Proof. For any open interval U with $U \cap C \neq \emptyset$, there is a smallest k such that U contains at least one interval of level k but does not contain any interval of level $k-1$. Let $G \in \mathcal{F}_k$ be an interval such that $G \subseteq U$.

- If U intersects only one level k interval, then it must be G and $U \cap C \subseteq G$. Letting $G_1 = G$ and $G_2, \cdots, G_{2N} = \emptyset$ we have

$$(1 - \frac{1}{2^N}) \sum_{j=1}^{2N} |G_j| \leq |G| \leq |U|.$$

- If U intersects more than one level k interval, let $A \in \mathcal{F}_k$ be the leftmost interval such that $U \cap A \neq \emptyset$ and V be the gap to the right of A. Let $U_1 := U \cap A$ and $U_2 = U \setminus (A \cup V)$.



Then U_1, U_2 satisfy the lemma above and we can find $G_1, \cdots G_{2N} \in \mathcal{G}$ so that

$$U_1 \cap C \subseteq G_1 \cap \cdots \cap G_N, U_2 \cap C \subseteq G_{N+1} \cap \cdots \cap G_{2N},$$

$$|G_1| + \cdots + |G_N| \leq (1 + \frac{1}{2^N - 1})|U_1|$$

and

$$|G_{N+1}| + \cdots + |G_{2N}| \leq (1 + \frac{1}{2^N - 1})|U_2|.$$

Hence

$$\sum_{j=1}^{2N} |G_j| \leq (1 + \frac{1}{2^N - 1})(|U_1| + |U_2|) \leq (1 + \frac{1}{2^N - 1})|U|$$

or, equivalently,

$$(1 - \frac{1}{2^N}) \sum_{j=1}^{2N} |G_j| \leq |U|$$

and

$$U \cap C = (U_1 \cap C) \cup (U_2 \cap C) \subseteq G_1 \cap \cdots \cap G_{2N}.$$

∎

Theorem 35. *Let C be a homogeneous Cantor set and $0 \leq \alpha \leq 1$. Then*

$$(1-\frac{1}{2^N})^\alpha \frac{(2N)^\alpha}{2N} H^\alpha_{\mathcal{G}}(C) \leq H^\alpha(C) \leq H^\alpha_{\mathcal{G}}(C)$$

for all $N \geq 1$.

Proof. In the definition of $H^\alpha(C)$, the infimum is taken over a larger collection of coverings than for $H^\alpha_{\mathcal{G}}(C)$, so the upper bound $H^\alpha(C) \leq H^\alpha_{\mathcal{G}}(C)$ is immediate. Let us look at the lower bound.

We can apply Corollary 34 and the concavity of x^α to see that for any δ-covering $\{U^i\}$ of C, we can obtain a $\mathcal{G}$-$(1+\frac{1}{2^N-1})\delta$-covering

$$\bigcup_i \{G^i_1, \cdots, G^i_{2N}\} \subseteq \mathcal{G}$$

of C such that

$$(1-\frac{1}{2^N})^\alpha \frac{(2N)^\alpha}{2N} \sum_{j=1}^{2N} |G^i_j|^\alpha \leq \left((1-\frac{1}{2^N}) \sum_{j=1}^{2N} |G^i_j|\right)^\alpha \leq |U^i|^\alpha$$

for all i. Hence

$$(1-\frac{1}{2^N})^\alpha \frac{(2N)^\alpha}{2N} \sum_i \sum_{j=1}^{2N} |G^i_j|^\alpha \leq \sum_i |U^i|^\alpha$$

and therefore

$$(1-\frac{1}{2^N})^\alpha \frac{(2N)^\alpha}{2N} H^\alpha_{\mathcal{G}}(C) \leq H^\alpha(C).$$

■

Corollary 36. *Let $0 \leq \alpha \leq 1$. Then*

$$\left(\sup_{N\geq 1}(1-\frac{1}{2^N})^\alpha \frac{(2N)^\alpha}{2N}\right) \liminf_{k\to\infty} N_k s_k^\alpha \leq H^\alpha(C) \leq \liminf_{k\to\infty} N_k s_k^\alpha.$$

Proof. This follows from (5.1), Proposition 32 and Theorem 35. ■

Remark. When $N=1$, $(1-\frac{1}{2^N})^\alpha \frac{(2N)^\alpha}{2N} = \frac{1}{2}$. The inequality (5.1) is improved.

Corollary 37. *For all homogeneous Cantor sets C,*

$$H^{\alpha}(C) \geq \frac{3^{\alpha}}{4} \liminf_{k\to\infty} N_k s_k^{\alpha}.$$

Proof. When $N = 2$, $(1 - \frac{1}{2^N})^{\alpha} \frac{(2N)^{\alpha}}{2N} = \frac{3^{\alpha}}{4}$. ■

Remark. If $\alpha > \frac{\log 2}{\log 3}$, then $\frac{3^{\alpha}}{4} > \frac{1}{2}$.

Corollary 38. *If $\alpha = 1$, then $H^{\alpha}(C) = \liminf_{k\to\infty} N_k s_k^{\alpha}$.*

Proof. When $\alpha = 1$, $\sup_{N\geq 1}(1 - \frac{1}{2^N})^{\alpha} \frac{(2N)^{\alpha}}{2N} = 1$. ■

5.2 Packing measures and lower densities

It follows from Corollary 8 in Chapter 2 that the packing dimension of a homogeneous Cantor set $C = C(\{n_k\}, \{r_k\})$ is

$$\alpha = \limsup_{k\to\infty} \frac{\log N_k}{-\log s_k}.$$

The exact packing measure of a central Cantor set is calculated to be

$$P^{\alpha}(C) = 2^{\alpha} \limsup_{k\to\infty} 2^k (s_k + y_k)^{\alpha}$$

under some separation condition in [14]. Here we extend the results to the case of homogeneous Cantor sets.

Throughout this subsection, C will be a homogeneous Cantor set with $M := \sup_k n_k < \infty$. In addition, we will require the following separation condition: There exists $0 < \beta < 1$ and some $K \in \mathbb{N}$ such that $n_k r_k \leq \beta$ for all $k \geq K$.

Recall that y_k is the length of a gap between two subintervals at level k within the same parent interval.

Lemma 39. *Any homogeneous Cantor set satisfying the separation condition has the property that there exists $L \in \mathbb{N}$ such that*

$$s_{k+l} + y_{k+l} \leq y_k$$

for all $l \geq L$.

Proof. First, note that

$$y_k = \frac{s_{k-1} - n_k s_k}{n_k - 1} = r_1 \cdots r_{k-1} \left(\frac{1 - n_k r_k}{n_k - 1} \right)$$

and

$$s_{k+l} + y_{k+l} = r_1 \cdots r_{k+l-1} \left(\frac{1 - r_{k+l}}{n_{k+l} - 1} \right).$$

The inequality $s_{k+l} + y_{k+l} \leq y_k$ is equivalent to

$$r_k \cdots r_{k+l-1} \left(\frac{1 - r_{k+l}}{n_{k+l} - 1} \right) \leq \left(\frac{1 - n_k r_k}{n_k - 1} \right).$$

