Gauge Theory Dynamics and Calabi-Yau Moduli

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We compute the exact partition function of two dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories on S^2. For theories with $SU(2|1)_A$ invariance, the partition function admits two equivalent representations corresponding to localization on the Coulomb branch or the Higgs branch, which includes vortex and anti-vortex excitations at the poles. For $SU(2|1)_B$ invariant gauge theories, the partition function is localized to the Higgs branch which is generically a Kähler quotient manifold. The resulting partition functions are invariant under the renormalization group flow. For gauge theories that flow in the infrared to Calabi-Yau nonlinear sigma models, the partition functions for the $SU(2|1)_A$ (resp. $SU(2|1)_B$) invariant theories compute the Kähler potential on the Kähler moduli (resp. complex structure moduli) of the Calabi-Yau manifold. We also compute the elliptic genus of such theories in the presence of Stückelberg fields and show that they are modular completions of mock Jacobi forms.

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Chapter 1

Introduction

Mankind's greatest achievements in the quest to understand and wield nature is in the makeup of the standard model of particle physics. Its stunning accuracy in describing a wide range of physical phenomena we observe in nature cannot be overemphasized and its predictive and explanatory power has led us to believe that all physical phenomena stem from a small number of fundamental constituents with the non-trivial dynamics captured – in the experimentally accessible regimes – by the standard model of particle physics. The picture portrayed by the standard model as a quantum field theory (QFT) however is far from complete. First, our conventional methods for quantitatively or qualitatively describing physical phenomena in the quantum regime break down when gravitational interaction is included forcing us to treat gravity on a different footing from the electroweak and the strong interactions. This is the infamous problem of quantum gravity which has been the subject of or a motivation for the majority of the programs in theoretical high energy physics.

In addition to lacking a quantum mechanical description of the gravitational interaction, the standard model of particle physics suffers from conceptual issues such as the hierarchy problem – the order of magnitude discrepancy in the strength of different interactions – and naturalness, or fine-tuning of the free parameters, costing the standard model some of its predictive power.

Most apparent in the shortcomings of the standard model, however, is that a quantitative description of the dynamics is only possible in a small corner of the parameter space where the effective coupling constants controlling the strength of the interactions are small and the perturbative approach using Feynman diagrams is valid. This severely limits our understanding of the physics of non-perturbative phenomena and the dynamics of strongly coupled systems such as quantum chromodynamics (QCD), the theory underlying the strong interactions. At low energies, QCD is strongly coupled and exhibits a number of non-perturbative phenomena such as instantons – solutions to the (Euclidean) classical equations of motion with non-trivial topology – and confinement. Furthermore, quantum gravity is considered to be inherently non-perturbative. A systematic study of non-perturbative aspects of quantum field theories and strongly coupled systems seems therefore a crucial step in completing the picture portrayed by the standard model.

In the past few decades, many approaches have been devised and many tools have been developed to facilitate a better understanding of quantum field theories beyond the perturbative regime. The study of scattering amplitudes in Yang-Mills gauge theories has, for instance, shed light on the large contrast in the complexity of the method of Feynman diagrams and the simplicity of the quantities they compute [1]. Additionally, applying the methods developed for integrable systems to (specific sectors of) these theories has led to many insightful results about the physics at strong coupling. [2]. Some of the most fruitful developments though have come from the study of extended objects and in connection with string theory [3, 4].

The study of extended objects is closely tied with the study of non-local observables in QFT and quantum gravity. Such observables correspond to gauge-invariant operators supported on submanifolds of the spacetime manifold. The simplest example of such operators is the Wilson loop operator in quantum electrodynamics associated to a closed curve C in space

$$W_C = \mathrm{Pexp}\left(iq \oint_C A \right) .$$

This operator measures the physical phase picked up by a particle of charge q when transported around the curve C. This phase is proportional to the net magnetic flux through the loop C. Such operators detect the presence of topological defects in the QFT and can therefore be used to probe the topological structure of the theory as well as the physics of topological defects.

Such defects are rarely rigid, *i.e.* unaffected by the dynamics of the ambient QFT. In fact, generically, defects behave like extended objects in the theory and have a set of collective coordinates, or moduli, along which they can fluctuate or be deformed such as their position and shape in space [5]. There is a rich interplay between the physics of extended objects and the physics of the field theory in which they are embedded. This facilitates a mutual study of the dynamics of extended objects and the ambient QFT. Extended objects support non-local operators of the ambient quantum field theory and can probe both local and non-local structure of the QFT which may not be visible to local observables. Likewise, the low-energy effective dynamics of the extended object can be inferred from the ambient QFT. This effective description is in general only valid below a natural scale associated with the object and the ambient quantum field theory.

Starting from the low-energy effective description of an extended object, it is natural to ask what class of ambient QFTs can give rise to such effective dynamics. That is, what are the possible UV completions of the low-energy description? Indeed a possible UV completion is provided by introducing more degrees of freedom via embedding the object in a UV complete QFT. In some cases there is a much more interesting alternative which is to view the extended object as a fundamental entity with a UV complete QFT description. Indeed such QFTs arise in string theory [6–10].

Consider the case of a $1+1$ dimensional object, *i.e.* a string, embedded in a $d+1$ dimensional spacetime M with the metric g. Let $\{\sigma^a\}_{a=0,1}$ be the coordinates on the string and let $\{X^\mu(\sigma)\}_{\mu=0,\dots,d}$ be the embedding functions. The embedding functions are therefore maps from the *string worldsheet* to the *target space* M. Most string-like objects that arise in nature, such as flux tubes in superconductors, have a low-energy worldsheet description captured by the Nambu-Goto action

$$S = -T \int \mathrm{d}^2\sigma \sqrt{-\det \gamma}\,,$$

where T is the string tension and γ is the metric on the worldsheet induced by the embedding X, that is

$$\gamma_{ab} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b}\,.$$

The Nambu-Goto action has a very intuitive interpretation: it evaluates the area that the string sweeps as it evolves in spacetime.[1] This description of the strings observed in nature is only valid well below an energy scale typically associated with the width of the string.

[1]This is the direct two dimensional analogue of the worldline formulation of quantum field theory where the action evaluates the proper time along the worldline of the particle.

Introducing an auxiliary worldsheet metric h the Nambu-Goto action can be rewritten as

$$S = -\frac{T}{2}\int \mathrm{d}^2\sigma\sqrt{-h}\, g_{\mu\nu}h^{ab}\,\partial_a X^\mu \partial_b X^\nu\,.$$

From the point of view of the worldsheet the embedding functions X^μ are simply scalar fields and the worldsheet theory is just a two dimensional scalar field theory or *non-linear sigma model* (NLSM).[2] Furthermore, analyzing the symmetries of this action reveals that this NLSM exhibits conformal symmetry. We may expect then that this NLSM is a consistent conformal quantum field theory (CFT) and does not require extra degrees of freedom to be included in the UV. This is only true if the symmetries of the classical theory are not anomalous at the quantum level. This can be accommodated and the corresponding NLSM is bosonic string theory.

Following the logic we just outlined, we can search for a QFT living in the target space that corresponds to string theory. To this end, we quantize the worldsheet string theory and look for the spectrum of states. Studying the spectrum of bosonic string theory leads to a remarkable surprise: the spectrum of closed string excitations contains a massless spin two state *i.e.* the spacetime theory is a quantum theory of gravity [11, 12]. Unfortunately however, the spectrum also contains a tachyon hinting that the naïve vacuum of the theory is unstable. Additionally, Weyl anomaly cancellation condition for theories that admit spacetime Poincaré invariance is only satisfied in the critical spacetime dimension $d = 26$. Lastly, the spectrum of this theory is devoid of fermionic states rendering the theory phenomenologically unappealing.

The issues mentioned above can all be remedied by considering supersymmetric extensions of the bosonic string theory [13–15]. Combined with the conformal symmetry of the worldsheet, the resulting theory is a superconformal field theory (SCFT) of strings or superstring theory. Supersymmetry removes the tachyon state from the spectrum[3] and reduces the critical dimension down to 10 spacetime dimensions. Furthermore, worldsheet supersymmetry transcends into spacetime supersymmetry introducing fermionic states in the spectrum. The amount of supersymmetry in the spacetime theory depends on the choice of boundary conditions. There are five sets of consistent boundary conditions corresponding to distinct superstring theories. These are the *type I* superstring theory arising

[2]Note that the target metric is in general a function $g = g(X)$. The special class of sigma models whose target space is flat $g = \eta$ are called *linear sigma models.*

[3]We restrict our discussion to superstring theories with (local) spacetime supersymmetry. The tachyon state in these theories is rendered unphysical by the GSO projection operator [16].

from the study of open strings, the *heterotic* superstring theories corresponding to the two possible choices of gauge groups, namely $SO(32)$ and $E_8 \times E_8$, and *type IIA* (non-chiral) and *type IIB* (chiral) theories arising from the study of closed strings. There are various dualities among these theories such as the S, T and U dualities and they are believed to arise as different vacua of a unique theory known as M-theory [17–19]. The massless sector of the type I and the heterotic superstring theories is described by the type I supergravity theories with 16 supercharges while the massless sectors of type IIA and type IIB superstring theories reduce to the corresponding supergravity theories with 32 supercharges.

To arrive at a phenomenologically viable theory, we need to dimensionally reduce superstring theory down to four spacetime dimensions. As such we consider superstring theory with a target manifold $M_{10} = \mathbb{R}^4 \times X_6$, where X_6 is a six dimensional compact manifold. For phenomenological reasons, we would like the four dimensional theory to exhibit $\mathcal{N} = 1$ supersymmetry (preserving 4 supercharges). This restricts us to manifolds X_6 of $SU(3)$ holonomy. Such manifolds are Calabi-Yau 3-folds which are equivalently defined as complex Kähler manifolds with vanishing first Chern class. Hence, type I or heterotic superstring theories, when compactified on a Calabi-Yau 3-fold, reduce to four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories with $SO(32)$ or $E_8 \times E_8$ gauge groups. Most notably, the heterotic superstring theory with $E_8 \times E_8$ gauge group compactified on a Calabi-Yau leads to a chiral quantum field theory with gauge interactions and with the gauge group E_6. The corresponding worldsheet SCFT exhibits $\mathcal{N} = (2, 2)$ superconformal symmetry. The supersymmetry enhancement stems from embedding the $SU(3)$ holonomy group of the Calabi-Yau 3-fold into the gauge group. The resulting theory also includes gravitational interactions making it an excellent candidate for the unified quantum theory of gravity and gauge interactions.

On the other hand, type IIA or type IIB superstring theories compactified of Calabi-Yau 3-folds reduce to four dimensional QFTs with $\mathcal{N} = 2$ supersymmetry or 8 supercharges. Such theories are too symmetric to be of phenomenological interest. Furthermore, the massless spectrum of these theories is devoid of vector bosons mediating non-abelian gauge interactions. The resolution of these shortcomings is provided by D-branes. These are dynamical extended objects characterized by the property that strings can end on them. Supergravity backgrounds including D-branes preserve less supersymmetry. Furthermore, D-branes host gauge degrees of freedom thereby introducing non-abelian gauge interactions into type II superstring theories.

The study of D-branes [20–22] has proved to be one of the most successful avenues leading to a better understanding of quantum field theories. The effective QFT on D-branes is typically a supersymmetric Yang-Mills theory enabling us to study such QFTs as embedded in string theory. This approach has provided a bird's-eye view of (a large portion of) the parameter space of quantum field theories revealing a lot of geometric structure on this space and has been central in unveiling many dualities – symmetries on the QFT parameter space. Such dualities can often be understood as a consequence or a special case of various string dualities that map different superstring theories into one another.

A large class of the dualities that have thus been uncovered are strong/weak coupling dualities of which the infamous holographic gauge/gravity duality [23–25] is an example. This is believed to be an exact duality between a quantum theory of gravity and a gauge theory on a fixed background. In particular in the regime where quantum gravity effects can be neglected, the classical theory of gravity on a space with a boundary and prescribed asymptotics (often asymptotically anti-de Sitter) captures the physics of a gauge theory living on the boundary at strong coupling (in the planar limit). Such dualities provide a probe of non-perturbative phenomena and physics of QFTs at strong coupling.

Almost all methods developed so far for studying QFT beyond perturbation theory rely heavily on the symmetries of the physical systems considered and are only applicable when the system exhibits powerful symmetries such as supersymmetry. In particular, supersymmetry imposes strong constraints resulting in a much more controlled dynamics in supersymmetric QFTs. One might worry that theories exhibiting supersymmetry are somewhat idealistic. Nonetheless it has been found that many non-perturbative aspects of gauge theories persist in the presence of such symmetries and are shared across dimensions.

The recurrence of non-perturbative phenomena in idealized models affirms their fundamental importance and offers a window through which they can be studied using the non-perturbative approaches mentioned earlier. In particular, the powerful technique of supersymmetric localization [26–29] has enabled us to perform many exact computations in supersymmetric gauge theories in various dimensions probing the dynamics of these theories deep in the quantum regime and at all values of the coupling constants. The study of supersymmetric observables using localization has led to many remarkable results such as the computation of the Witten index, or a generalization thereof, for various models. This enables us to study the spectrum of these models and their phases in different regions of the parameter space.

Exact computations in supersymmetric gauge theories have also provided strong checks of various dualities such as the gauge/gravity duality, Seiberg duality and mirror symmetry. Seiberg duality [30, 31] relates two supersymmetric gauge theories on the same manifold whose gauge symmetries are related in a non-trivial way. Mirror symmetry [32, 33] is a pairwise correspondence between Calabi-Yau n-folds and was first discovered in the context of string theory [34,35] where it was discovered that the worldsheet theories describing the dynamics of strings compactified on mirror Calabi-Yau three-folds are identical. These dualities are all strong/weak dualities offering a window to the non-perturbative dynamics of quantum field theories in various dimensions.