Since $n_k \geq 2$ and the separation condition gives $n_k r_k \leq \beta$,

$$r_k \cdots r_{k+l-1} \left(\frac{1 - r_{k+l}}{n_{k+l} - 1} \right) \leq \frac{1}{n_k \cdots n_{k+l-1}} \leq \frac{1}{n_k 2^{l-1}}$$

and

$$\frac{1 - n_k r_k}{n_k - 1} \geq \frac{1 - \beta}{n_k}.$$

So it suffices to find l such that $2^{1-l} \leq 1 - \beta$ and hence we may take

$$L \geq \log_2 (\frac{1}{1 - \beta}) + 1.$$

■

Lemma 40 ([14])**.** *Let $a_j, b_j > 0$ for $j \in \{1, \cdots, L\}$ and $0 < \alpha < 1$. Then*

$$\min \left\{ \frac{a_j}{b_j^\alpha} : 1 \leq j \leq L \right\} \leq \frac{a_1 + \cdots + a_L}{(b_1 + \cdots + b_L)^\alpha}.$$

Proof. Let $m = \min_j \left\{ \frac{a_j}{b_j^\alpha} \right\}$. Then $m b_j^\alpha \leq a_j$ for $1 \leq j \leq L$ and

$$m(b_1 + \cdots + b_L)^\alpha \leq m(b_1^\alpha + \cdots + b_L^\alpha) \leq a_1 + \cdots + a_L$$

by concavity of the function $h(x) := x^\alpha$. ■

To obtain the packing measure, we will calculate the lower density of the uniform Cantor measure μ on C. The lower density of a measure ν is defined as

$$\Theta^\alpha(\nu, x) := \liminf_{r\to 0} \frac{\nu(B(x,r))}{(2r)^\alpha}.$$

When $\nu = \mu$ is the uniform Cantor measure, we just write $\Theta^\alpha(x) = \Theta^\alpha(\mu, x)$. We can obtain the packing measure through the following proposition, obtained by [14] for central Cantor sets. The following proof is almost the same as that for central Cantor sets.

Proposition 41. *Let $C = C(\{n_k\}, \{r_k\})$ be a homogeneous Cantor set such that $0 < P^\alpha(C) < \infty$. Then*

$$\Theta^\alpha(x) = (P^\alpha(C))^{-1}$$

for μ a.e. $x \in C$.

Proof. For each $k \geq 1$ and $\sigma \in D_k$, the set $I_\sigma \cap C$ is a translation of $I_{0^k} \cap C$. By translation invariance of the packing measure, $P^\alpha(C) = N^k P^\alpha(C \cap I_\sigma)$. If we define $\nu = (P^\alpha(C))^{-1} P^\alpha|_C$, then ν and the uniform Cantor measure μ coincide on each I_σ with measure $\frac{1}{N_k}$. By regularity, the two measures are identical. Since $\Theta^\alpha(P^\alpha|_C, x) = 1$ for P^α a.e. $x \in C$ by [23, Theorem 6.10], $\Theta^\alpha(\mu, x) = (P^\alpha(C))^{-1}$ for μ a.e. $x \in C$. ■

We first consider the lower bound of the lower density.

Theorem 42. *Let $C = C(\{n_k\}, \{r_k\})$ be a homogeneous Cantor set such that $P^\alpha(C) < \infty$. Then*

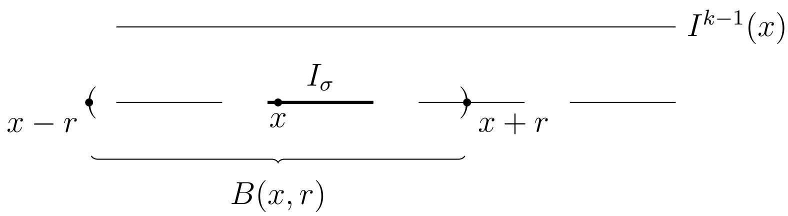
$$\Theta^\alpha(x) \geq \left(2^\alpha \limsup_{k\to\infty} N_k(s_k + y_k)^\alpha\right)^{-1}$$

for μ a.e. $x \in C$.

Proof. Let $B_\alpha = \limsup_{k\to\infty} N_k(s_k + y_k)^\alpha$. Given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$N_k(s_k + y_k)^\alpha < B_\alpha + \varepsilon.$$

Fix $x \in C \setminus \{0, 1\}$. For each $r > 0$, choose k such that $I^k(x) \subseteq B(x, r)$ but $I^{k-1}(x) \nsubseteq B(x, r)$. Let $r > 0$ be small enough so that $x + r < 1$ and $k \geq k_0$. Assume $I^k(x) = I_\sigma$ for $|\sigma| = k$.



We remark that either $x+r \in I^{k-1}(x)$ or $x-r \in I^{k-1}(x)$, for otherwise the ball $B(x,r)$ will contain $I^{k-1}(x)$.

First, suppose $x+r \in I^{k-1}(x)$. Note that $x+r \notin I_\sigma$ since $I_\sigma \subseteq B(x,r)$. As before, we use the notation $I_\sigma = [a(\sigma), b(\sigma)]$. We start our estimation by observing that $\mu(B(x,r)) \geq \mu([a(\sigma), x+r))$.

Subcase 1. Suppose either $x+r \notin C$ or $x+r$ is the endpoint of a basic interval. If $a(\sigma)+r \leq x+r \leq a(\sigma)+s_k+y_k$, then $r \leq s_k+y_k$. Since $I_\sigma \subseteq B(x,r)$, $\mu(B(x,r)) \geq \mu(I_\sigma) = \frac{1}{N_k}$. Consequently

$$\frac{\mu(B(x,r))}{(2r)^\alpha} \geq \frac{1}{2^\alpha (s_k+y_k)^\alpha} \frac{1}{N_k} \geq \frac{1}{2^\alpha (B_\alpha + \varepsilon)}.$$

If $a(\sigma)+s_k+y_k < x+r < 1$, then $x+r$ must lie within the closure of a gap of the following form: [1]

$$x+r \in [b(\sigma|_{k-1}\tau_k\tau_{k+1}\cdots\tau_{k+L}), a(\sigma|_{k-1}\tau_k\tau_{k+1}\cdots(\tau_{k+L}+1))]$$

for some word $\tau = \tau_k\tau_{k+1}\cdots\tau_{k+L}$, where $L \geq 0$, $\tau_{k+i} \in \{0, \cdots, n_{k+i}-1\}$, $\tau_k \geq \sigma_k + 1$ (because $I_\sigma \subseteq B(x,r)$) and $\tau_{k+L}+1 \in \{1, \cdots, n_{k+L}-1\}$.

Let us relabel the indices as follows: let $k_0 = k$ and

$$k_j = \min\{i : k_{j-1} < i < k+L \text{ and } \tau_i \neq 0\}.$$

This contruction will stop in finitely many steps, and if it stops at $j = L_0 - 1$, then let $k_{L_0} = k+L$. (Here $L_0 \leq L$.) Let

$$A_0 = \tau_k - \sigma_k,\ A_j = \tau_{k_j},\ A_{L_0} = \tau_{k+L}+1$$

where $1 \leq j \leq L_0 - 1$. Thus, A_j is the number of intervals at level k_j contained in $[a(\sigma), x+r)$ and all $A_j > 0$. This gives the following estimate:

$$\mu(B(x,r)) \geq \sum_{j=0}^{L_0} \frac{A_j}{n_1 \cdots n_{k_j}}.$$

[1] It may be helpful for the reader to review the example given following the proof.