The case of two dimensional supersymmetric gauge theories or gauged linear sigma models (GLSM) which is the subject of this text is exceptionally rich. The space of these theories exhibits many dualities including the aforementioned Seiberg duality and mirror symmetry. In addition to exhibiting such phenomena as dimensional transmutation, chiral symmetry breaking and non-perturbative corrections due to vortices, under the renormalization group (RG) flow these theories flow in the infrared to non-linear sigma models (NLSM). Furthermore, for a subset of these GLSMs the infrared non-linear sigma model is a superconformal field theory (SCFT). In particular, one can engineer GLSMs whose infrared fixed-point corresponds to non-linear sigma models with a compact Calabi-Yau n-fold target manifold. Of special interest is the case $n = 3$. The resulting SCFTs correspond to Calabi-Yau compactification of heterotic or type II superstring theories and have $\mathcal{N} = (2, 2)$ superconformal symmetry. At low energies, these theories have an effective description as four dimensional field theories. Exactly marginal operators in the worldsheet SCFT correspond to massless scalar fields in the effective 4 dimensional field theory. The dynamics of these fields are dictated by worldsheet correlation functions in the SCFT.

In the worldsheet SCFT of a string theory compactified on a Calabi-Yau 3-fold (CY_3), exactly marginal operators stem from the moduli of the CY_3. Most notable among these are the moduli of metric deformations, that is, the moduli of complex structure deformations and the moduli of Kähler structure deformations. These can be thought of as deformations in the shape and the size of the manifold respectively. The dimension of these moduli are given by the Betti numbers $h^{1,2}$ and $h^{1,1}$ for the complex moduli and the Kähler moduli respectively. The set of operators corresponding to each moduli form a ring in the CFT.

These are the so called chiral and twisted chiral ring

$$\begin{aligned} \text{complex moduli} \;&\longleftrightarrow\; \qquad\text{chiral ring} \qquad = \{\mathcal{O}_a \,|\, a = 1, \ldots, h^{1,2}\}\,, \\ \text{Kähler moduli} \;&\longleftrightarrow\; \text{twisted chiral ring} \;= \{\mathcal{O}_i \,|\, i = 1, \ldots, h^{1,1}\}\,. \end{aligned}$$

Furthermore, the (complexified) Kähler moduli and complex moduli each carry Kähler structure. The Kähler metric on these moduli can be identified [36] with the corresponding Zamolodchikov metric

$$G^K_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} \mathcal{K}^K = \langle i|\bar{j}\rangle_{\text{CFT}}\,, \tag{1.1}$$

$$G^C_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} \mathcal{K}^C = \langle a|\bar{b}\rangle_{\text{CFT}}\,, \tag{1.2}$$

where $\mathcal{K}^K$ and $\mathcal{K}^C$ are the corresponding Kähler potentials. The metrics G^K and G^C have straightforward interpretation in the four dimensional effective field theory: together they define a metric on the field space for the massless scalar fields arising from the CY_3 metric moduli.

While the complex structure moduli metric does not receive quantum corrections, the Kähler moduli metric is corrected by the worldsheet instantons. These are holomorphic maps from the worldsheet to the target CY_3. Summing over instanton corrections is of interest to mathematicians and physicists alike as they define a class of topological invariants known as the integral Gromov-Witten invariants, and they enable us to compute exact Yukawa couplings of the corresponding heterotic string compactification. Instanton corrections, however, are non-perturbative and very hard to compute in general. For an $\mathcal{N} = (2,2)$ superconformal non-linear sigma model that admits a GLSM description, we may avoid this issue by reintroducing the ultraviolet degrees of freedom back into the path integrals (1.1) and (1.2). As we shall see in chapter 2 and chapter 3, the Kähler and complex structure moduli metrics are guaranteed to be invariant under the RG flow. Moreover, invoking the superconformal symmetry of the non-linear sigma models we conclude that the two sets of marginal operators, the chiral ring and the twisted chiral ring, decouple. This is due to the invariance of these operators under different supercharges and R-symmetries. While the operators in the chiral ring are invariant under the “B-type” supersymmetry generated by $\mathcal{Q}^B$ and the axial $U(1)$ R-symmetry, the operators in the twisted chiral ring are invariant under the “A-type” supersymmetry generated by $\mathcal{Q}^A$ and the vector $U(1)$

R-symmetry,

$$[\mathcal{Q}^A, \mathcal{O}_i] = [\mathrm{R}, \mathcal{O}_i] = 0\,, \tag{1.3}$$

$$[\mathcal{Q}^B, \mathcal{O}_a] = [\mathrm{A}, \mathcal{O}_a] = 0\,. \tag{1.4}$$

Exploiting the decoupling of the two sectors, we construct $\mathcal{N} = (2,2)$ supersymmetric GLSMs realizing the $SU(2|1)_A$ (resp. $SU(2|1)_B$) supersymmetry to compute the Kähler moduli (resp. complex moduli) metric.

As was speculated in [37], formed into a sharp conjecture in [38] and proved in [39], the metric on the Kähler moduli space is given by the sphere two-point function of exactly marginal operators in the twisted chiral and twisted anti-chiral rings inserted at polar opposite points on the sphere,

$$G^K_{i\bar{j}} = \langle \mathcal{O}_i(N)\mathcal{O}_{\bar{j}}(S)\rangle_{S^2}\,, \tag{1.5}$$

while the metric on the complex moduli is given by the sphere two-point function of exactly marginal operators in the chiral and anti-chiral rings

$$G^C_{a\bar{b}} = \langle \mathcal{O}_a(N)\mathcal{O}_{\bar{b}}(S)\rangle_{S^2}\,. \tag{1.6}$$

As we remarked above, the Kähler (resp. complex) moduli carries a Kähler structure and the metric G^K (resp. G^C) is a Kähler metric arising from the Kähler potential $\mathcal{K}_K$ (resp. $\mathcal{K}_C$). As a matter of fact we can push the field theory/CY$_3$ correspondence further and give a direct field theory interpretation to the Kähler potential: the Kähler potential is the logarithm of the partition function. As was conjectured in [38] – and established in [39] – for the $SU(2|1)_A$ supersymmetric theory we have

$$\mathcal{Z}^A_{S^2} = e^{-\mathcal{K}_K}\,, \tag{1.7}$$

and for the $SU(2|1)_B$ invariant theory

$$\mathcal{Z}^B_{S^2} = e^{-\mathcal{K}_C}\,. \tag{1.8}$$

In collaboration with Jaume Gomis, Bruno Le Floch and Sungjay Lee in [37] (see also [40]) and in collaboration with Jaume Gomis in [41] we have constructed gauged linear sigma models with $SU(2|1)_A$ and $SU(2|1)_B$ supersymmetry on a sphere. We have

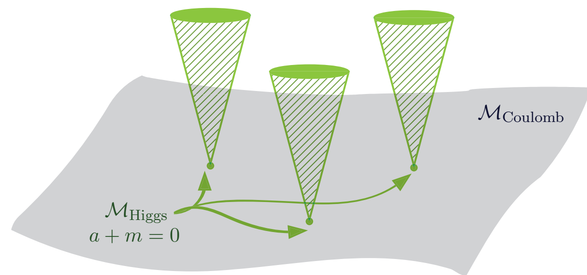

Figure 1.1: Supersymmetric vacua of the $SU(2|1)_A$ invariant theory.

shown that the supersymmetric vacua of the $SU(2|1)_A$ invariant theory consists of both Coulomb and Higgs branches as depicted in figure 1. Through direct computation using different supersymmetric localization schemes, we have shown that the exact partition function for the A-type theory admits two equivalent representations: first as an integral over the Coulomb branch

$$\mathcal{Z}^A_{S^2}(\tau,\bar{\tau}) = \sum_B \int \mathrm{d}\sigma\, \mathcal{Z}_{\text{cl}}(\sigma, B, \tau, \bar{\tau}) \mathcal{Z}_{\text{one-loop}}(\sigma, B)\,, \tag{1.9}$$

where B is the total flux on the sphere and the Coulomb parameter σ is the background value for a real scalar in the vector multiplet. The second representation of the partition function is as a sum over the Higgs vacua with vortex and anti-vortex configurations supported at the poles

$$\mathcal{Z}^A_{S^2}(\tau,\bar{\tau}) = \sum_{v\in\text{Higgs vacua}} \mathcal{Z}_{\text{cl}}(v,\tau,\bar{\tau})\mathcal{Z}_{\text{one-loop}}(v)\mathcal{Z}_{\text{vortex}}(v,\tau)\mathcal{Z}_{\text{anti-vortex}}(v,\bar{\tau})\,. \tag{1.10}$$

The partition function depends on the twisted superpotential couplings parameterizing the Kähler moduli.[4] The $\mathcal{Z}^A_{S^2}$ of a CY_3 GLSM computes the exact Kähler potential, including all worldsheet instanton corrections, for the Kähler moduli of the CY_3 without using mirror symmetry. With the Jockers *et. al.* prescription, this enables us to extract the quantum corrected Gromov-Witten invariants without invoking mirror symmetry. We can also extract the coefficients of three point functions of the four dimensional effective field theory involving the massless scalars arising from the Kähler moduli. For heterotic string compactifications, these correspond to the Yukawa couplings.

[4]One can also turn on twisted mass parameters in a supersymmetric way.

For GLSMs with B-type supersymmetry, we have shown, by localizing the path integral to the supersymmetric Higgs vacua, that the exact sphere partition function for a Kähler quotient Calabi-Yau M takes the form

$$\mathcal{Z}^{B}_{S^2}(W,\overline{W}) = i^{\dim M}\int_M \Omega(W)\wedge\overline{\Omega}(\overline{W})\,, \tag{1.11}$$

where Ω is the nowhere vanishing top holomorphic form. This is indeed the Kähler potential for the complex structure moduli of the Calabi-Yau manifold M in the form proposed in [42, 43].

Another interesting observable of two dimensional $\mathcal{N} = (2,2)$ gauge theories which is invariant under the renormalization group flow is the elliptic genus [44–48]. This observable has recently been computed for a large class of theories using the powerful technique of supersymmetric localization [49–51]. The resulting elliptic genera are holomorphic Jacobi forms with weight zero and index determined by the central charge. More generally, when there is a continuous spectrum in the IR fixed point, non-holomorphicity of the elliptic genus is a measure of the difference in spectral densities for bosonic and fermionic right-moving primaries [52, 53].[5] This difference is determined in terms of the asymptotic supercharge [52, 54] and the continuum contribution is dictated by the asymptotic geometry. This yields a real Jacobi form [53, 55–58]. A known example of this phenomenon is the cigar coset CFT whose gauged linear sigma model description includes a Stückelberg field linearly transforming under gauge transformations, and rendering the two-dimensional gauge field massive [59, 60].

The results of [49–51] can be generalized by considering abelian two-dimensional gauge theories including Stückelberg fields. As was shown in [61], the elliptic genus of these theories are real Jacobi forms.

[5]This is the analogue of the temperature dependence of the Witten index in sypersymmetric quantum mechanics.

The dissertation is organized as follows:

Chapter 2 is devoted to the construction of gauge theories with with $SU(2|1)_A$ supersymmetry on the sphere. To this end, we present the realization of the $SU(2|1)_A$ algebra on various multiplets and construct the corresponding supersymmetric actions. By studying the cohomology of different supercharges, we establish that the sphere partition function is independent of the superpotential terms and prove that it is invariant under the renormalization group flow. We then compute the sphere partition function using supersymmetric localization on the Coulomb branch and demonstrate that it can be factorized into north and south pole contributions. Following a different route, we localize the partition function to Higgs branch of the supersymmetric vacua which consists of vortex and anti-vortex configurations at the poles of the sphere. This yields a direct derivation of the factorized form of the partition function. We end this chapter by a discussion on some of the applications of the results. The contents of this chapter were first presented in[6] [37] by N.D., Jaume Gomis, Bruno Le Floch and Sungjay Lee.

In Chapter 3 we focus on gauge theories with $SU(2|1)_B$ supersymmetry on the sphere. We present the realization of the $SU(2|1)_B$ algebra on vector and chiral multiplets and construct the supersymmetric actions. We then prove that the sphere partition function for an $SU(2|1)_B$ invariant GLSM is independent of the Kähler structure parameters and that it is invariant under the renormalization group flow. We present a direct evaluation of the path integral via localization and we close this chapter by studying a large class of examples. This Chapter is based on [41] by N.D. and Jaume Gomis.

Chapter 4 focuses on elliptic genera in $(2,2)$ supersymmetric GLSMs. Here we present the results that first appeared in [61] by Sujay Ashok, N.D. and Jan Troost, including the GLSM derivation of a real Jacobi form as the elliptic genus of an NLSM with a non-compact Calabi-Yau target space with an asymptotically linear Dilaton direction. The GLSM derivation has obvious generalizations which we present at end of this chapter.

We conclude this dissertation with a summary of the results followed by a discussion on applications and future directions.

[6] See also [40].

Chapter 2

$SU(2|1)_A$ Invariant Gauge Theories

In this chapter we focus on two dimensional gauge theories with $SU(2|1)_A$ supersymmetry. First we explicitly construct the Lagrangian of such $\mathcal{N} = (2,2)$ supersymmetric gauge theories on S^2. We then compute the sphere partition function of these theories exactly using supersymmetric localization.

The basic multiplets of two dimensional $\mathcal{N} = (2,2)$ supersymmetry are the vector multiplet, the chiral multiplet, the twisted vector multiplet and the twisted chiral multiplet. In this chapter we focus on gauge theories with vector and chiral multiplets. Gauge theories with twisted vector and twisted chiral multiplets are studied in chapter 3.