We also need an upper bound of r. For this we note that

$$\begin{aligned}a(\sigma)+r \le x+r &\le a(\sigma|_{k-1}\tau_k\tau_{k+1}\cdots(\tau_{k+L}+1))\\ &\le a(\sigma)+\sum_{j=0}^{L_0} A_j(s_{k_j}+y_{k_j}),\end{aligned}$$

which gives

$$r \le \sum_{j=0}^{L_0} A_j(s_{k_j}+y_{k_j}).$$

Therefore,

$$\begin{aligned}\frac{\mu(B(x,r))}{(2r)^\alpha} &\ge \frac{1}{2^\alpha}\left(\sum_{j=0}^{L_0} A_j(s_{k_j}+y_{k_j})\right)^{-\alpha}\sum_{j=0}^{L_0}\frac{A_j}{n_1\cdots n_{k_j}}\\ &\ge \frac{1}{2^\alpha}\min_j\left\{\frac{A_j(n_1\cdots n_{k_j})^{-1}}{A_j^\alpha(s_{k_j}+y_{k_j})^\alpha}\right\} && \text{(by Lemma 40)}\\ &\ge \frac{1}{2^\alpha}\min_j\left\{\frac{1}{N_{k_j}(s_{k_j}+y_{k_j})^\alpha}\right\} && (\text{ as } A_j^{1-\alpha}\ge 1)\\ &\ge \frac{1}{2^\alpha(B_\alpha+\varepsilon)}.\end{aligned}$$

Subcase 2. If $x+r\in C$, but is not an endpoint of a basic interval, then there is an infinite word $\omega\in W^\infty$ such that $x+r\in I_{\omega|_i}$ for all $i\ge 1$. Again we let

$$\begin{aligned}k_0 &= k,\\ k_j &= \min\{i : k_{j-1}<i \text{ and } \omega_i\ne 0\}\end{aligned}$$

and

$$A_0=\omega_k-\sigma_k,\ A_j=\omega_{k_j}$$

for $j\ge 1$. By similar reasoning, for any L we have

$$\mu(B(x,r))\ge \mu([a(\sigma),x+r))\ge \sum_{j=0}^{L}\frac{A_j}{n_1\cdots n_{k_j}}.$$

As $x+r\in I_{\omega|_{k_L}}$ for all L,

$$a(\sigma)+r\le x+r\le b(\omega|_{k_L}) = a(\sigma)+\sum_{j=0}^{L} A_j(s_{k_j}+y_{k_j})+s_{k_L},$$

and this gives

$$r \le \sum_{j=0}^{L} A_j(s_{k_j} + y_{k_j}) + s_{k_L}$$

for all L. Since $\lim_{L\to\infty} s_{k_L} = 0$, given any $\varepsilon > 0$, for sufficiently large L we have

$$\begin{aligned}\frac{\mu(B(x,r))}{(2r)^\alpha} &\ge \frac{1}{2^\alpha}\left(\sum_{j=0}^{L} A_j(s_{k_j} + y_{k_j})\right)^{-\alpha} \sum_{j=0}^{L} \frac{A_j}{n_1 \cdots n_{k_j}} - \varepsilon \\ &\ge \frac{1}{2^\alpha(B_\alpha + \varepsilon)} - \varepsilon\end{aligned}$$

by the same argument as in the previous subcase.

The other case, $x - r \in I^{k-1}(x)$, can be done similarly. Therefore, we have shown that

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{(2r)^\alpha} \ge \frac{1}{2^\alpha B_\alpha}.$$

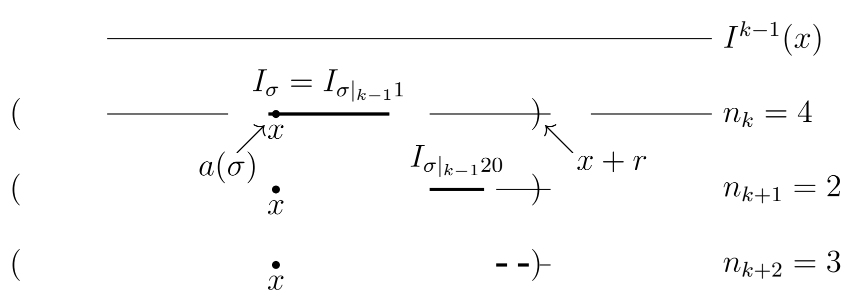
■

Example. Let us illustrate the proof above with an example, in the case where $x + r \notin C$ or $x + r$ is the endpoint of a basic interval.

Let the number of divisions at level $k, k+1$ and $k+2$ be $n_k = 4$, $n_{k+1} = 2$ and $n_{k+2} = 3$. Suppose $x \in I_\sigma = [a(\sigma), b(\sigma)]$ where $\sigma_k = 1$ and

$$x + r \in [b(\sigma|_{k-1}211), a(\sigma|_{k-1}212)],$$

i.e. $\tau_k\tau_{k+1}\tau_{k+2} = 211$ and $L = 2$.



The numbers of intervals at level $k_0 = k$, $k_1 = k+1$ and $k_2 = k+2$ contained in $[a(\sigma), x+r) \subseteq B(x,r)$ are

$$\begin{aligned}
A_0 &= \tau_k - \sigma_k = 2 - 1 = 1,\\
A_1 &= \tau_{k+1} = 1,\\
A_2 &= \tau_{k+2} + 1 = 1 + 1 = 2.
\end{aligned}$$

(See the diagram.)

From

$$\mu(B(x,r)) \geq \frac{1}{n_1 \cdots n_k} + \frac{1}{n_1 \cdots n_{k+1}} + \frac{2}{n_1 \cdots n_{k+2}}$$

and

$$r \leq (s_k + y_k) + (s_{k+1} + y_{k+1}) + 2(s_{k+2} + y_{k+2})$$

we can then estimate $\mu(B(x,r))(2r)^{-\alpha}$ as in the proof.

We consider the upper bound next.

Theorem 43. *Let C be a homogeneous Cantor set such that $P^\alpha(C) < \infty$ and the separation condition holds. Then*

$$\Theta^\alpha(x) \leq \left(2^\alpha \limsup_{k\to\infty} N_k(s_k + y_k)^\alpha\right)^{-1}$$

for μ a.e. $x \in C$.

Proof. Fix L as in Lemma 39. Take a subsequence (k_j) such that

$$\lim_{j\to\infty} N_{k_j}(s_{k_j} + y_{k_j})^\alpha = \limsup_{k\to\infty} N_k(s_k + y_k)^\alpha,$$

$k_j > L$ and $k_{j+1} - k_j > j$ for all $j \in \mathbb{N}$. For each $j \geq 1$, choose i such that $M^i \leq j < M^{i+1}$.

Define a sequence of sets

$$A_j := \{x \in C : \sigma_{k_j - L}(x) = 1, \sigma_{k_j - L + 1}(x) = \cdots = \sigma_{k_j - L + i}(x) = 0\}.$$

They will be used to construct a subset of μ-measure 1 satisfying the bound in the statement. Note that

$$\mu(A_j) = \frac{1}{n_{k_j - L} \cdots n_{k_j - L + i}} \geq \frac{1}{M^{i+1}}$$

because A_j consists of the union of one Cantor interval of level $k_j - L + i$ in each Cantor interval of level $k_j - L - 1$, and

$$\sum_{j=M^i}^{M^{i+1}-1} \mu(A_j) \geq M^i(M-1)\frac{1}{M^{i+1}} = 1 - \frac{1}{M}.$$

This implies

$$\sum_{j=1}^{\infty} \mu(A_j) = \infty.$$

Note that the levels defining A_j's are distinct for different j:

$$k_{j+1} - (k_j - L + i) > j + L - i \geq M^i + L - i > 0,$$

so the events A_j's are independent with respect to the probability measure μ. Let

$$A = \bigcap_{l \geq 1} \bigcup_{j \geq l} A_j.$$

By the Borel-Cantelli Lemma, $\mu(A) = 1$.

We want to estimate the lower density at points $x \in A$. If $x \in A$, then $x \in A_j$ for infinitely many j. For such j, let

$$r_j := s_{k_j} + y_{k_j} - s_{k_j - L + i}$$

where i is the integer given by $M^i \leq j < M^{i+1}$.