2.1 Vector and Chiral Multiplets

The vector and chiral multiplets of $\mathcal{N} = (2,2)$ supersymmetry in two dimensions arise by dimensional reduction of the familiar four dimensional $\mathcal{N} = 1$ supersymmetry multiplets. The field content is therefore

$$\begin{aligned} \text{vector multiplet: } & (A_\mu, \sigma_1, \sigma_2, \lambda, \bar\lambda, \mathrm{D}) \\ \text{chiral multiplet: } & (\phi, \bar\phi, \psi, \bar\psi, F, \bar F)\,. \end{aligned} \tag{2.1}$$

The fields $(\lambda, \bar\lambda, \psi, \bar\psi)$ are two component complex Dirac spinors,[1] $(\phi, \bar\phi, F, \bar F)$ are complex scalar fields while $(\sigma_1, \sigma_2, \mathrm{D})$ are real scalar fields.[2] The fields in the vector multiplet

[1]Our conventions for spinors are listed in appendix B.

[2]The reality of the auxiliary field D is altered when coupled with matter fields.

transform in the adjoint representation of the gauge group G while the chiral multiplet fields transform in a representation $\mathbf{R}$ of G. The field content of an arbitrary $\mathcal{N}=(2,2)$ supersymmetric gauge theory admitting a Lagrangian description is captured by these multiplets by letting G be a product gauge group and $\mathbf{R}$ a reducible representation.

While it is well known how to construct the Lagrangian of $\mathcal{N}=(2,2)$ supersymmetric gauge theories in $\mathbb{R}^2$ (i.e. flat space), constructing supersymmetric theories on S^2 requires some thought, as S^2 does not admit covariantly constant spinors. Indeed, we must first characterize the $\mathcal{N}=(2,2)$ supersymmetry algebra on S^2. This is the subalgebra of the two dimensional $\mathcal{N}=(2,2)$ superconformal algebra on S^2 that generates the isometries of S^2, but none of the conformal transformations of S^2. There are two such algebras corresponding to the $U(1)_V$ and $U(1)_A$ R-symmetries. In this chapter we restrict our discussion to theories with vectorial R-symmetry. The $\mathcal{N}=(2,2)$ supersymmetry algebra on S^2 thus defined obeys the (anti)commutation relations of the $SU(2|1)_A$ superalgebra[3]

$$\begin{aligned} &[J_m,J_n]=i\epsilon_{mnp}J_p && [J_m,Q_\alpha]=-\frac{1}{2}\gamma_m{}^{\beta}{}_{\alpha}Q_\beta && [J_m,S_\alpha]=-\frac{1}{2}\gamma_m{}^{\beta}{}_{\alpha}S_\beta \\ &\{S_\alpha,Q_\beta\}=\gamma^m_{\alpha\beta}J_m-\frac{1}{2}C_{\alpha\beta}R && [R,Q_\alpha]=-Q_\alpha && [R,S_\alpha]=S_\alpha\,. \end{aligned} \tag{2.2}$$

The supercharges Q_α and S_α are two dimensional Dirac spinors generating the supersymmetry transformations, J_m are the $SU(2)$ charges generating the isometries of S^2 while R is a $U(1)$ R-symmetry charge. This supersymmetry algebra is the S^2 counterpart of the $\mathcal{N}=(2,2)$ super-Poincaré algebra in flat space.

Constructing a supersymmetric Lagrangian on S^2 requires finding supersymmetry transformations on the vector and chiral multiplet fields that represent the $SU(2|1)_A$ algebra. We construct these by restricting the $\mathcal{N}=(2,2)$ superconformal transformations to those corresponding to the $SU(2|1)_A$ subalgebra. The $\mathcal{N}=(2,2)$ superconformal transformations on the fields are easily obtained by combining the $\mathcal{N}=(2,2)$ super-Poincaré transformations in flat space (with the flat metric replaced by an arbitrary metric), with additional terms that are uniquely fixed by demanding that the supersymmetry transformations are covariant under Weyl transformations.[7] Given the $SU(2|1)_A$ supersymmetry transformations on the vector and chiral multiplet fields constructed this way and shown below, it is straightforward to construct the corresponding $SU(2|1)_A$ invariant Lagrangian. The

[3]See appendix C for details.

supersymmetry transformations and action may equivalently be obtained by "twisted" dimensional reduction from three dimensional $\mathcal{N}=2$ gauge theories on $S^1 \times S^2$, considered in [62].

Without further ado, we write down the renormalizable $SU(2|1)_A$ invariant action of an arbitrary gauge theory on S^2

$$S = S_{\text{v.m.}} + S_{\text{top}} + S_{\text{FI}} + S_{\text{c.m.}} + S_{\text{mass}} + S_{\text{W}} \,. \tag{2.3}$$

The vector multiplet action is given by

$$S_{\text{v.m.}} = \frac{1}{2g^2}\int \mathrm{d}^2x\sqrt{h}\ \operatorname{Tr}\left\{V_\mu V^\mu + V_3 V^3 + \mathrm{D}^2 + i\lambda\left(\not{D}\bar{\lambda} - \left[\sigma_1,\bar{\lambda}\right] - i\left[\sigma_2,\gamma^{\hat{3}}\bar{\lambda}\right]\right)\right\}\,, \tag{2.4}$$

where

$$\begin{aligned} V^\mu &= \varepsilon^{\mu\nu} D_\nu \sigma_2 + D^\mu \sigma_1 \,, \\ V^3 &= \frac{1}{2}\varepsilon^{\mu\nu} F_{\mu\nu} + i\left[\sigma_1,\sigma_2\right] + \frac{1}{r}\sigma_1 \,. \end{aligned} \tag{2.5}$$

The bosonic part of the action can also be written as

$$\frac{1}{2g^2}\int \mathrm{d}^2x\sqrt{h}\ \operatorname{Tr}\left\{\left(F_{\hat{1}\hat{2}} + \frac{1}{r}\sigma_1\right)^2 + \left(D_\mu\sigma_1\right)^2 + \left(D_\mu\sigma_2\right)^2 - \left[\sigma_1,\sigma_2\right]^2 + \mathrm{D}^2\right\}\,. \tag{2.6}$$

In the vector multiplet action g denotes the super-renormalizable gauge coupling[4], h is the round metric on S^2 and r is its radius.

For each $U(1)$ factor in G, the gauge field action in two dimensions can be enriched by the addition of the topological term

$$S_{\text{top}} = -i\frac{\vartheta}{2\pi}\int \operatorname{Tr} F \,, \tag{2.7}$$

and of a supersymmetric Fayet-Iliopoulos (FI) D-term on S^2

$$S_{\text{FI}} = -i\xi\int \mathrm{d}^2x\sqrt{h}\ \operatorname{Tr}\left(\mathrm{D} - \frac{\sigma_2}{r}\right)\,. \tag{2.8}$$

The couplings ϑ and ξ are classically marginal, and can be combined into a complex gauge coupling

$$\tau = \frac{\vartheta}{2\pi} + i\xi \tag{2.9}$$

[4]For a product gauge group, there is an independent gauge coupling for each factor in the gauge group.

for each $U(1)$ factor in the gauge group. Quantum mechanically, the coupling τ depends on the energy scale, and can be traded with the dynamically generated, renormalization group invariant scale Λ.[5] We will return to this dynamical transmutation in section 2.3.

The action for the chiral multiplet coupled to the vector multiplet is[6]

$$S_{\text{c.m.}} = \int \mathrm{d}^2x\sqrt{h}\, \Big\{ \bar\phi \left(-D_\mu^2 + \sigma_1^2 + \sigma_2^2 + i\mathrm{D} + i\frac{q-1}{r}\sigma_2 - \frac{q^2-2q}{4r^2} \right)\phi + \bar F F \\ - i\bar\psi\left(\not{D} - \sigma_1 - i\sigma_2\gamma^{\hat 3} + \frac{q}{2r}\gamma^{\hat 3}\right)\psi + i\bar\psi\lambda\phi - i\bar\phi\bar\lambda\psi \Big\}. \tag{2.10}$$

Here q denotes the $U(1)$ R-charge of the chiral multiplet, which takes the value $q=0$ for the canonical chiral multiplet.[7] In a theory with flavour symmetry G_F, the $U(1)$ R-charges take values in the Cartan subalgebra of G_F (see discussion below).

In two dimensions, it is possible to turn on in a supersymmetric way twisted masses for the chiral multiplet. These supersymmetric mass terms are obtained by first weakly gauging the flavour symmetry group G_F acting on the theory, coupling the matter fields to a vector multiplet for G_F, and then turning on a supersymmetric background expectation value for the fields in that vector multiplet. For $\mathcal{N}=(2,2)$ gauge theories on S^2, unbroken $SU(2|1)$ supersymmetry (see equations (2.17) and (2.18)) implies that the mass parameters are given by a constant background expectation value for the scalar field σ_2 in the vector multiplet for G_F. This can be taken in the Cartan subalgebra of the flavour symmetry group G_F. Therefore, the supersymmetric twisted mass terms on S^2 are obtained by substituting

$$\sigma_2 \to \sigma_2 + m \tag{2.11}$$

in (2.10), with m in the Cartan subalgebra of G_F

$$S_{\text{mass}} = \int \mathrm{d}^2x\sqrt{h}\, \left\{ \bar\phi\left(m^2 + 2m\sigma_2 + i\frac{q-1}{r}m\right)\phi - \bar\psi m\gamma^{\hat 3}\psi \right\}. \tag{2.12}$$

Likewise, the $U(1)$ R-charge parameters q introduced in (2.10) can be obtained by turning on an imaginary expectation value for the scalar field σ_2 in the vector multiplet for G_F.

[5]The dynamical scale is given by $\Lambda^{b_0} = \mu^{b_0}e^{2\pi i\tau(\mu)}$, where $\beta(\xi) \equiv \frac{b_0}{2\pi}$ and μ is the floating scale.

[6]The representation matrices of G in the representation $\mathbf{R}$, which we do not write explicitly to avoid clutter, intertwine the vector multiplet and chiral multiplet fields in the usual way.

[7]q also determines the Weyl weight of the fields in the chiral multiplet. The Weyl weight of a field can be read from the commutator of two superconformal transformations (see appendix C), which represents the two dimensional $\mathcal{N}=(2,2)$ superconformal algebra on the fields.

The corresponding supersymmetric terms in the action are obtained by shifting the action in (2.10) for $q = 0$ by

$$\sigma_2 \to \sigma_2 + \frac{i}{2r} q \,. \tag{2.13}$$

The flavour symmetry G_F is determined by the representation $\mathbf{R}$ under which the chiral multiplet transforms and by the choice of superpotential, as this can break the group of transformations rotating the chiral multiplets down to the actual G_F symmetry of the theory. If $\mathbf{R}$ contains N_F copies of an irreducible representation $\mathbf{r}$ and the theory has a trivial superpotential, then the theory has $U(N_F)$ as part of its flavour symmetry group and gives rise to N_F twisted mass parameters $m = (m_1, \ldots, m_{N_F})$ and N_F $U(1)$ R-charges $q = (q_1, \ldots, q_{N_F})$. Occasionally, we will find it convenient to combine these parameters into the holomorphic combination

$$\mathrm{M}_I = m_I + \frac{i}{2r} q_I \,. \tag{2.14}$$

Finally, we can add in a supersymmetric way a superpotential for the chiral multiplet

$$S_{\mathrm{W}} = \int \mathrm{d}^2 x \sqrt{h} \left\{ F_W + \bar{F}_{\bar{W}} \right\}, \tag{2.15}$$

whenever the total $U(1)$ R-charge of the superpotential is $-q_W = -2$. F_W is the gauge invariant auxiliary component of the superpotential chiral multiplet.[8] Under these conditions, the Lagrangian in (2.15) transforms into a total derivative under the $SU(2|1)$ supersymmetry transformations below.

A few brief remarks about the $\mathcal{N} = (2,2)$ gauge theories in S^2 thus constructed are in order. The action (and supersymmetry transformations) can be organized in a power series expansion in $1/r$, starting with the covariantized $\mathcal{N} = (2,2)$ gauge theory action in flat space. The action is deformed by terms of order $1/r$ and $1/r^2$, with terms proportional to $1/r$ not being reflection positive. These features are consistent with the general arguments in [63]. The theory on S^2 breaks the classical[9] $U(1)_A$ R-symmetry of the corresponding $\mathcal{N} = (2,2)$ gauge theory in flat space. This can be observed in the asymmetry between the scalar fields σ_1 and σ_2 in the action on S^2, which are otherwise rotated into each other

[8] In terms of the ϕ chiral multiplet, $F_W = \frac{\partial W}{\partial \phi} F - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^2} \psi\psi$. Invariance of (2.15) under supersymmetry when $q_W = 2$ follows from equations (2.28) and (2.29).

[9] This classical symmetry of the flat space theory, being chiral, can be anomalous.

by the $U(1)_A$ symmetry of the flat space theory. This asymmetry is also manifested in the twisted masses m being real on S^2, while they are complex in flat space.[10] The real twisted masses m on S^2, however, combine with the $U(1)$ R-charges q into the holomorphic parameters $\mathrm{M} = m + \frac{i}{2r}q$ introduced in (2.14).

The gauge theory action we have written down is invariant under the $SU(2|1)$ supersymmetry algebra. The supersymmetry transformations are parametrized by conformal Killing spinors[11] ϵ and $\bar{\epsilon}$ on S^2. These can be taken to obey

$$\begin{aligned}\nabla_\mu \epsilon &= +\frac{1}{2r}\gamma_\mu \gamma^{\hat{3}}\epsilon \\ \nabla_\mu \bar{\epsilon} &= -\frac{1}{2r}\gamma_\mu \gamma^{\hat{3}}\bar{\epsilon}\,,\end{aligned} \tag{2.16}$$

where ϵ and $\bar{\epsilon}$ are complex Dirac spinors in two dimensions and r is the radius of the S^2. The spinors ϵ_α and $\bar{\epsilon}_\alpha$ are the supersymmetry parameters associated to the supercharges Q_α and S_α respectively. More details about the supersymmetry transformations can be found in appendix C.