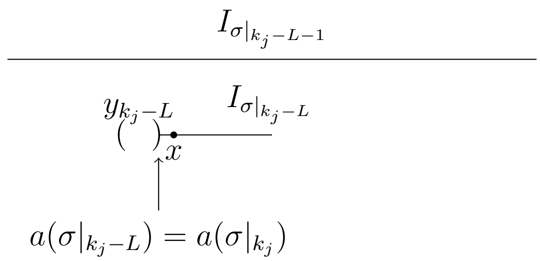
Let $\sigma = \sigma(x) \in W^\infty$ be an infinite word such that $x \in I_{\sigma|_k}$ for all k. Claim: $B(x, r_j) \cap C \subseteq (I_{\sigma|_{k_j}} \cap C) \cup \{a(\sigma|_{k_j - 1}1)\}$ for large j.

When j is large enough, then $i \geq L$, hence $k_j - L + i \geq k_j$ and $a(\sigma|_{k_j - L + i}) = a(\sigma|_{k_j})$. On one hand,

$$\begin{aligned}
x + r_j &= x - s_{k_j - L + i} + s_{k_j} + y_{k_j} \\
&\leq a(\sigma|_{k_j - L + i}) + s_{k_j} + y_{k_j} \\
&= a(\sigma|_{k_j}) + s_{k_j} + y_{k_j} \\
&= a(\sigma|_{k_j - 1}1).
\end{aligned}$$

On the other hand, by Lemma 39,

$$\begin{aligned}
x - r_j &= x + s_{k_j - L + i} - s_{k_j} - y_{k_j} \\
&> a(\sigma|_{k_j - L}) - y_{k_j - L}.
\end{aligned}$$



Thus $(x - r_j, x]$ is contained in the union of $I_{\sigma|_{k_j}}$ and the gap of level $k_j - L$ immediately to the left of $I_{\sigma|_{k_j}}$. Therefore, $B(x, r_j) \cap C \subseteq (I_{\sigma|_{k_j}} \cap C) \cup \{a(\sigma|_{k_j-1}1)\}$.

It follows that

$$\mu(B(x, r_j)) \leq \frac{1}{n_1 \cdots n_{k_j}}$$

and

$$\begin{aligned}\frac{\mu(B(x, r_j))}{(2r_j)^\alpha} &\leq \frac{1}{n_1 \cdots n_{k_j}} \frac{1}{2^\alpha (s_{k_j} + y_{k_j} - s_{k_j-L+i})^\alpha} \\ &= \frac{1}{2^\alpha n_1 \cdots n_{k_j} (s_{k_j} + y_{k_j})^\alpha} \frac{1}{(1 - \frac{s_{k_j-L+i}}{s_{k_j}+y_{k_j}})^\alpha}.\end{aligned}$$

Since

$$\begin{aligned}0 \leq \frac{s_{k_j-L+i}}{s_{k_j} + y_{k_j}} &\leq \frac{1}{n_{k_j-L+i} \cdots n_{k_j+1}} \frac{s_{k_j}}{s_{k_j} + y_{k_j}} \\ &\leq \frac{1}{M^{i-L}} \frac{s_{k_j}}{s_{k_j} + y_{k_j}} \leq \frac{1}{M^{i-L}} \to 0\end{aligned}$$

as $j \to \infty$, we deduce that for any $x \in A$, and hence for μ a.e. $x \in C$,

$$\liminf_{j\to\infty} \frac{\mu(B(x, r_j))}{(2r_j)^\alpha} \leq \frac{1}{2^\alpha \lim_{j\to\infty} N_{k_j} (s_{k_j} + y_{k_j})^\alpha}.$$

■

Corollary 44. *If C is a homogeneous Cantor set satisfying the separation condition, then*

$$P^\alpha(C) = 2^\alpha \limsup_{k\to\infty} N_k (s_k + y_k)^\alpha.$$

Proof. If $P^\alpha(C) = 0$ or ∞, then by Theorem 1 and Corollary 4, $\limsup_{k\to\infty} N_k s_k^\alpha = 0$ or ∞ respectively. Since

$$N_k s_k^\alpha \leq N_k(s_k + y_k)^\alpha \leq M N_{k-1} s_{k-1}^\alpha,$$

the theorem holds in these two cases.

If $0 < P^\alpha(C) < \infty$, from Theorem 42 and 43 we know that the lower density is

$$\Theta^\alpha(x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{(2r)^\alpha} = \left(2^\alpha \limsup_{k\to\infty} N_k(s_k + y_k)^\alpha\right)^{-1}$$

for μ a.e. $x \in C$. Therefore, by Proposition 41,

$$P^\alpha(C) = 2^\alpha \limsup_{k\to\infty} N_k(s_k + y_k)^\alpha.$$

■

Chapter 6

L^p-improving property

A measure μ on $[0,1]$ is said to be L^p-improving ([29, 8]) if and only if there exist $q > p$ and some constant $B > 0$ such that

$$\|\mu * f\|_q \leq B\|f\|_p$$

for all $f \in L^p([0,1])$, where by $[0,1]$ we mean the group under addition mod 1. An interpolation argument shows that if there exists one such pair, $q_0 > p_0$, then for all $1 < p < \infty$ there exists $q > p$ such that μ acts as a bounded operator from L^p to L^q. It was proved [25] that the uniform Cantor measure on the middle third Cantor set is L^p-improving. The result was later extended [8], with a different technique, to the uniform Cantor measures on the central Cantor sets with ratios of dissection bounded away from 0. In this final chapter, we will prove that under certain assumptions, a **p**-Cantor measure on a homogeneous Cantor set is also L^p-improving. Our method is based on [8].

Let $S_\mu f := \mu * f$ be the convolution operator. Since $\widehat{S_\mu f} = \hat{\mu}\hat{f}$ after taking Fourier transform, we can study the convolution operator as a multiplier operator. In general, if $m : \mathbb{Z} \to \mathbb{C}$, the multiplier operator is defined by $\widehat{T_m f} = m\hat{f}$. We also write $mf = T_m f$. Sometimes it is convenient to view m as a function defined on $\mathbb{R}$. If $1 \leq p \leq q \leq \infty$, the multiplier norm of m is defined as

$$\|m\|_{p,q} = \sup_{f \neq 0} \|mf\|_q / \|f\|_p.$$

We need some lemmas to estimate the norm.

Lemma 45 ([8])**.** *Let $m : \mathbb{Z} \to \mathbb{C}$. Let $\{I_j : 1 \leq j \leq L\}$ be disjoint intervals and $m_j = m\chi_{I_j}$. Suppose $m = \sum m_j$ and $2 \leq q < \infty$. Then there exists $A_1 = A_1(q, L)$ such*

that
$$\|m\|_{2,q} \leq A_1 \max \|m_j\|_{2,q}.$$
If L is fixed, then $A_1 \to 1$ as $q \to 2$.

Proof. Let $2 \leq q < \infty$. Fix a number t such that $q < t$. By the Cauchy-Schwartz inequality,

$$|mf|^t = |\sum_{j=1}^{L} m_j f|^t \leq \left((\sum_{j=1}^{L} |m_j f|^2)^{1/2} L^{1/2} \right)^t.$$

Integrating gives
$$\|mf\|_t \leq L^{1/2} \| \left(\sum |m_j f|^2 \right)^{1/2} \|_t.$$
By Parseval's identity, as the functions $\{m_j f\}$ are orthogonal,

$$\|mf\|_2 = \| \left(\sum |m_j f|^2 \right)^{1/2} \|_2.$$

By the vector-valued version of the Riesz-Thorin interpolation theorem [1, Theorem 4.1.2, 5.1.1, 5.1.2],
$$\|mf\|_q \leq L^{\theta/2} \| \left(\sum |m_j f|^2 \right)^{1/2} \|_q$$
where θ is given by
$$\frac{1}{q} = \frac{\theta}{t} + \frac{1-\theta}{2}.$$

Let $f_j = \chi_{I_j} f$. Since $2 \leq q$, by Minkowski's inequality,

$$\begin{aligned} \|mf\|_q &\leq L^{\theta/2} \left(\sum \|m_j f_j\|_q^2 \right)^{1/2} \\ &\leq L^{\theta/2} \max_j \|m_j\|_{2,q} \left(\sum \|f_j\|_2^2 \right)^{1/2} \\ &= L^{\theta/2} \max_j \|m_j\|_{2,q} \|f\|_2. \end{aligned}$$

Therefore
$$\|m\|_{2,q} \leq L^{\theta/2} \max_j \|m_j\|_{2,q}.$$
As $q \to 2$, $\theta \to 0$, thus $A_1 := L^{\theta/2} \to 1$. ■

Definition 8. Let $\sigma > 1$. A strictly increasing sequence $\{n_j\}_j$ of integers is said to be σ-lacunary if

$$(n_{j+1} - n_j) \geq \sigma(n_j - n_{j-1})$$

for all $j \geq 1$.

Given a σ-lacunary sequence define the multipliers Δ_j by

$$(\widehat{\Delta_j f})(n) = \begin{cases} \hat{f}(n) & \text{if } n_j \leq n < n_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 46 ([8]). *Suppose $2 \leq q < \infty$ and $\sigma > 1$. Then there exists $A_2 = A_2(q, \sigma)$ such that for any σ-lacunary sequence $\{n_j\}_j$ and any $m : \mathbb{Z} \to \mathbb{C}$ with $m(n) = 0$ for all $n < n_0$, we have*

$$\|m\|_{2,q} \leq A_2 \sup_j \|\Delta_j m\|_{2,q}.$$

If σ is fixed, then $A_2 \to 1$ as $q \to 2$.

Proof. Let $1 < q < \infty$. By the Littlewood-Paley theory [9], there exists C_1 so that if $\hat{f}(n) = 0$ for all $n < n_0$ then

$$\|f\|_q \leq C_1 \| \sum (|\Delta_j f|^2)^{1/2} \|_q.$$

By using the vector-valued version of the Riesz-Thorin interpolation theorem, as in Lemma 45, we have $C_1 \to 1$ as $q \to 2$. Hence when $2 \leq q$,

$$\begin{aligned} \|mf\|_q &\leq C_1 \| \sum (|\Delta_j m f|^2)^{1/2} \|_q \\ &\leq C_1 \sum (\|\Delta_j m f\|_q^2)^{1/2} \\ &\leq C_1 \sup \|\Delta_j m\|_{2,q} \sum (\|\Delta_j f\|_2^2)^{1/2} \\ &= C_1 \sup \|\Delta_j m\|_{2,q} \|f\|_2 \end{aligned}$$

where the second inequality follows from Minkowski's inequality. If $A_2 = C_1$, then $A_2 \to 1$ as $q \to 2$.

■

Let $C = C(\{n_k\}, \{r_k\})$ be a fixed homogeneous Cantor set. Recall that $n_k \geq 2$ and $n_k r_k \leq 1$. Let c_k be the sum of the lengths of a level k Cantor interval and a gap, i.e.

$$c_k = s_k + y_k = \frac{s_{k-1}(1 - r_k)}{n_k - 1} \text{ for } k \geq 1,$$

where $s_0 = 1$ and $c_0 = 1$. In the following, we will focus on a $\mathbf{p}$-Cantor measure with $\mathbf{p} = \{p_{kj}\}$, not necessarily uniform weights, on C.

Lemma 47. *The* $\mathbf{p}$*-Cantor measure* μ *is the weak* $*$ *limit of the discrete measures*

$$\mu_N = *_{k=1}^{N} \sum_{j=0}^{n_k - 1} p_{kj} \delta_{jc_k}.$$

Remark. We write

$$\mu = *_{k=1}^{\infty} \sum_{j=0}^{n_k - 1} p_{kj} \delta_{jc_k}. \tag{6.1}$$

Proof. The measure μ_N can be written in the form of

$$\mu_N = \sum_{\substack{u \in W \\ |u| = N}} \prod_{k=1}^{N} p_{ku_k} \delta_{x_u}$$

where $x_u = \sum_{k=1}^{N} u_k c_k$ is the left endpoint of the Cantor interval I_u. Let ν be a weak $*$ limit of μ_N, which exists by Banach-Alaoglu theorem. It suffices to check that $\nu(I_w) = \prod_{k=1}^{N_0} p_{kw_k}$ for $w \in W$ of length $|w| = N_0$ and ν is 0 on the complement of C. It then follows from Caratheodory extension theorem that ν is equal to the $\mathbf{p}$-Cantor measure μ.

Let $I_w = [a, b]$. Let $\iota_0 \geq 1$ be such that $[a - \frac{1}{\iota_0}, b + \frac{1}{\iota_0}]$ does not intersect any Cantor interval at level N_0 other than I_w. For $\iota \geq 1$, define $g_\iota(x) := 1$ on $[a, b]$, $g_\iota(x) = 0$ on $(-\infty, a - \frac{1}{\iota + \iota_0}] \cup [b + \frac{1}{\iota + \iota_0}, \infty)$ and extend it to a piecewise linear function.

If $N > N_0$,

$$\begin{aligned} \int g_\iota d\mu_N &= \sum_{\substack{u \in W \\ |u| = N}} \prod_{k=1}^{N} p_{ku_k} \delta_{x_u}(I_w) \\ &= \sum_{\substack{|u| = N \\ u|_{N_0} = w}} \prod_{k=1}^{N_0} p_{kw_k} \prod_{k=N_0+1}^{N} p_{ku_k} = \prod_{k=1}^{N_0} p_{kw_k} \end{aligned}$$

because I_w contains all those left endpoints x_u with $u|_{N_0} = w$. Since g_ι is continuous, by weak $*$ convergence of μ_N,

$$\int g_\iota d\nu = \lim_{N\to\infty} \int g_\iota d\mu_N = \prod_{k=1}^{N_0} p_{kw_k}$$

for all ι.

On the other hand, notice that $|g_\iota| \leq 1$ and $g_\iota \to \chi_{I_w}$ pointwise as $\iota \to \infty$. By the dominated convergence theorem,

$$\nu(I_w) = \int \chi_{I_w} d\nu = \lim_\iota \int g_\iota d\nu = \prod_{k=1}^{N_0} p_{kw_k}.$$

We can check that $\nu = 0$ on C^c by applying a similar approximation process on the open intervals in the complement of C. ■

From here on, μ will be the **p**-Cantor measure as in (6.1). We make the following two assumptions:

1. $r = \inf_k r_k > 0$.
2. $p = \inf_{k,j} p_{kj} > 0$.