As mentioned earlier, the explicit supersymmetry transformations can be found by restricting the $\mathcal{N} = (2,2)$ superconformal transformations to the $SU(2|1)_A$ subalgebra. The $SU(2|1)_A$ supersymmetry transformations of the vector multiplet fields are

$$\delta\lambda = \left(iV_m\gamma^m - \mathrm{D}\right)\epsilon \tag{2.17}$$

$$\delta\bar{\lambda} = \left(i\bar{V}_m\gamma^m + \mathrm{D}\right)\bar{\epsilon} \tag{2.18}$$

$$\delta A_\mu = -\frac{i}{2}\left(\bar{\epsilon}\gamma_\mu\lambda + \epsilon\gamma_\mu\bar{\lambda}\right) \tag{2.19}$$

$$\delta\sigma_1 = \frac{1}{2}\left(\bar{\epsilon}\lambda - \epsilon\bar{\lambda}\right) \tag{2.20}$$

$$\delta\sigma_2 = -\frac{i}{2}\left(\bar{\epsilon}\gamma_{\hat{3}}\lambda + \epsilon\gamma_{\hat{3}}\bar{\lambda}\right) \tag{2.21}$$

$$\begin{aligned}\delta\mathrm{D} = &-\frac{i}{2}\bar{\epsilon}\left(\not{D}\lambda + [\sigma_1, \lambda] - i\left[\sigma_2, \gamma^{\hat{3}}\lambda\right]\right) \\ &+\frac{i}{2}\epsilon\left(\not{D}\bar{\lambda} - [\sigma_1, \bar{\lambda}] - i\left[\sigma_2, \gamma^{\hat{3}}\bar{\lambda}\right]\right)\,,\end{aligned} \tag{2.22}$$

[10] Where twisted masses correspond to background values of σ_1, σ_2 in the vector multiplet for G_F.

[11] Thus named since the defining equation $\nabla_\mu\epsilon = \gamma_\mu\tilde{\epsilon}$ is conformally invariant.

with V_m and $\bar{V}_m$ defined by

$$\begin{aligned} V^\mu &= \varepsilon^{\mu\nu} D_\nu \sigma_2 + D^\mu \sigma_1 \,, \quad V^3 = \frac{1}{2}\varepsilon^{\mu\nu} F_{\mu\nu} + i\,[\sigma_1, \sigma_2] + \frac{1}{r}\sigma_1 \\ \bar{V}^\mu &= \varepsilon^{\mu\nu} D_\nu \sigma_2 - D^\mu \sigma_1 \,, \quad \bar{V}^3 = \frac{1}{2}\varepsilon^{\mu\nu} F_{\mu\nu} - i\,[\sigma_1, \sigma_2] + \frac{1}{r}\sigma_1 \,. \end{aligned} \tag{2.23}$$

The transformations of the massless chiral multiplet fields are

$$\delta\phi = \bar{\epsilon}\psi \tag{2.24}$$

$$\delta\bar{\phi} = \epsilon\bar{\psi} \tag{2.25}$$

$$\delta\psi = i\left(\not{D}\phi + \sigma_1\phi - i\sigma_2\phi\gamma^{\hat{3}} + \frac{q}{2r}\phi\gamma^{\hat{3}}\right)\epsilon + \bar{\epsilon}F \tag{2.26}$$

$$\delta\bar{\psi} = i\left(\not{D}\bar{\phi} + \bar{\phi}\sigma_1 + i\bar{\phi}\sigma_2\gamma^{\hat{3}} - \frac{q}{2r}\bar{\phi}\gamma^{\hat{3}}\right)\bar{\epsilon} + \epsilon\bar{F} \tag{2.27}$$

$$\delta F = -i\left(D_\mu\psi\gamma^\mu + \sigma_1\psi - i\sigma_2\psi\gamma^{\hat{3}} + \lambda\phi + \frac{q}{2r}\psi\gamma^{\hat{3}}\right)\epsilon \tag{2.28}$$

$$\delta\bar{F} = -i\left(D_\mu\bar{\psi}\gamma^\mu + \bar{\psi}\sigma_1 + i\bar{\psi}\sigma_2\gamma^{\hat{3}} - \bar{\phi}\bar{\lambda} - \frac{q}{2r}\bar{\psi}\gamma^{\hat{3}}\right)\bar{\epsilon}\,. \tag{2.29}$$

The supersymmetry transformations of the theory with twisted masses are obtained from equations (2.24–2.29) by shifting $\sigma_2 \to \sigma_2 + m$ as in (2.11).

With these transformations, the $SU(2|1)_A$ supersymmetry algebra (2.2) is realized off-shell on the vector multiplet and chiral multiplets fields. Splitting $\delta \equiv \delta_\epsilon + \delta_{\bar{\epsilon}}$, we find that this representation of $SU(2|1)_A$ on the fields obeys

$$[\delta_\epsilon, \delta_\epsilon] = 0 \qquad\qquad [\delta_{\bar{\epsilon}}, \delta_{\bar{\epsilon}}] = 0\,, \tag{2.30}$$

and[12]

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \delta_{SU(2)}(v) + \delta_R(\alpha) + \delta_G(\Lambda) + \delta_{G_F}(\Lambda_m)\,, \tag{2.31}$$

thus generating an infinitesimal $SU(2)\times R\times G\times G_F$ transformation. When localizing the path integral of $\mathcal{N}=(2,2)$ gauge theories on S^2, we will choose a particular supercharge $\mathcal{Q}$ in $SU(2|1)_A$. The $SU(2)\times R\times G\times G_F$ transformation it generates will play an important role in our computation of the partition function.

[12]The explicit form of the commutator of supersymmetry transformations on the vector multiplet and chiral multiplet fields can be found in appendix C.

The $SU(2)$ isometry transformation induced by the commutator of supersymmetry transformations is parametrized by the Killing vector field[13]

$$v^\mu = i\bar{\epsilon}\gamma^\mu\epsilon \,. \tag{2.32}$$

It acts on the bosonic fields via the usual Lie derivative and on the fermions via the Lie-Lorentz derivative

$$\mathcal{L}_v \equiv v^\mu \nabla_\mu + \frac{1}{4}\nabla_\mu v_\nu \gamma^{\mu\nu} \,. \tag{2.33}$$

The $U(1)$ R-symmetry transformation generated by the commutator of the supersymmetry transformations is parametrized by

$$\alpha = -\frac{1}{2r}\bar{\epsilon}\gamma^{\hat{3}}\epsilon \,. \tag{2.34}$$

It acts on the fields by multiplication by the corresponding charge. The $U(1)$ R-symmetry charges of the various fields, supercharges and parameters are given by:

supersymmetry				vector multiplet						chiral multiplet					
ϵ	$\bar{\epsilon}$	Q	S	A_μ	σ_1	σ_2	λ	$\bar{\lambda}$	D	ϕ	ψ	F	$\bar{\phi}$	$\bar{\psi}$	$\bar{F}$
1	-1	-1	1	0	0	0	1	-1	0	$-q$	$-(q-1)$	$-(q-2)$	q	$q-1$	$q-2$

Since the action of R on the fields is non-chiral, this classical symmetry is not spoiled by quantum anomalies and is an exact symmetry of the $\mathcal{N}=(2,2)$ gauge theories we have constructed.

The commutator of two supersymmetry transformations generates a field dependent gauge transformation, taking values in the Lie algebra of the gauge group G. The induced gauge transformation is labeled by the gauge parameter

$$\Lambda = (\bar{\epsilon}\epsilon)\sigma_1 - i(\bar{\epsilon}\gamma^{\hat{3}}\epsilon)\sigma_2 - v^\mu A_\mu \,, \tag{2.35}$$

which acts on the various fields by the standard gauge redundancy transformation laws. On the gauge field it acts by

$$\delta_\Lambda A_\mu = D_\mu \Lambda \tag{2.36}$$

[13]The fact that v is a Killing vector, that it obeys $\nabla^\mu v^\nu + \nabla^\nu v^\mu = 0$, is a consequence of the choice of conformal Killing spinors in (2.16). As desired, it does not generate conformal transformations of S^2.

while on a field φ it acts by

$$\delta_\Lambda \varphi = i\Lambda \cdot \varphi\,, \tag{2.37}$$

where Λ acts on φ in the corresponding representation of G.

Finally, in the presence of twisted masses m, a G_F flavour symmetry rotation on the chiral multiplet fields is generated by $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$. The induced flavour symmetry transformation acts on the chiral multiplet fields in the fundamental representation of G_F, and is parametrized by

$$\Lambda_m = -i(\bar{\epsilon}\gamma^{\hat{3}}\epsilon)m\,, \tag{2.38}$$

with m taking values in the Cartan subalgebra of G_F. It acts trivially on the vector multiplet fields.

2.2 Localization of the Path Integral

In this section our goal is to perform the exact computation of the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2. The powerful tool that allow us to achieve this goal is supersymmetric localization.

The central idea of supersymmetric localization [64] is that the path integral – possibly decorated with the insertion of observables or boundary conditions invariant under a supercharge $\mathcal{Q}$ – localizes to the $\mathcal{Q}$-invariant field configurations. If the orbit of $\mathcal{Q}$ in the space of fields is non-trivial,[14] then the path integral vanishes upon integrating over the associated Grassmann collective coordinate. Therefore, the non-vanishing contributions to the path integral can only arise from the trivial orbits, i.e. the fixed points of supersymmetry. These fixed point field configurations are the solutions to the supersymmetry variation equations generated by the supercharge $\mathcal{Q}$, which we denote by

$$\delta_{\mathcal{Q}}\,\text{fermions} = 0\,. \tag{2.39}$$

In the path integral we must integrate over the moduli space of solutions of the partial differential equations implied by supersymmetry fixed point equations (2.39).

Under favorable asymptotic behavior, integration by parts implies that the result of the path integral does not depend on the deformation of the original supersymmetric

[14]By definition of $\mathcal{Q}$-invariance of the path integral, the space of fields admits the action of $\mathcal{Q}$.

Lagrangian by a $\mathcal{Q}$-exact term[15]

$$\mathcal{L} \to \mathcal{L} + t\, \mathcal{Q} \cdot V\,, \tag{2.40}$$

as long as V is invariant under the bosonic transformations generated by $\mathcal{Q}^2$. Obtaining a sensible path integral requires that the action is nondegenerate and that the path integral is convergent in the presence of the deformation term $\mathcal{Q} \cdot V$.

In the $t \to \infty$ limit, the semiclassical approximation with respect to $\hbar_{\text{eff}} \equiv 1/t$ is exact. In this limit, only the saddle points of $\mathcal{Q} \cdot V$ can contribute and, moreover, the path integral is dominated by the saddle points with vanishing action. However, of all the saddle points of $\mathcal{Q} \cdot V$, only the $\mathcal{Q}$-supersymmetric field configurations give a non-vanishing contribution. Therefore, we must integrate over the intersection of the supersymmetric field configurations and the saddle points of $\mathcal{Q} \cdot V$. We denote this intersection by $\mathcal{F}$.

Using the saddle point approximation, the path integral in the $t \to \infty$ limit can be calculated by restricting the original Lagrangian $\mathcal{L}$ to $\mathcal{F}$,[16] integrating out the quadratic fluctuations of all the fields in the deformation $\mathcal{Q} \cdot V$ expanded around a point in $\mathcal{F}$, and integrating the combined expression over $\mathcal{F}$.[17] Of course, even though the path integral is one-loop exact with respect to t, it yields exact results with respect to the original coupling constants and parameters of the theory.

The final result of the localization computation does not depend on the choice of deformation $\mathcal{Q} \cdot V$. One may add to $\mathcal{Q} \cdot V$ another $\mathcal{Q}$-exact term, and the result of the path integral will not change as long as the new $\mathcal{Q}$-exact term is non-degenerate, and no new supersymmetric saddle points are introduced that can flow from infinity. This can be accomplished by choosing the deformation term such that it does not change the asymptotic behavior of the potential in the space of fields. We will take advantage of this freedom and choose a deformation term $\mathcal{Q} \cdot V$ that makes computations most tractable.

Since our aim is to localize the path integral of gauge theories, some care has to be taken to localize the gauge fixed theory. This requires combining in a suitable way the deformed action $\mathcal{Q} \cdot V$ and gauge fixing terms $\mathcal{L}_{\text{g.f.}}$ into a $\hat{\mathcal{Q}} = \mathcal{Q} + Q_{\text{BRST}}$ exact term $\hat{\mathcal{Q}} \cdot \hat{V}$, where $\hat{V} = V + V_{\text{ghost}}$. This refinement, while technically important, does not modify the

[15] $\mathcal{Q} \cdot V$ denotes the supersymmetry transformation of V generated by $\mathcal{Q}$ (see also (2.67)).

[16] The deformation term $\mathcal{Q} \cdot V$ vanishes on $\mathcal{F}$ since it is a linear combination of the supersymmetry equations.

[17] The original Lagrangian $\mathcal{L}$ is irrelevant for the localization one-loop analysis.

fact that the gauge fixed path integral localizes to $\mathcal{F}$. The inclusion of the gauge fixing term, however, plays an important role in the evaluation of the one-loop determinants in the directions normal to $\mathcal{F}$.

2.2.1 Choice of Supercharge

In this section we choose a particular supersymmetry generator $\mathcal{Q}$ in the $SU(2|1)_A$ supersymmetry algebra with which to localize the path integral of $\mathcal{N}=(2,2)$ gauge theories on S^2. We consider[18]

$$\mathcal{Q} = S_1 + Q_2\,. \tag{2.41}$$

This supercharge generates an $SU(1|1)$ subalgebra of $SU(2|1)_A$, given by

$$\mathcal{Q}^2 = J + \frac{R}{2} \qquad\qquad \left[J + \frac{R}{2}, \mathcal{Q}\right] = 0\,, \tag{2.42}$$

where J is the charge corresponding to a $U(1)$ subgroup of the $SU(2)$ isometry group of the S^2 while R is the R-symmetry generator in $SU(2|1)$. In terms of embedding coordinates where S^2 is parametrized by

$$X_1^2 + X_2^2 + X_3^2 = r^2\,, \tag{2.43}$$

J acts under an infinitesimal transformation, as follows

$$\begin{aligned} X_1 \to & X_1 - \varepsilon X_2 \\ X_2 \to & X_2 + \varepsilon X_1\,. \end{aligned} \tag{2.44}$$

Geometrically, the action of J has two antipodal fixed points on S^2, which can be used to define the north and south poles of S^2. These are located at $(0,0,r)$ and $(0,0,-r)$ in the embedding coordinates (2.43). In terms of the coordinates of the round metric on S^2

$$\mathrm{d}s^2 = r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \tag{2.45}$$

the corresponding Killing vector is

$$i\frac{\partial}{\partial\varphi}\,, \tag{2.46}$$

[18] In section 2.3 we also analyze localization of the path integral with respect to both Q_1 and Q_2. The analysis leads directly to the Coulomb branch representation of the partition function. On the other hand, this other choice does not allow non-trivial field configurations in the Higgs branch, and therefore cannot give rise to the Higgs branch representation of the partition function.

with the north and south poles corresponding to $\theta = 0$ and $\theta = \pi$ respectively. The supersymmetry algebra (2.42) is the same used in [29] in the computation of the partition function of four dimensional $\mathcal{N} = 2$ gauge theories on S^4.