The Fourier-Stieltjes transform of μ is given by

$$\begin{aligned}\hat{\mu}(\xi) &= \lim_{K\to\infty} \left(*_{k=1}^{K} \sum_{j=0}^{n_k-1} p_{kj}\delta_{jc_k} \right)^{\wedge}(\xi) = \lim_{K\to\infty} \prod_{k=1}^{K} \left(\sum_{j=0}^{n_k-1} p_{kj}\delta_{jc_k} \right)^{\wedge}(\xi) \\ &= \prod_{k=1}^{\infty} \left(\sum_{j=0}^{n_k-1} p_{kj} e^{-2\pi i \xi c_k j} \right).\end{aligned}$$

For $\xi \in \mathbb{R}$, let $f_k(\xi) = \sum_{j=0}^{n_k-1} p_{kj} e^{-2\pi i \xi c_k j}$. Note that $f_k(\xi + \frac{1}{c_k}) = f_k(\xi)$. If α is the period of f_k, then

$$1 = f_k(0) = f_k(\alpha) = \sum_{j=0}^{n_k-1} p_{kj} e^{-2\pi i \alpha c_k j}.$$

By the strict convexity of the unit ball, $e^{-2\pi i\alpha c_k j} = 1$ for $1 \le j \le n_k - 1$ and hence $\alpha = \frac{z}{c_k}$ for some integer z. Thus f_k is indeed $\frac{1}{c_k}$-periodic. In particular, $|f_k(\xi)| = 1$ if and only if $\xi = \frac{z}{c_k}$ for some $z \in \mathbb{Z}$.

First, we want to see how the function f_k stays away from 1.

Lemma 48. *Let $0 < \delta < p^2$. Then there exists $\eta = \eta(p, \delta) > 0$ (independent of k) such that if $d(c_k\xi, \mathbb{Z}) \ge \eta$, then $|f_k(\xi)| \le 1 - \delta$. (Here $d(x, \mathbb{Z}) = \min\{|x - n| : n \in \mathbb{Z}\}$.) Moreover, $\eta \to 0$ as $\delta \to 0$.*

Remark. The assumption $\inf_k r_k > 0$ is not required in this lemma, but it is needed in everything afterwards.

Proof. Choose $\eta > 0$ such that if $d(x, \mathbb{Z}) \ge \eta$, then $1 - \cos 2\pi x \ge \frac{\delta}{p^2}$. Certainly, $\eta \to 0$ as $\delta \to 0$.

Assume $z_0 \in \mathbb{Z}$ is such that $d(c_k\xi, \mathbb{Z}) = |c_k\xi - z_0| \ge \eta$. Put $\theta = -(c_k\xi - z_0)$. One can see that

$$\begin{aligned}
|f_k(\xi)| = |f_k(\xi - \frac{z_0}{c_k})| &= \left|\sum_{j=0}^{n_k-1} p_{kj} e^{2\pi i\theta j}\right| \\
&\le \sum_{j=2}^{n_k-1} |p_{kj} e^{2\pi i\theta j}| + |p_{k0} + p_{k1} e^{2\pi i\theta}| \\
&= 1 - (p_{k0} + p_{k1}) + (p_{k0} + p_{k1})\sqrt{1 - \frac{2p_{k0}p_{k1}}{(p_{k0} + p_{k1})^2}(1 - \cos 2\pi\theta)}.
\end{aligned}$$

As $p \le p_{k0}, p_{k1} \le 1 - p$,

$$\frac{(p_{k0} + p_{k1})^2}{2p_{k0}p_{k1}} = \frac{1}{2}(\frac{p_{k0}}{p_{k1}} + \frac{p_{k1}}{p_{k0}}) + 1 \le \frac{1-p}{p} + 1 = \frac{1}{p}.$$

Since $d(\theta, \mathbb{Z}) \ge \eta$, it follows that

$$1 - \frac{2p_{k0}p_{k1}}{(p_{k0} + p_{k1})^2}(1 - \cos 2\pi\theta) \le 1 - p \cdot \frac{\delta}{p^2} \le (1 - \frac{\delta}{2p})^2.$$

Therefore,

$$|f_k(\xi)| \le 1 - (p_{k0} + p_{k1}) + (p_{k0} + p_{k1})(1 - \frac{\delta}{2p}) = 1 - (p_{k0} + p_{k1})\frac{\delta}{2p} \le 1 - \delta.$$

■

Let $l_0 = 1$ and

$$l_k = \frac{1}{s_{k-1}(1-r_k)} = \frac{1}{(n_k-1)c_k} \geq \frac{1}{(M-1)c_k}$$

for $k \geq 1$. We will be concerned with the ratio $\frac{l_{k-1}}{l_k}$.

Lemma 49. *The ratio $\frac{l_{k-1}}{l_k}$ is uniformly bounded away from 0 and 1.*

Proof. If $k = 1$, $\frac{l_0}{l_1} = 1 - r_1$ is strictly between 0 and 1. If $k > 1$, as $s_{k-2}r_{k-1} = s_{k-1}$,

$$\frac{l_{k-1}}{l_k} = \frac{s_{k-1}(1-r_k)}{s_{k-2}(1-r_{k-1})} = \frac{r_{k-1}(1-r_k)}{1-r_{k-1}}.$$

Since $0 < r \leq r_k \leq \frac{1}{2}$,

$$0 < \frac{1}{2}\frac{r}{1-r} \leq \frac{l_{k-1}}{l_k} \leq (1-r_k) \leq 1-r < 1,$$

i.e. the ratio $\frac{l_{k-1}}{l_k}$ is bounded away from 0 and 1. ■

Lemma 50. *There exists $\delta > 0$ such that for all $k \geq 1$ and for any interval I with length $|I| \leq \frac{1}{2}l_k$, we can find a subinterval $J \subseteq I$ satisfying the following:*

$$|J| \leq \frac{1}{2}l_{k-1},$$
$$|I \setminus J| \leq \frac{1}{2}(l_k - l_{k-1}),$$
$$|f_k(\xi)| = \left|\sum_{j=0}^{n_k-1} p_{kj}e^{-2\pi i\xi c_k j}\right| \leq 1-\delta$$

for any $\xi \in I \setminus J$. Moreover, each endpoint of J either coincides with an endpoint of I or stays away from the endpoints of I at a distance greater than δl_k.

Proof. Let $e_k := (M-1)\eta l_k \geq \frac{\eta}{c_k}$ where $\eta = \eta(p,\delta)$ is obtained from Lemma 48. By Lemma 49, $\frac{l_{k-1}}{l_k}$ is uniformly bounded away from 0 and 1, thus we can choose $\delta > 0$ small enough that for all k,

$$e_k \leq \min\{\frac{l_k}{4}, \frac{l_{k-1}}{4} - 2\delta l_k\}$$

and

$$\frac{l_{k-1}}{l_k} \leq 1 - 2\delta,$$

or, equivalently, $\delta l_k \leq \frac{1}{2}(l_k - l_{k-1})$. We note that $e_k > 0$ implies $l_{k-1} > 8\delta l_k$.

Fix k. Recall that $|f_k(\xi)| = 1$ if only if $\xi = \frac{z}{c_k} = zl_k$ for some integer z. Suppose ξ_0 is the point closest to I such that $|f_k(\xi_0)| = 1$.

If $|I| \leq \frac{1}{2}l_{k-1}$, simply take $J = I$. Then $I \setminus J = \emptyset$ and the conclusion holds.

Now suppose $|I| > \frac{1}{2}l_{k-1}$. Let $l = \frac{1}{2}l_{k-1}$.

Case 1: Suppose ξ_0 is outside I, say to the left of I.