In order to derive the supersymmetry fixed point equations (2.39) generated by the supercharge $\mathcal{Q}$, first we need to construct the conformal Killing spinors associated to it, which we denote by $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$. The conformal Killing spinors on S^2 obeying (2.16) are explicitly given by[19]

$$\begin{aligned} \epsilon &= \exp\left(-\frac{i\theta}{2}\gamma^{\hat{2}}\right) \exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right) \epsilon_\circ \\ \bar{\epsilon} &= \exp\left(+\frac{i\theta}{2}\gamma^{\hat{2}}\right) \exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right) \bar{\epsilon}_\circ \,, \end{aligned} \tag{2.47}$$

where $\epsilon_\circ$ and $\bar{\epsilon}_\circ$ are constant, complex Dirac spinors. The conformal Killing spinors $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$ are given by (2.47), with $\epsilon_\circ$ and $\bar{\epsilon}_\circ$ being chiral spinors of opposite chirality, that is

$$\begin{aligned} \gamma^{\hat{3}}\epsilon_\circ &= +\epsilon_\circ \\ \gamma^{\hat{3}}\bar{\epsilon}_\circ &= -\bar{\epsilon}_\circ \,. \end{aligned} \tag{2.48}$$

Therefore, explicitly

$$\begin{aligned} \epsilon_{\mathcal{Q}} &= e^{i\varphi/2} \exp\left(-\frac{i\theta}{2}\gamma^{\hat{2}}\right) \epsilon_\circ \\ \bar{\epsilon}_{\mathcal{Q}} &= e^{-i\varphi/2} \exp\left(+\frac{i\theta}{2}\gamma^{\hat{2}}\right) \bar{\epsilon}_\circ \,. \end{aligned} \tag{2.49}$$

We note that at the north and the south poles of the S^2 the conformal Killing spinors $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$ have definite chirality, and that the chirality at the north pole is opposite to that at the south pole

$$\begin{aligned} \gamma^{\hat{3}}\epsilon_{\mathcal{Q}}(N) &= \epsilon_{\mathcal{Q}}(N) \qquad & \gamma^{\hat{3}}\epsilon_{\mathcal{Q}}(S) &= -\epsilon_{\mathcal{Q}}(S) \\ \gamma^{\hat{3}}\bar{\epsilon}_{\mathcal{Q}}(N) &= -\bar{\epsilon}_{\mathcal{Q}}(N) \qquad & \gamma^{\hat{3}}\bar{\epsilon}_{\mathcal{Q}}(S) &= \bar{\epsilon}_{\mathcal{Q}}(S) \,. \end{aligned} \tag{2.50}$$

As we shall see, the fact that $\mathcal{Q}$ is chiral at the poles implies that the corresponding chiral field configurations – vortices localized at the north pole and anti-vortices at the south pole – may contribute to the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2.

[19] In the vielbein basis $e^{\hat{1}} = r\mathrm{d}\theta$ and $e^{\hat{2}} = r\sin\theta\mathrm{d}\varphi$. For details, please refer to appendix C.

We note that the circular Wilson loop operator supported on a latitude angle θ_0

$$W_{\theta_\circ} = \operatorname{Tr}\operatorname{Pexp}\oint_{\theta_\circ}[-iA_\mu \mathrm{d}x^\mu + ir(\sigma_1\cos\theta_\circ - i\sigma_2)\mathrm{d}\varphi] \tag{2.51}$$

is invariant under the action of $\mathcal{Q}$. Therefore the expectation value of these operators can be computed when localizing with respect to the supercharge $\mathcal{Q}$.

Given our choice of supercharge $\mathcal{Q}$, we can explicitly determine the infinitesimal $J\times R\times G\times G_F$ transformation that $\mathcal{Q}^2$ generates when acting on the fields. The spinor bilinears constructed from $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$ in section 2.1 evaluate to[20]

$$\begin{aligned} \bar{\epsilon}_{\mathcal{Q}}\epsilon_{\mathcal{Q}} &= i\cos\theta & \qquad v &= \frac{i}{r}\partial_\varphi \\ \bar{\epsilon}_{\mathcal{Q}}\gamma^{\hat{3}}\epsilon_{\mathcal{Q}} &= i & \qquad \alpha &= -\frac{i}{2r}\,. \end{aligned} \tag{2.52}$$

Therefore, in view of (2.44), $\mathcal{Q}^2$ generates $J+R/2$, i.e. a simultaneous infinitesimal rotation and R-symmetry transformation with parameter

$$\varepsilon = \frac{1}{r}\,, \tag{2.53}$$

and a gauge transformation with gauge parameter

$$\Lambda = i\cos\theta\sigma_1 + \sigma_2 - \frac{i}{r}A_2\,. \tag{2.54}$$

On the chiral multiplet fields, $\mathcal{Q}^2$ also induces a G_F flavour symmetry rotation parametrized by the twisted masses m.

2.2.2 Localization Equations

Here we present the key steps in the derivation of the set of partial differential equations that characterize the vector multiplet and chiral multiplet field configurations that are invariant under the action of $\mathcal{Q}$. The details of the derivation are omitted here and can be found in appendix 2.A.

[20]By fixing the overall normalization $\bar{\epsilon}_\circ\epsilon_\circ = i$.

We must identify the partial differential equations implied by (2.39)

$$\delta_{\mathcal{Q}}\lambda = \delta_{\mathcal{Q}}\bar{\lambda} = 0 \tag{2.55}$$

$$\delta_{\mathcal{Q}}\psi = \delta_{\mathcal{Q}}\bar{\psi} = 0\,, \tag{2.56}$$

where $\delta_{\mathcal{Q}} \equiv \delta_{\epsilon_{\mathcal{Q}}} + \delta_{\bar{\epsilon}_{\mathcal{Q}}}$, from the explicit supersymmetry transformations given in equations (2.17, 2.18) and (2.26, 2.27) for the choice of conformal Killing spinors $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$ in (2.49). The moduli space of solutions to these equations, once intersected with the saddle points of our choice of $\mathcal{Q}$-exact deformation term, determines the space of field configurations that need to be integrated over in the path integral.

Given a choice of deformation term, in order for the path integral to converge we need to impose reality conditions on the fields. These reality conditions restrict the contour of path integration so that the integrand falls of sufficiently fast in the asymptotic region in the space of field configurations. The residual freedom in the choice of contour *i.e.* deformations of the contour which do not change the asymptotic behavior of the integrand, is then used to make sure that the contour of integration includes the saddle points of the deformed action.

We are interested in deformation terms that do not alter the asymptotic behavior of the original action (2.3). We may therefore extract the reality conditions by requiring the original path integral for some effective couplings to be convergent.

From the kinetic terms in the bosonic part of the action (2.3) we conclude that the scalar fields σ_1, σ_2 and the connection A_i in the vector multiplet are hermitian while the chiral multiplet complex scalars ϕ and $\bar{\phi}$ satisfy $\bar{\phi} = \phi^\dagger$. Next we note that the path integration over the chiral multiplet auxiliary fields F, $\bar{F}$ is just a Gaussian integral and we simply require $\bar{F} = F^\dagger$. For the convergence of the path integral, one should choose the contour of integration for the auxiliary field D such that $\mathrm{D} + i g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1})$ is hermitian. In other words

$$\operatorname{Im}\mathrm{D} + \mathrm{g}_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1}) = 0\ , \tag{2.57}$$

where the explicit form of the coupling constants g_{eff}^2 and ξ_{eff} are determined by choice of $\mathcal{Q}$-exact deformation terms.

The supersymmetry fixed point equations for the vector multiplet fields (2.55) are given by

$$D_{\hat{2}}\sigma_1 = D_{\hat{i}}\sigma_2 = 0 \qquad\qquad D_{\hat{1}}\sigma_1 + g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1})\sin\theta = 0 \tag{2.58}$$

$$\operatorname{Re}\mathrm{D} = [\sigma_1, \sigma_2] = 0 \qquad\qquad F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} + g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1})\cos\theta = 0\,, \tag{2.59}$$

while the supersymmetry equations for the chiral multiplet fields (2.56) reduce to

$$\cos\frac{\theta}{2}\left(D_{\hat{1}} + iD_{\hat{2}}\right)\phi + \sin\frac{\theta}{2}\left(\sigma_1 - \frac{q}{2r}\right)\phi = 0 \qquad\qquad F = 0 \tag{2.60}$$

$$\sin\frac{\theta}{2}\left(D_{\hat{1}} - iD_{\hat{2}}\right)\phi + \cos\frac{\theta}{2}\left(\sigma_1 + \frac{q}{2r}\right)\phi = 0 \qquad\qquad (\sigma_2 + m)\,\phi = 0\,. \tag{2.61}$$

These differential equations on S^2 are a supersymmetric extension of classic differential equations in physics. Our equations interpolate between BPS vortex equations at the north pole ($\theta = 0$)

$$\begin{aligned}
\left(D_{\hat{1}} + iD_{\hat{2}}\right)\phi &= 0 & D_{\hat{i}}\left(\sigma_1 + i\sigma_2\right) &= 0 \\
F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} + g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1}) &= 0 & \operatorname{Re}\mathrm{D} = [\sigma_1, \sigma_2] &= 0 \\
\left(\sigma_1 + \frac{q}{2r}\right)\phi &= 0 & (\sigma_2 + m)\,\phi &= 0\,,
\end{aligned} \tag{2.62}$$

and BPS anti-vortex equations at the south pole ($\theta = \pi$)

$$\begin{aligned}
\left(D_{\hat{1}} - iD_{\hat{2}}\right)\phi &= 0 & D_{\hat{i}}\left(\sigma_1 + i\sigma_2\right) &= 0 \\
F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} - g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1}) &= 0 & \operatorname{Re}\mathrm{D} = [\sigma_1, \sigma_2] &= 0 \\
\left(\sigma_1 - \frac{q}{2r}\right)\phi &= 0 & (\sigma_2 + m)\,\phi &= 0\,.
\end{aligned} \tag{2.63}$$

This system of differential equations is akin to the one found in [65] in the localization computation of four dimensional $\mathcal{N} = 2$ gauge theories on S^4. We return later to the study of the supersymmetry equations at the poles, which play a crucial role in our analysis, yielding the Higgs branch representation of the gauge theory partition function on S^2.

2.2.3 Vanishing Theorem

As explained previously, the path integral localizes to the space $\mathcal{F}$ of supersymmetric field configurations which are also saddle points of the localizing deformation term. In this section, we consider the supersymmetry equations in the absence of effective FI parameters and we write down the most general *smooth* solutions to the supersymmetry equations for generic values of the R-charges. These solutions are parametrized by the expectation value of fields in the vector multiplet, thus, we denote this space of solutions by $\mathcal{F}_{\text{Coul}}$. In section 2.3 we localize the path integral to $\mathcal{F}_{\text{Coul}}$ and derive the Coulomb branch representation of the partition function.

With $\xi_{\text{eff}} = 0$ and for generic R-charges, the most general smooth solution to the equations (2.58),(2.59),(2.60) and (2.61) is given by[21]

$$\begin{aligned} &A = \frac{B}{2}\left(\kappa - \cos\theta\right)\mathrm{d}\varphi && \sigma_1 = -\frac{B}{2r} && \phi = 0 \\ &\mathrm{D} = 0 && \sigma_2 = a && F = 0\,, \end{aligned} \tag{2.64}$$

where a and B are constant commuting matrices which live in the gauge Lie algebra and its Cartan subalgebra respectively. The matrix B is further restricted by the first Chern class quantization to have integer eigenvalues. The constant κ parametrizes a pure gauge background which is necessary in any coordinate patch which includes one of the poles and can be gauged away in the coordinate patch which excludes the poles.

It is interesting to note that if the R-charge is tuned to be a negative integer or zero, then there are nontrivial solutions of the form

$$\phi = e^{\frac{i}{2}(\kappa B - q)\varphi}\frac{(\sin\frac{\theta}{2})^{\frac{B-q}{2}}}{(\cos\frac{\theta}{2})^{\frac{B+q}{2}}}\phi_\circ \tag{2.65}$$

with $\phi_\circ$ being a constant in the kernel of $a+m$. Imposing regularity at the poles restricts the allowed value of q and B as follows: $q+|B|$ must be even and non-positive integers. In such a case, the above field configuration can be written in terms of the magnetic flux B monopole scalar harmonics $Y^{\frac{B}{2}}_{j,m}$ as

$$\phi = Y^{\frac{B}{2}}_{-\frac{q}{2},-\frac{q}{2}}\phi_\circ\,. \tag{2.66}$$

It is worth mentioning that these field configurations are also supersymmetric configurations in the localization computation of the partition function of three dimensional $\mathcal{N}=2$ gauge theories on $S^1\times S^2$ [62], which computes the superconformal index of these theories.

[21]A detailed derivation of this result is presented in appendix 2.A.

In our computations, we can ignore these discrete, tuned solutions to the supersymmetry equations: for theories flowing to superconformal theories in the infrared, unitarity constrains the R-charges to be non-negative. Furthermore, as will be explained in section 2.3, these solutions are not saddle points of the localized path integral.

We note that even though our choice of $\mathcal{Q}$ breaks the $SU(2)$ symmetry of S^2, the $\mathcal{Q}$-invariant field configurations (2.64) are $SU(2)$ invariant. Later on, we take an alternative approach in which the Coulomb branch is lifted and the saddle point equations admit singular solutions at the poles thereby breaking the $SU(2)$ symmetry. We will consider the physics behind singular solutions localized at the north and south poles of S^2 in section 2.4.

2.3 Coulomb Branch

In order to evaluate the path integral of an $\mathcal{N} = (2,2)$ gauge theory on S^2 using supersymmetric localization, we must choose a deformation of the original supersymmetric Lagrangian by a $\mathcal{Q}$-exact term (2.40)

$$\mathcal{L} \to \mathcal{L} + t\, \delta_{\mathcal{Q}} V\,. \tag{2.67}$$

The deformation term $\delta_{\mathcal{Q}} V$ defines the measure of integration through the associated one-loop determinant. In this section we calculate the contribution to the path integral due to the smooth field configurations (2.64). This yields the Coulomb branch representation of the path integral, as an integral over the Coulomb branch saddle points $\mathcal{F}_{\text{Coul}}$.

A calculation shows that the vector multiplet action (2.4) and the chiral multiplet action (2.10) are $\mathcal{Q}$-exact with respect to our choice of supercharge (2.41). Specifically,

$$(\bar{\epsilon}_{\mathcal{Q}} \gamma^{\hat{3}} \epsilon_{\mathcal{Q}})\, g^2\, \mathcal{L}_{\text{v.m.}} = \delta_{\mathcal{Q}} \delta_{\bar{\epsilon}_{\mathcal{Q}}} \operatorname{Tr} \left(\frac{1}{2} \bar{\lambda} \gamma^{\hat{3}} \lambda - 2i \mathrm{D} \sigma_2 + \frac{i}{r} \sigma_2^2 \right)\,, \tag{2.68}$$

and

$$-\,(\bar{\epsilon}_{\mathcal{Q}} \gamma^{\hat{3}} \epsilon_{\mathcal{Q}})\, (\mathcal{L}_{\text{c.m.}} + \mathcal{L}_{\text{mass}}) = \delta_{\mathcal{Q}} \delta_{\bar{\epsilon}_{\mathcal{Q}}} \operatorname{Tr} \left(\bar{\psi} \gamma^{\hat{3}} \psi - 2\bar{\phi} \left(\sigma_2 + m + i\frac{q}{2r} \right) \phi + \frac{i}{r} \bar{\phi} \phi \right)\,, \tag{2.69}$$

where $\delta_{\mathcal{Q}} \equiv \delta_{\epsilon_{\mathcal{Q}}} + \delta_{\bar{\epsilon}_{\mathcal{Q}}}$. This implies that correlation functions of $\mathcal{Q}$-closed observables in an $\mathcal{N} = (2,2)$ gauge theory on S^2 are independent of g, the Yang-Mills coupling constant.

Despite being g independent, these correlators are nontrivial functions of the renormalized FI parameter $\xi_{\rm ren}$ for each $U(1)$ factor in the gauge group, and of the twisted masses m.

We now turn to the choice of deformation term $\delta_{\mathcal{Q}}V$. The most canonical choice would be to take

$$V_{\rm can} = (\delta_{\mathcal{Q}}\lambda)^\dagger\,\lambda + \left(\delta_{\mathcal{Q}}\bar\lambda\right)^\dagger\bar\lambda + (\delta_{\mathcal{Q}}\psi)^\dagger\,\psi + \left(\delta_{\mathcal{Q}}\bar\psi\right)^\dagger\bar\psi\,. \tag{2.70}$$

For this choice, the bosonic part of the deformation term $\delta_{\mathcal{Q}}V_{\rm can}$ is manifestly non-negative. It is therefore guaranteed that all $\mathcal{Q}$-invariant field configurations are the saddle points of $\delta_{\mathcal{Q}}V_{\rm can}$ with minimal (zero) action. The disadvantage of such a deformation term is that the resulting action $\delta_{\mathcal{Q}}V_{\rm can}$ does not necessarily preserve the $SU(2)$ symmetries of S^2, thus technically complicating the computation of the one-loop determinants in the directions transverse to the $\mathcal{Q}$-invariant field configurations. But as we argued in section 2.2, the result is largely insensitive to the choice of deformation, as long as it is non-degenerate and does not change the asymptotics of the potential in the space of fields. Therefore, we will instead use as the deformation term the technically simpler, $SU(2)$ symmetric, vector multiplet and chiral multiplet actions $\delta_{\mathcal{Q}}V = \mathcal{L}_{\rm v.m.} + \mathcal{L}_{\rm c.m.} + \mathcal{L}_{\rm mass}$. Contrarily to the canonical choice $\delta_{\mathcal{Q}}V_{\rm can}$, the saddle points of $\delta_{\mathcal{Q}}V$ do not coincide with the supersymmetric configurations and thus fully localize the path integral to the intersection.

It is straightforward to show that all Coulomb branch field configurations in $\mathcal{F}_{\rm Coul}$ are saddle points of $\delta_{\mathcal{Q}}V$ and must be integrated over. However, the solutions to the vortex and anti-vortex equations we found at the poles are not saddle points of $\delta_{\mathcal{Q}}V$. This can be demonstrated using both the supersymmetry and the saddle point equations at the poles as follows.[22] Since we are taking the masses to be non-degenerate, it follows from the equations

$$(\sigma_2 + m_I)\phi_I = 0 \tag{2.71}$$

that any pair of distinct non-vanishing vectors ϕ_I and ϕ_J have to be independent. In addition, the above equation combined with the covariant constancy of σ_2 and its equation of motion imply

$$\sum_I (q_I - 1)\phi_I\bar\phi_I = 0\,, \tag{2.72}$$

[22]With some more effort it is possible to prove using only the equation of motion for D that the vortex and anti-vortex configurations are not saddle points of the action in the limit in which the coefficient of the deformation term $\delta_{\mathcal{Q}}V$ goes to infinity.

while the equation of motion for D yields

$$i\mathrm{D} - \sum_I \phi_I \bar{\phi}_I = 0\,. \tag{2.73}$$

However, since all non-vanishing ϕ_I are independent, we can conclude[23] from (2.72) that $\phi_I \bar{\phi}_I$ vanishes for each I. It therefore excludes the aforementioned supersymmetric solutions (2.66) with fine-tuned values of q from the set of saddle points. Combined with (2.73), it also sets $\mathrm{D} = 0$. Plugging this result in the supersymmetry equations fixes $F = -\sigma_1/r = B/2r^2$ and $\sigma_2 = a$ and we recover the Coulomb branch field configurations spanning $\mathcal{F}_{\text{Coul}}$, thus eliminating the vortex and anti-vortex configurations.

The conclusion that the path integral can be written as a integral over just $\mathcal{F}_{\text{Coul}}$ can also be derived as follows. As we remarked earlier, the path integral does not depend on the choice of supercharge $\mathcal{Q}$ used in the localization computation. Therefore, we may instead try to localize the partition function with respect to the supercharges Q_1 and Q_2. This, however, requires finding a deformation term which is Q_1 and Q_2 exact. Such a deformation term is provided by the following terms in the action

$$\mathcal{L}_{\text{v.m.}} + \mathcal{L}_{\text{c.m.}} + \mathcal{L}_{\text{mass}} = \delta_{\epsilon_1}\delta_{\epsilon_2} V'\,, \tag{2.74}$$

with $V' = 1/2\,\mathrm{Tr}(\lambda\lambda + \bar{\phi}F)$, which are exact with respect to both supercharges since $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 0$. In this approach the path integral localizes to the Q_1 and Q_2 invariant field configurations, which are the solutions to the equations

$$\begin{aligned} \delta_{\epsilon_1}\lambda &= \delta_{\epsilon_2}\lambda = 0 \\ \delta_{\epsilon_1}\psi &= \delta_{\epsilon_2}\psi = 0 \\ \delta_{\epsilon_1}\bar{\psi} &= \delta_{\epsilon_2}\bar{\psi} = 0\,. \end{aligned} \tag{2.75}$$

These equations directly lead[24] to the Coulomb branch field configurations (2.64) parametrizing $\mathcal{F}_{\text{Coul}}$ while immediately rendering the vortex and anti-vortex configurations non-supersymmetric. Note that this conclusion is reached by considering the supersymmetry equations alone, contrary to localization with respect to $\mathcal{Q}$, where the saddle point equations

[23]This step requires us to assume that none of the R-charges is 1.

[24]Supersymmetry implies that $V_1 = V_2 = V_3 = \mathrm{D} = 0$. The fact that the solutions to these equations are the Coulomb branch field configurations (2.64) follows by using the equality of actions in (2.4) and (2.6), derived by integrating by parts. Non-trivial chiral multiplet configuration are manifestly non-supersymmetric.

of $\delta_{\mathcal{Q}}V$ also need to be invoked to show that vortex and anti-vortex configurations do not contribute. Since the saddle points and deformation term (2.74) are precisely the same as the one for $\mathcal{Q}$, this guarantees that we obtain the same Coulomb branch representation of the path integral. A drawback of localizing with respect to Q_1 and Q_2 is that we cannot study the expectation value of the circular Wilson loop (2.51) since it is not Q_1 and Q_2 invariant.

In section 2.4 we will obtain the payoff of using the supercharge $\mathcal{Q}$. As we have shown in section 2.2, supersymmetry leads to the vortex and anti-vortex equations at the poles. In that section, we will argue that localizing the path integral $\mathcal{Q}$ in a different limit yields the Higgs branch representation of the partition function.

2.3.1 Integral Representation of the Partition Function

We now can write down the expression of the partition as an integral over the Coulomb branch field configurations $\mathcal{F}_{\text{Coul}}$. The Coulomb branch representation of the partition function is thus given by[25]

$$Z_{\text{Coulomb}}(m,\tau) = \sum_B \int_{\mathfrak{t}} da\, Z_{\text{cl}}(a,B,\tau)\, Z_{\text{one-loop}}(a,B,m)\,, \tag{2.76}$$

where the integral over a has been reduced to the Cartan subalgebra $\mathfrak{t}$ of G. The first factor arises from evaluating the renormalized gauge theory action on the smooth supersymmetric field configurations (2.64)

$$Z_{\text{cl}}(a,B,\tau) = e^{-4\pi i r \xi_{\text{ren}} \operatorname{Tr} a + i\vartheta \operatorname{Tr} B}\,, \tag{2.77}$$

and the one-loop determinant $Z_{\text{one-loop}}(a,B,m)$ specifies the measure of integration over a, which is determined by the deformation term $\delta_{\mathcal{Q}}V$.

Some care has been taken to ensure that the computation, including the regularization of the one-loop determinants $Z_{\text{one-loop}}(a,B,m)$, is $\mathcal{Q}$-invariant. Even though the FI parameter ξ is classically marginal, it runs quantum mechanically according to the renormalization

[25]The partition function has an anomalous dependence on the radius r of the S^2 due to the conformal anomaly in two dimensions. We do not retain this factor throughout our formulae, which can be extracted from our one-loop determinants.

group equation

$$\xi(\mu) = \xi + \frac{1}{2\pi}\sum_j Q_j \ln\left(\frac{\mu}{M_{\rm UV}}\right) = \frac{1}{2\pi}\sum_j Q_j \ln\left(\frac{\mu}{\Lambda}\right) , \tag{2.78}$$

where Q_j is the charge of the j-th chiral multiplet under the $U(1)$ gauge group corresponding to ξ, $M_{\rm UV}$ is the ultraviolet cutoff, μ is the floating scale and Λ is the renormalization group invariant scale. A simple way of performing this renormalization in a $\mathcal{Q}$-invariant way, is to enrich the theory one is interested in with an “expectator” chiral multiplet of mass M and charge $-Q = -\sum_j Q_j$, so that in the enriched theory the FI parameter does not run. Now, to extract the result for the theory of interest, we take the answer of the finite theory in the limit where M is very large, thereby decoupling the expectator chiral multiplet. This procedure results in a $\mathcal{Q}$-invariant ultraviolet cutoff M for the theory under study. As shown in appendix 2.C, taking M large in the one-loop determinant (2.82) for the expectator chiral multiplet precisely reproduces the running of the FI parameter (2.78) with $M_{\rm UV} = M$ and $\mu = \varepsilon = 1/r$. That is, the renormalized coupling obtained in this way is evaluated at the inverse radius of the S^2, which is the infrared scale of S^2

$$\xi_{\rm ren} \equiv \xi\left(\mu = 1/r\right)\big|_{M_{\rm UV}=M} = \xi + \frac{1}{2\pi}\sum_i Q_i \ln\left(\frac{\varepsilon}{M}\right) . \tag{2.79}$$

The one-loop factor in the localization computation $Z_{\text{one-loop}}(a, B, m)$ takes the form

$$Z_{\text{one-loop}}(a, B, m) = Z^{\text{v.m.}}_{\text{one-loop}}(a, B) \cdot Z^{\text{c.m.}}_{\text{one-loop}}(a, B, m) \cdot \mathcal{J}(a, B) , \tag{2.80}$$

where the Jacobian factor $\mathcal{J}(a, B)$ accounts for the reduction of the integral over all a such that $[a, B] = 0$ to an integral over the Cartan subalgebra $\mathfrak{t}$. The magnetic flux B over the S^2 breaks the gauge symmetry G down to a subgroup $H_B = \{g \in G \mid gBg^{-1} = B\}$. Therefore, the associated Jacobian factor is

$$\mathcal{J}(a, B) = \frac{1}{|\mathcal{W}(H_B)|}\prod_{\substack{\alpha\in\Delta^+\\ \alpha\cdot B=0}} (\alpha\cdot a)^2 , \tag{2.81}$$

where $\alpha \in \Delta^+$ are positive roots of the Lie algebra of G and $|W(H_B)|$ is the order of the Weyl group of H_B.