(i) If $a+l < b \leq a+l+\delta l_k$, take $J = [a, b-\delta l_k]$. Note that $b-\delta l_k-a > \frac{1}{2}l_{k-1}-\delta l_k > 0$, so

$$|I \setminus J| = b - (b - \delta l_k) = \delta l_k \leq \frac{1}{2}(l_k - l_{k-1}).$$

(ii) If $a + l + \delta l_k < b$, take $J = [a, a + l]$ and then

$$|I \setminus J| = |I| - \frac{1}{2}l_{k-1} \leq \frac{1}{2}(l_k - l_{k-1}).$$

In either of these situations, let ξ_1 be the closest point to the right of b so that $|f_k(\xi_1)| = 1$. Note that $\xi_1 - \xi_0 = l_k$. Since $b - a \leq \frac{1}{2}l_k$ and ξ_0 is closer to a than ξ_1 is to b, we have $|b - \xi_1| \geq \frac{1}{4}l_k \geq e_k$. Thus if $\xi \in I \setminus J$, then

$$|\xi - \xi_1| \geq e_k.$$

At the same time, if $\xi \in I \setminus J$, then $|\xi - \xi_0| \geq |J|$, so either

(i)

$$|\xi - \xi_0| \geq b - \delta l_k - \xi_0 > |I| - \delta l_k \geq \frac{l_{k-1}}{2} - \delta l_k \geq e_k$$

or

(ii)

$$|\xi - \xi_0| \geq l = \frac{1}{2}l_{k-1} \geq e_k.$$

Furthermore, $|\xi - \xi_1| \geq |b - \xi_1| \geq e_k$. It follows that $|\xi - \frac{z}{c_k}| \geq e_k \geq \frac{\eta}{c_k}$ for all $z \in \mathbb{Z}$. Hence $|f_k(\xi)| \leq 1 - \delta$ by Lemma 48.

Case 2: Suppose $\xi_0 \in I$. Without loss of generality, assume a is closer to ξ_0 than b. If $\xi_1 \neq \xi_0$ is any other point such that $|f_k(\xi_1)| = 1$, then $|\xi_1 - \xi_0| \geq l_k$. Since $b - a \leq \frac{1}{2}l_k$, a and b will be at least $\frac{1}{2}l_k(\geq e_k)$ away from ξ_1. In consequence, if $\xi \in I$, then $|\xi - \xi_1| \geq e_k$.

(i) If $a < \xi_0 - \frac{l}{2} - \delta l_k$, then $b > \xi_0 + \frac{l}{2} + \delta l_k$ since b is further away from ξ_0. Take $J = [\xi_0 - \frac{l}{2}, \xi_0 + \frac{l}{2}]$. Then

$$|I \setminus J| = |I| - l \leq \frac{1}{2}(l_k - l_{k-1}).$$

If $\xi \in I \setminus J$, then $|\xi - \xi_0| > \frac{1}{2}l \geq e_k$ and consequently an application of Lemma 48 shows $|f_k(\xi)| \leq 1 - \delta$.

(ii) If $\xi_0 - \frac{l}{2} - \delta l_k \leq a \leq \xi_0$, then $b - \xi_0 = b - a + a - \xi_0 \geq l - (\frac{l}{2} + \delta l_k) = \frac{l}{2} - \delta l_k$. If $a + l < b \leq a + l + \delta l_k$, take $J = [a, b - \delta l_k]$ and then

$$|I \setminus J| = b - (b - \delta l_k) = \delta l_k \leq \frac{1}{2}(l_k - l_{k-1}).$$

As $b - \xi_0 \geq \frac{l}{2} - \delta l_k \geq \delta l_k$, $\xi_0 \in J$. Hence, if $\xi \in I \setminus J$, then

$$|\xi - \xi_0| \geq b - \delta l_k - \xi_0 \geq a + l - \delta l_k - \xi_0 \geq \frac{l}{2} - 2\delta l_k \geq e_k$$

and therefore $|f_k(\xi)| \leq 1 - \delta$ by Lemma 48.
If $a + l + \delta l_k < b$, take $J = [a, a + l]$ and then

$$|I \setminus J| = |I| - l \leq \frac{1}{2}(l_k - l_{k-1}).$$

One can again see that $\xi_0 \in J$, thus, if $\xi \in I \setminus J$, then

$$|\xi - \xi_0| \geq a + l - \xi_0 \geq \frac{l}{2} - \delta l_k \geq e_k$$

and hence $|f_k(\xi)| \leq 1 - \delta$ by Lemma 48.

■

Lemma 51. *Let $m_k : \mathbb{Z} \to \mathbb{C}$ be the multiplier given by*

$$m_k(\xi) = \prod_{j=1}^{k} (\sum_{j=0}^{n_k-1} p_{kj} e^{-2\pi i \xi c_k j}).$$

Fix $B > 1$. There exists $q > 2$ such that for all $K \geq 1$, if I is any interval with $|I| \leq \frac{1}{2} l_K$, then

$$\|m_K \chi_I\|_{2,q} \leq B.$$

Remark. m_k is the multiplier T_{μ_k}, where μ_k is the finitely supported, discrete measure given in Lemma 47.

Proof. First, we fix a few constants which we will use later. Let D be such that

$$\frac{1}{D+1} \leq \frac{1}{2} \frac{r}{1-r} \leq \frac{l_{k-1}}{l_k}$$

for all k. The consequence $\frac{1}{2D}(l_k - l_{k-1}) \leq \frac{1}{2} l_{k-1}$ will be used.

Choose $\delta > 0$ as in the previous lemma. Notice that

$$\begin{aligned} \frac{l_k}{l_{k+N}} &= \frac{s_{k+N-1}(1 - r_{k+N})}{s_{k-1}(1 - r_k)} = r_{k+1} \cdots r_{k+N-1}(1 - r_{k+N}) \frac{r_k}{1 - r_k} \\ &\leq r_{k+1} \cdots r_{k+N-1}(1 - r) \leq \frac{1}{2^{N-1}}, \end{aligned}$$

thus we can take N independent of k such that

$$l_k \leq \delta l_{k+N} \tag{6.2}$$

for all k.

Applying Lemma 45 with the L of that lemma equal to $\max\{D, N, 3\}$, we get $A_1 = A_1(q, L)$. The numbers D and N are independent of k, so A_1 is also independent of k.

Recall that in Lemmas 45 and 46, the constants A_1 and $A_2 = A_2(q, 2)$ tend to 1 as $q \to 2$. Choose $q > 2$ such that $A_1 \leq B$ and $(1 - \delta) A_2 A_1^4 \leq 1$. Then

$$\max\{A_1, (1 - \delta) A_2 A_1^4 B\} \leq B.$$

We will prove our conclusion for this q.

We proceed by induction on K. When $K = 1$ and $|I| \leq \frac{1}{2}l_1 = \frac{1}{2(1-r_1)} \leq 1$, the interval I contains at most one integer, say n_0. Then

$$\begin{aligned}\|m_1\chi_I f\|_q &= \|m_1(n_0)\hat{f}(n_0)e^{2\pi i n_0 x}\|_q = |m_1(n_0)||\hat{f}(n_0)| \\ &\leq \|m_1\|_\infty \|f\|_2 \leq \|f\|_2.\end{aligned}$$

Hence

$$\|m_1\chi_I\|_{2,q} \leq 1 < B.$$

Assume the statement is true for all $K \leq k-1$. When $K = k$, given an interval I with $|I| \leq \frac{1}{2}l_k$, we can obtain by Lemma 50 a sequence of subintervals

$$J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{k-1} \subseteq I$$

such that $|J_i| \leq \frac{1}{2}l_i$, with the conclusion of the lemma.

Partition $I \setminus J_{k-1}$ into $D+2$ subintervals, $\{I_d\}_d$, so that each I_d has length at most

$$\frac{1}{D}|I \setminus J_{k-1}| \leq \frac{1}{2D}(l_k - l_{k-1}) \leq \frac{1}{2}l_{k-1}.$$

By Lemma 45 and the induction hypothesis,

$$\begin{aligned}\|m_{k-1}\chi_{I\setminus J_{k-1}}\|_{2,q} &\leq A_1 \max_d \|m_{k-1}\chi_{I_d}\|_{2,q} \\ &\leq A_1 B.\end{aligned}$$

On $I \setminus J_{k-1}$, $|m_k\chi_{I\setminus J_{k-1}}| \leq (1-\delta)|m_{k-1}\chi_{I\setminus J_{k-1}}|$ and this implies

$$\begin{aligned}\|m_k\chi_{I\setminus J_{k-1}}\|_{2,q} &\leq (1-\delta)\|m_{k-1}\chi_{I\setminus J_{k-1}}\|_{2,q} \\ &\leq (1-\delta)A_1 B. \qquad (6.3)\end{aligned}$$

Similarly,

$$\|m_k\chi_{J_i\setminus J_{i-1}}\|_{2,q} \leq (1-\delta)A_1 B \qquad (6.4)$$

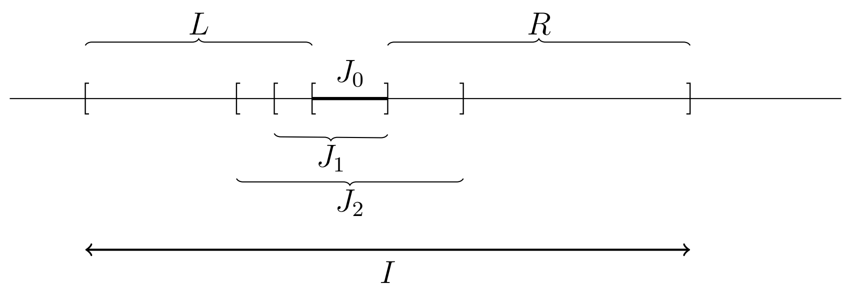
for $1 \leq i \leq k-1$. On the other hand,

$$\|m_k\chi_{J_0}\|_{2,q} \leq 1$$

by the same proof as in the base case since $|J_0| < 1$. The next step is to piece them together.

Let R and L be the right and left part of $I \setminus J_0$ respectively. (See diagram for an example of what this might look like.) Let $\{n_j\}_j$ be the finite sequence of distinct right endpoints of the intervals J_i in ascending order. We want to pick a lacunary subsequence as follows.

Example



Consider $\{n_j : j \equiv 0 \mod N\}$, i.e. $\{n_{Nj}\}_j$. There will be distinct i such that $n_{Nj} - n_{N(j-1)} \leq \frac{1}{2}l_i$. By the construction of Lemma 50 and (6.2),

$$n_{N(j+1)} - n_{Nj} \geq \delta(l_{i+k_1} + \cdots + l_{i+k_N}) \geq \delta l_{i+N} \geq l_i.$$

Thus

$$\frac{n_{N(j+1)} - n_{Nj}}{n_{Nj} - n_{N(j-1)}} \geq \frac{l_i}{\frac{1}{2}l_i} = 2,$$

so $\{n_{Nj}\}_j$ is 2-lacunary.

On each $[n_{Nj}, n_{N(j+1)}]$, since $\|m_k\chi_{[n_{Nj+x}, n_{Nj+x+1}]}\|_{2,q} = \|m_k\chi_{J_i\setminus J_{i-1}}\|_{2,q}$ for suitable i, (6.4) and Lemma 45 implies

$$\|m_k\chi_{[n_{Nj}, n_{N(j+1)}]}\|_{2,q} \leq A_1 \max_{0\leq x\leq N-1} \|m_k\chi_{[n_{Nj+x}, n_{Nj+x+1}]}\|_{2,q} \leq (1-\delta)A_1^2 B.$$

We can then apply Lemma 46 and get

$$\begin{aligned}\|m_k\chi_{(J_{k-1}\setminus J_1)\cap R}\|_{2,q} &\leq A_2 \sup_j \|m_k\chi_{[n_{Nj}, n_{N(j+1)}]}\|_{2,q} \\ &\leq A_2(1-\delta)A_1^2 B.\end{aligned}$$

By (6.3) and Lemma 45 again,

$$\begin{aligned}\|m_k\chi_R\|_{2,q} &\leq A_1 \max\{\|m_k\chi_{R\setminus J_{k-1}}\|_{2,q}, \|m_k\chi_{(J_{k-1}\setminus J_1)\cap R}\|_{2,q}\} \\ &\leq A_1 \max\{(1-\delta)A_1 B, A_2(1-\delta)A_1^2 B\} \\ &\leq (1-\delta)A_2 A_1^3 B.\end{aligned}$$

Similarly,

$$\|m_k\chi_L\|_{2,q} \leq (1-\delta)A_2 A_1^3 B.$$

Finally, combining the three pieces,

$$\begin{aligned}\|m_k\chi_I\|_{2,q} &\le A_1 \max\{\|m_k\chi_{J_0}\|_{2,q}, \|m_k\chi_R\|_{2,q}, \|m_k\chi_L\|_{2,q}\} \\ &\le A_1 \max\{1, (1-\delta)A_2A_1^3B\} \\ &\le B.\end{aligned}$$

■

We can now obtain the main theorem of this chapter.

Theorem 52. *Let $C = C(\{n_k\}, \{r_k\})$ be a homogeneous Cantor set with $\inf r_k > 0$. Assume μ is a $\mathbf{p}$-Cantor measure on C with $p = \inf_{k,j} p_{kj} > 0$. If $B > 1$, then there exists $q > 2$ such that*

$$\|S_\mu\|_{2,q} \le B.$$

Proof. Fix $B > 1$. Assume $r = \inf r_k$. With our usual notation, the Fourier-Stieltjes transform of the Cantor measure $\mu = *_{k=1}^{\infty} \sum_{j=0}^{n_k-1} p_{kj}\delta_{jc_k}$ is

$$\hat{\mu}(\xi) = \prod_{k=1}^{\infty}(\sum_{j=0}^{n_k-1} p_{kj}e^{-2\pi i\xi c_k j}).$$

Let $m_k(\xi) = \prod_{j=1}^{k}(\sum_{j=0}^{n_k-1} p_{kj}e^{-2\pi i\xi c_k j})$. Obtain $q > 2$ from Lemma 51 and let q' be the conjugate of q, i.e. $\frac{1}{q'} + \frac{1}{q} = 1$.

For any trigonometric polynomial f, $\text{supp}\hat{f} \subseteq I$ for some interval I, with $|I| \le \frac{1}{2}l_k$ for some k. A duality argument shows $\|m_k\chi_I\|_{2,q} = \|m_k\chi_I\|_{q',2}$. Thus by Lemma 51, $\|m_k\chi_I\|_{q',2} \le B$ for any interval I with $|I| \le \frac{1}{2}l_k$, i.e.,

$$\|m_kf\|_2 = \|m_k\chi_If\|_2 \le B\|f\|_{q'}.$$

Since $|\hat{\mu}(\xi)| \le |m_k(\xi)|$ for all k,

$$\|\mu * f\|_2 \le \|m_kf\|_2 \le B\|f\|_{q'}.$$

Therefore,

$$\|S_\mu\|_{2,q} = \|S_\mu\|_{q',2} \le B.$$

■

If μ is the uniform Cantor measure on a homogeneous Cantor set with $n_k \leq M$ for all k, then each $p_{kj} = \frac{1}{n_k} \geq \frac{1}{M}$, so μ satisfies the hypothesis of the theorem. Thus we obtain the following corollary.

Corollary 53. *Let C be a homogeneous Cantor set with $\inf r_k > 0$. If μ is the uniform Cantor measure on C, then μ is L^p-improving.*

Another special case is a **p**-Cantor measure on a central Cantor set.

Corollary 54. *Let $0 < p < 1$ and $\mathbf{p} = \{p, 1-p\}$. If μ is a $\mathbf{p}$-Cantor measure on a central Cantor set with ratios of dissection bounded away from 0, then μ is L^p-improving.*

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