The one-loop determinants for our choice of deformation term $\delta_{\mathcal{Q}} V$, which is the sum of (2.68) and (2.69), are computed in appendix 2.B. For a chiral multiplet in a reducible representation $\mathbf{R} = \oplus_I \mathbf{r}_I$ we obtain

$$Z^{\text{c.m.}}_{\text{one-loop}}(a, B, m) = \prod_I \prod_{w_I \in \mathbf{r_I}} (-i)^{w_I \cdot B} (-1)^{|w_I \cdot B|/2} \frac{\Gamma\left(\frac{q_I}{2} - ir(w_I \cdot a + m_I) + \frac{|w_I \cdot B|}{2}\right)}{\Gamma\left(1 - \frac{q_I}{2} + ir(w_I \cdot a + m_I) + \frac{|w_I \cdot B|}{2}\right)}, \tag{2.82}$$

where w_I are the weights of the representation $\mathbf{r}_I$ and $\Gamma(x)$ is the Euler gamma function. The twisted masses and R-charges m_I and q_I of the chiral multiplets, which take values in the Cartan subalgebra of the flavour symmetry G_F, combine into the holomorphic combination $M = m + \frac{i}{2r} q$ introduced in (2.14).

For the vector multiplet contribution we obtain

$$Z^{\text{v.m.}}_{\text{one-loop}}(a, B) = \prod_{\substack{\alpha \in \Delta^+ \\ \alpha \cdot B \neq 0}} \left[\left(\frac{\alpha \cdot B}{2r} \right)^2 + (\alpha \cdot a)^2 \right]. \tag{2.83}$$

We note that the Jacobian factor and the vector multiplet determinant combine nicely into an unconstrained product over the positive roots of the Lie algebra

$$Z^{\text{v.m.}}_{\text{one-loop}}(a, B) \cdot J(a, B) = \frac{1}{|\mathcal{W}(H_B)|} \prod_{\alpha \in \Delta^+} \left[\left(\frac{\alpha \cdot B}{2r} \right)^2 + (\alpha \cdot a)^2 \right]. \tag{2.84}$$

The Coulomb branch representation of the partition function of an $\mathcal{N} = (2, 2)$ gauge theory on S^2 is thus given by

$$\begin{aligned} Z_{\text{Coulomb}}(m, \tau) = \sum_B \frac{1}{|\mathcal{W}(H_B)|} \int_{\mathfrak{t}} da \, e^{-4\pi i \xi_{\text{ren}} r \operatorname{Tr} a + i\vartheta \operatorname{Tr} B} \prod_{\alpha \in \Delta^+} \left[\left(\frac{\alpha \cdot B}{2r} \right)^2 + (\alpha \cdot a)^2 \right] \\ \times \prod_{I, w_I} \left[(-i)^{w_I \cdot B} (-1)^{|w_I \cdot B|/2} \frac{\Gamma\left(-ir(w_I \cdot a + \mathrm{M}_I) + \frac{|w_I \cdot B|}{2}\right)}{\Gamma\left(1 + ir(w_I \cdot a + \mathrm{M}_I) + \frac{|w_I \cdot B|}{2}\right)} \right]. \end{aligned} \tag{2.85}$$

The expectation value of the circular Wilson loop (2.51) is obtained enriching the integrand in (2.85) with the insertion of

$$\operatorname{Tr} e^{2\pi a - i\pi B}. \tag{2.86}$$

2.3.2 Factorization of the Partition Function

We show in this subsection that the Coulomb branch representation of the partition function (2.85) can be written as a discrete sum, whose summand factorizes into the product of two functions. A related factorization was found previously by Pasquetti [66] when evaluating the partition function of three dimensional $\mathcal{N}=2$ abelian gauge theories on the squashed S^3.[26]

We recognize the expression we obtain as the sum over Higgs vacua of the product of the vortex partition function due to vortices at the north pole with the anti-vortex partition function due to the anti-vortices at the south pole. This result is interpreted in section 2.4 as a direct path integral evaluation of the partition function, where the path integral is argued to localize on vortices and anti-vortices in the Higgs branch.

Let us consider for definiteness the case of two dimensional $\mathcal{N}=(2,2)$ SQCD. This theory has $G=U(N)$ gauge group and N_F fundamental chiral multiplets and $\widetilde{N}_F$ anti-fundamental chiral multiplets. The partition function (2.85) of this theory is[27]

$$Z_{\text{SQCD}}^{U(N)} = \frac{1}{N!}\sum_{B_1,\ldots,B_N\in\mathbb{Z}}\int \mathrm{d}a_1\cdots\mathrm{d}a_N\Bigg\{e^{-4\pi i\xi\,\mathrm{Tr}\,a}e^{i\vartheta\,\mathrm{Tr}\,B}\prod_{i<j}\left[(a_i-a_j)^2+\left(\frac{B_i-B_j}{2}\right)^2\right]$$
$$\cdot\prod_{s=1}^{N_F}\prod_{i=1}^{N}\frac{(-1)^{\frac{|B_i|+B_i}{2}}\Gamma(-ia_i-i\mathrm{M}_s+|B_i|/2)}{\Gamma(1+ia_i+i\mathrm{M}_s+|B_i|/2)}\prod_{s=1}^{\widetilde{N}_F}\prod_{i=1}^{N}\frac{(-1)^{\frac{|B_i|-B_i}{2}}\Gamma(ia_i-i\widetilde{\mathrm{M}}_s+|B_i|/2)}{\Gamma(1-ia_i+i\widetilde{\mathrm{M}}_s+|B_i|/2)}\Bigg\}. \tag{2.87}$$

In the large a limit, the integrand is of order $|a|^{N(N-1)+N\sum_I(q_I-1)}$, hence this N-dimensional integral is convergent as long as

$$\sum_{s=1}^{N_F}q_s+\sum_{s=1}^{\widetilde{N}_F}\widetilde{q}_s<N_F+\widetilde{N}_F-N\,. \tag{2.88}$$

In the cases where $N_F>\widetilde{N}_F$, or $N_F=\widetilde{N}_F$ and $\xi>0$, the contour can be closed towards $ia_i\to+\infty$, enclosing poles of the fundamental multiplets' one-loop determinants; the contour must be chosen to enclose poles of the anti-fundamental multiplets' one-loop

[26] The partition function of three dimensional gauge theories on $S^2\times S^1$ can also be factorized [67].

[27] Without loss of generality we set $r=1$ to unclutter formulas. It can easily be restored by dimensional analysis.

determinants in cases where $N_F < \widetilde{N}_F$, or $N_F = \widetilde{N}_F$ and $\xi < 0$. Assuming that all R-charges are positive, or deforming the integration contour to ensure that we enclose the same set of poles, this expresses the Coulomb branch integral as a sum of the residues at combined poles

$$ia_i = -i\mathrm{M}_{p_i} + n_i + \frac{|B_i|}{2} \quad \text{for all } 1 \le i \le N, \tag{2.89}$$

with $1 \le p_1, \dots, p_N \le N_F$ and $n_1, \dots, n_N \ge 0$ labelling the poles. The resulting ratios of Gamma functions in the integrand can be recast in terms of Pochhammer raising factorials $(x)_n = x(x+1)\cdots(x+n-1)$ as

$$\frac{\Gamma(i\mathrm{M}_{p_i} - i\mathrm{M}_s - n_i)}{\Gamma(1 + i\mathrm{M}_s - i\mathrm{M}_{p_i} + |B_i| + n_i)} = \frac{\gamma(i\mathrm{M}_{p_i} - i\mathrm{M}_s)(-1)^{n_i}}{(1 + i\mathrm{M}_s - i\mathrm{M}_{p_i})_{n_i}(1 + i\mathrm{M}_s - i\mathrm{M}_{p_i})_{n_i + |B_i|}}\,, \tag{2.90}$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,, \tag{2.91}$$

and similarly for the ratios of Gamma functions coming from the anti-fundamental chiral multiplets.

The symmetry between n_i and $n_i + |B_i|$ in (2.90) leads us to introduce new coordinates

$$k_i^{\pm} = n_i + [B_i]^{\pm} = n_i + |B_i|/2 \pm B_i/2 \ge 0 \tag{2.92}$$

on the summation lattice, such that $\{n_i, n_i + |B_i|\} = \{k_i^{\pm}\}$. In section 2.4, the N integers k_i^+ will be interpreted as labelling vortices located at the north pole, and k_i^- anti-vortices at the south pole. More precisely, $k_i^{\pm}$ measures the amount of vortex and anti-vortex charge carried by the i-th Cartan generator in $U(N)$: note that the flux $B_i = k_i^+ - k_i^-$.

This change of coordinates decouples the sums over $k^+ \ge 0$ and $k^- \ge 0$ and yields the following expression after converting signs to a shift in the theta angle

$$\begin{aligned} Z_{\mathrm{SQCD}}^{U(N)} = \frac{(2\pi)^N}{N!} \sum_{p_1,\dots,p_N=1}^{N_F} \Bigg[& e^{4\pi\xi \sum_j i\mathrm{M}_{p_j}} \prod_{i=1}^{N} \frac{\prod_{s=1}^{\widetilde{N}_F} \gamma(-i\widetilde{\mathrm{M}}_s - i\mathrm{M}_{p_i})}{\prod_{s\neq p_i}^{N_F} \gamma(1 + i\mathrm{M}_s - i\mathrm{M}_{p_i})} \\ & \cdot \sum_{k_i^+ \ge 0} \left[e^{(2\pi i\tau + i\pi N_F)\sum_i k_i^+} \prod_{i<j}^{N} \left(\mathrm{M}_{p_j} - \mathrm{M}_{p_i} + ik_j^+ - ik_i^+\right) \prod_{i=1}^{N} \frac{\prod_{s=1}^{\widetilde{N}_F}(-i\widetilde{\mathrm{M}}_s - i\mathrm{M}_{p_i})_{k_i^+}}{\prod_{s=1}^{N_F}(1 + i\mathrm{M}_s - i\mathrm{M}_{p_i})_{k_i^+}} \right] \\ & \cdot \sum_{k_i^- \ge 0} \left[e^{(-2\pi i\bar\tau + i\pi \widetilde{N}_F)\sum_i k_i^-} \prod_{i<j}^{N} \left(\mathrm{M}_{p_j} - \mathrm{M}_{p_i} + ik_j^- - ik_i^-\right) \prod_{i=1}^{N} \frac{\prod_{s=1}^{\widetilde{N}_F}(-i\widetilde{\mathrm{M}}_s - i\mathrm{M}_{p_i})_{k_i^-}}{\prod_{s=1}^{N_F}(1 + i\mathrm{M}_s - i\mathrm{M}_{p_i})_{k_i^-}} \right] \Bigg]. \end{aligned} \tag{2.93}$$

Terms with $p_a = p_b$ for some $a \neq b \leq N$ vanish, because the sum over k^+ is then antisymmetric under the exchange of k_a^+ and k_b^+. We can thus normalize the series as

$$\begin{aligned} f(\{p_i\}, \mathrm{M}, z) &= \sum_{k_i \geq 0} \left[z^{\sum_i k_i} \prod_{i<j}^{N} \frac{i\mathrm{M}_{p_j} - i\mathrm{M}_{p_i} + k_i - k_j}{i\mathrm{M}_{p_j} - i\mathrm{M}_{p_i}} \frac{\prod_{s=1}^{\widetilde{N}_F} \prod_{i=1}^{N} (-i\widetilde{\mathrm{M}}_s - i\mathrm{M}_{p_i})_{k_i}}{\prod_{s=1}^{N_F} \prod_{i=1}^{N} (1 + i\mathrm{M}_s - i\mathrm{M}_{p_i})_{k_i}} \right] \\ &= \sum_{k_i \geq 0} \left[\frac{z^{\sum_i k_i}}{\prod_i k_i!} \frac{\prod_{s=1}^{\widetilde{N}_F} \prod_{i=1}^{N} (-i\mathrm{M}_{p_i} - i\widetilde{\mathrm{M}}_s)_{k_i}}{\prod_{i \neq j}^{N} (i\mathrm{M}_{p_j} - i\mathrm{M}_{p_i} - k_j)_{k_i} \prod_{s \notin \{p\}}^{N_F} \prod_{i=1}^{N} (1 + i\mathrm{M}_s - i\mathrm{M}_{p_i})_{k_i}} \right], \end{aligned} \tag{2.94}$$

which as we will see in the next section, corresponds to the vortex partition function studied in [68], with $z = \exp(2\pi i \tau)$ playing the role of the vortex fugacity. Note that this series converges for all z (all ξ) if $N_F > \widetilde{N}_F$, and for $|z| < 1$ (that is, $\xi > 0$) if $N_F = \widetilde{N}_F$, consistent with the constraints required by our choice of contour. All in all, the partition function factorizes as

$$Z_{\mathrm{SQCD}}^{U(N)} = \sum_{\substack{v_i = -\mathrm{M}_{p_i} \\ 1 \leq p_1 < \ldots < p_N \leq N_F}} Z_{\mathrm{cl}}(v, 0, \tau) \operatorname*{res}_{a=v} Z_{\text{one-loop}}(a, 0, \mathrm{M})\, f(\{p_i\}, \mathrm{M}, (-1)^{N_F} z)\, f(\{p_i\}, \mathrm{M}, (-1)^{\widetilde{N}_F} \bar{z}) \tag{2.95}$$

with

$$\operatorname*{res}_{a_i = -\mathrm{M}_{p_i}} Z_{\text{one-loop}}(a, 0, \mathrm{M}) = \prod_{i=1}^{N} \frac{\prod_{s=1}^{\widetilde{N}_F} \gamma(-i\widetilde{\mathrm{M}}_s - i\mathrm{M}_{p_i})}{\prod_{s \notin \{p\}}^{N_F} \gamma(1 + i\mathrm{M}_s - i\mathrm{M}_{p_i})} \tag{2.96}$$

up to a constant factor. In the next section we obtain this result directly by localizing the path integral to Higgs branch configurations with vortices and anti-vortices. In the matching, some care must be taken when comparing the mass parameters of the gauge theory on the sphere with the parameters describing the theory in the Ω-background used to evaluate the vortex partition function.

The final expression we find is reminiscent of the discrete sums of the product of holomorphic and anti-holomorphic conformal blocks that appear in correlators of the A_{N_F-1} Toda CFT in the presence of completely degenerate fields. A precise matching between the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2 and correlators in Toda is provided in the abelian case in [37], and in the case of $U(N)$ in [69].

Note that this factorization result applies to any gauge group G with an abelian factor and matter representation **R**.[28] This yields a representation of the path integral that can

[28] See [37] for details.

be interpreted as a sum over Higgs vacua of terms factorized into holomorphic and anti-holomorphic contributions, corresponding to vortices and anti-vortices respectively. These formulas motivate natural conjectures for the vortex partition functions corresponding to gauge theories with gauge group G. In the absence of U(1) factors in the gauge group, the factorization can be carried out formally, but the two factors may be divergent series.

Note that this factorization result applies to any group of the form $U(1)\times G$. This yields a representation of the path integral that can be interpreted as a sum over Higgs vacua of terms factorized into holomorphic and anti-holomorphic contributions, corresponding to vortices and anti-vortices respectively. These formulas motivate natural conjectures for the vortex partition functions corresponding to gauge theories with gauge group $U(1) \times G$. In the absence of $U(1)$ factors in the gauge group, the factorization can be carried out formally, and the two factors collapse into two identical, possibly divergent series.

2.4 Higgs Branch Representation

The localization principle, under mild conditions, guarantees that the path integral does not depend either on the choice of supercharge $\mathcal{Q}$ or on the choice of V in the deformation term. But different choices can lead to different representations of the same path integral and therefore to non-trivial identities.

In section 2.3 we have derived a representation of the partition function as an integral over Coulomb branch vacua. In section 2.3.2, by explicitly evaluating the integral, we have demonstrated that the partition function also has an alternative representation as a sum – in the Higgs phase – over vortex and anti-vortex field configurations localized at the poles.

This section aims to derive from path integral localization arguments the Higgs branch representation of the partition function. This representation should have a direct derivation using localization. The appropriate choice of supercharge to use to obtain this representation is the same supercharge $\mathcal{Q}$ introduced in (2.41), since it has the elegant feature of giving rise to the vortex equations at the north pole

$$\begin{aligned} (D_{\hat{1}} + iD_{\hat{2}})\,\phi &= 0 & \qquad D_{\hat{i}}\,(\sigma_1 + i\sigma_2) &= 0 \\ F_{\hat{1}\hat{2}} + \sigma_1 + g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1}) &= 0 & \qquad \operatorname{Re}\mathrm{D} = [\sigma_1, \sigma_2] &= 0 \\ \left(\sigma_1 + \frac{q}{2}\right)\phi &= 0 & \qquad (\sigma_2 + m)\,\phi &= 0\,, \end{aligned} \tag{2.97}$$

and anti-vortex equations at the south pole

$$\begin{aligned} (D_{\hat{1}} - iD_{\hat{2}})\,\phi &= 0 & \qquad D_{\hat{i}}\,(\sigma_1 + i\sigma_2) &= 0 \\ F_{\hat{1}\hat{2}} + \sigma_1 - g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1}) &= 0 & \qquad \operatorname{Re}\mathrm{D} = [\sigma_1, \sigma_2] &= 0 \\ \left(\sigma_1 - \frac{q}{2}\right)\phi &= 0 & \qquad (\sigma_2 + m)\,\phi &= 0\,. \end{aligned} \tag{2.98}$$

We remark that when the effective Fayet-Iliopoulos parameters are non-vanishing, these equations admit solutions with non-vanishing ϕ. These solutions then restrict σ_2 to be a diagonal matrix with the masses of the excited chiral fields on the diagonal and the Coulomb branch configurations (2.64) parametrizing $\mathcal{F}_{\text{Coul}}$ are lifted. The $\mathcal{Q}$-invariant field configurations admitted by (2.97) and (2.98) are vortex and anti-vortex configurations at the north and south pole of the S^2. Since vortices and anti-vortices exist in the Higgs phase, we denote this space of supersymmetric field configurations that must be integrated over by $\mathcal{F}_{\text{Higgs}}$.

2.4.1 Localizing onto the Higgs Branch

In this subsection we present a heuristic argument to introduce non-zero FI parameters in the localization computation, which as explained above yields to a representation of the path integral as a sum over vortex and anti-vortex configurations. For the purpose of this argument, we take all the R-charges to be zero.

Recall that our choice of deformation term $\delta_{\mathcal{Q}}V = \mathcal{L}_{\text{v.m.}} + \mathcal{L}_{\text{c.m.}} + \mathcal{L}_{\text{mass}}$ does not include a FI term. In section 2.3, we performed the saddle point approximation after taking the $t \to \infty$ limit. In this limit, the effective FI parameter vanishes and the saddle point equations forbid vortices, hence the path integral localizes to $\mathcal{F}_{\text{Coul}}$. Instead, we assume here that there is another choice of $\mathcal{Q}$-exact deformation terms V' leading to a non-vanishing effective FI parameter $\xi_{\text{eff}} \neq 0$ in the $t \to \infty$ limit.

The equation of motion for the D field arising from the deformed action $S + t\delta_{\mathcal{Q}}V'$ is

$$ig_{\text{eff}}^{-2}\mathrm{D} + \xi_{\text{eff}} - \sum_I \phi_I\bar{\phi}_I = 0. \tag{2.99}$$

On the space of $\mathcal{Q}$-supersymmetric field configurations (see section 2.2.3), D vanishes in the bulk and we conclude that

$$\sum_I \phi_I\bar{\phi}_I = \xi_{\text{eff}}\mathbb{1}_N\,, \tag{2.100}$$

which, together with $(a+m_I)\phi_I = 0$ imply that the Coulomb branch is lifted, localizing instead to the Higgs branch. Moreover the supersymmetry equations at the poles yield

$$\sigma_1 \phi_I \bar{\phi}_I = -\frac{B}{2}\phi_I \bar{\phi}_I = 0 \qquad (2.101)$$

which by virtue of (2.100) imply $B = \sigma_1 = 0$. This leads us directly to the vortex and anti-vortex equations at the north and the south poles.

The contribution of vortices and anti-vortices to the partition function of an $\mathcal{N} = (2,2)$ gauge theory on S^2 can be obtained as follows. Since the vortices and anti-vortices are localized at the poles, these can be studied by restricting the $\mathcal{N} = (2,2)$ gauge theory to the local $\mathbb{R}^2$ flat space near the north and south poles of S^2. Asymptotic infinity of each $\mathbb{R}^2$ is identified with a small latitude circle on S^2 close to the north and south pole respectively. Therefore, the contribution of vortices and anti-vortices is captured by the vortex/anti-vortex partition function of the gauge theory obtained by restricting our $\mathcal{N} = (2,2)$ gauge theory at the poles. As we will see in section 2.4.2, integrating over vortex and anti-vortex configurations for all Higgs branch vacua exactly reproduces the partition function computed by integrating over the Coulomb branch found in section 2.3.2.

A more precise and complete approach to obtain a finite FI parameter is to choose V' such that $\delta_{\mathcal{Q}} V'$ reintroduces a linear D-term into the new deformation action $\delta_{\mathcal{Q}}(V+V')$.[29] In the $t \to \infty$ limit, saddle points of this deformation action would lead directly to vortex and anti-vortex equations. It would be interesting to find this alternative and more rigorous way to localize to the Higgs branch.

2.4.2 Vortex Partition Function

Following the discussion in the last subsection, in the planes glued to the poles and in the presence of the FI parameter, the supersymmetry equations reduce to

$$(D_1 + iD_2)\,\phi_I = 0\,, \qquad (\sigma_2 + m_I)\phi_I = 0\,, \qquad F_{12} + \sum_I \phi_I \bar{\phi}_I - \xi_{\text{eff}} = 0\,, \qquad (2.102)$$

in the plane attached to the north pole, and

$$(D_1 - iD_2)\,\phi_I = 0\,, \qquad (\sigma_2 + m_I)\phi_I = 0\,, \qquad F_{12} - \sum_I \phi_I \bar{\phi}_I + \xi_{\text{eff}} = 0\,, \qquad (2.103)$$

[29] See [40] for a choice of V'.

in the copy of $\mathbb{R}^2$ attached to the south pole. These equations can be recognized as the differential equations describing supersymmetric vortices and anti-vortices in $\mathcal{N} = (2,2)$ supersymmetric gauge theories. Therefore, in our localization computation we must integrate over the moduli space of solutions of vortices at the north pole and anti-vortices at the south pole. For simplicity, we discuss their contribution to the partition function for $\mathcal{N} = (2,2)$ SQCD with $U(N)$ gauge group and N_F fundamental chiral multiplets and $\widetilde{N}_F$ anti-fundamental chiral multiplets.

Since the vortices and anti-vortices exist only in the Higgs phase, let us first work out the vacuum structure in the Higgs phase. We first note that vortices can only exist in vacua in which the anti-fundamental fields vanish. This follows from the known mathematical result that the vortex equations for an anti-fundamental field have no non-zero smooth solution when the background field is a connection of a bundle with positive first Chern class $c_1 = k > 0$. The vortex equations (2.102) and (2.103) then imply that exactly N chiral multiplets take non-zero values, and diagonalizing $\sigma_2 = \mathrm{diag}(a_1, \cdots, a_N)$, one obtains that each Higgs branch of solutions to these equations is labelled by a set of distinct integers $1 \le p_1 < \cdots < p_N \le N_F$, with

$$a_i + m_{p_i} = 0 \qquad i = 1, \ldots, N\,, \tag{2.104}$$

up to permutations of integers p_i. The contribution from vortices and anti-vortices depends on the choice of Higgs branch components. In each of these components, the $U(N) \times S[U(N_F) \times U(\widetilde{N}_F)]$ symmetry of the theory is broken to

$$S[U(N)_{\mathrm{diag}} \times U(N_F - N)] \times U(1) \times SU(\widetilde{N}_F)\,, \tag{2.105}$$

where $U(1)$ rotates fundamental and anti-fundamental chiral multiplets equally.

For a given Higgs branch component labeled by $\{p_i\}$, the familiar vortex equations (2.102) admit a multidimensional moduli space of solutions which we denote by $\mathcal{M}^{\{p_i\}}_{\mathrm{vortex}}$. Since the vorticity

$$k = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{Tr} F \tag{2.106}$$

is quantized, this moduli space splits into disconnected components $\mathcal{M}^{\{p_i\},k}_{\mathrm{vortex}}$, each of which is a Kähler manifold, of dimension $2kN_F$. Taking into account the south pole anti-vortex contributions, we find that the solutions of the localization equations on S^2 span the moduli space

$$\mathcal{F}_{\text{Higgs}} = \bigsqcup_{\{p_i\}} \left[\cup_{k=0}^{\infty} \mathcal{M}_{\text{vortex}}^{\{p_i\},k}\right] \oplus \left[\cup_{l=0}^{\infty} \mathcal{M}_{\text{anti-vortex}}^{\{p_i\},l}\right] . \tag{2.107}$$

We now argue that the vortex partition function at the poles is captured by the partition function of the $\mathcal{N} = (2,2)$ gauge theory in the Ω-background, which is a supersymmetric deformation of the $\mathcal{N} = (2,2)$ gauge theory in $\mathbb{R}^2$ by a $U(1)_\varepsilon$ equivariant rotation parameter ε. Let us recall that the supercharge with which we localize an $\mathcal{N} = (2,2)$ gauge theory on S^2 obeys

$$\mathcal{Q}^2 = J + \frac{1}{2}R\,. \tag{2.108}$$

The key observation is to note that (2.108) is precisely the supersymmetry preserved by an $\mathcal{N} = (2,2)$ gauge theory in $\mathbb{R}^2$ when placed in the Ω-background. The rotation generator in the Ω-background corresponds to $J + \frac{1}{2}R$, thus giving rise to the scalar supercharge under $U(1)_\varepsilon$ preserved by an $\mathcal{N} = (2,2)$ theory in the Ω-background. Therefore, the contribution to the partition function of an $\mathcal{N} = (2,2)$ gauge theory on S^2 due to vortices and anti-vortices localized at the poles is captured by the vortex/anti-vortex partition function of the same gauge theory placed in the Ω-background originally studied by Shadchin [68] (see also [70–74]).

The vortex partition function in the Higgs branch component $\{p_i\}$ of an $\mathcal{N} = (2,2)$ gauge theory in the Ω-background is obtained by performing the functional integral of that theory around the background field configuration of k vortices, and summing over all k. It admits an expansion

$$Z_{\text{vortex}}(\{p_i\}, M^\Omega, \widetilde{M}^\Omega, z_\Omega) = \sum_{k=0}^{\infty} z_\Omega^k Z_k(\{p_i\}, M^\Omega, \widetilde{M}^\Omega)\,, \tag{2.109}$$

where $z_\Omega = \exp(2\pi i \tau_\Omega)$ is the vortex fugacity and $Z_k(\{p_i\}, M^\Omega, \widetilde{M}^\Omega)$ is the equivariant volume of the moduli space of k vortices. The volume is given by

$$Z_k(\{p_i\}, M^\Omega, \widetilde{M}^\Omega) = \int_{\mathcal{M}_{\text{vortex}}^{\{p_i\},k}} e^{\hat{\omega}}\,, \tag{2.110}$$

where $\hat{\omega}$ is the $U(1)_\varepsilon$ equivariant closed Kähler form[30] on $\mathcal{M}_{\text{vortex}}^{\{p_i\},k}$. Our computations of the supersymmetry transformations on S^2 in section 2.2.1 imply that the equivariant rotation

[30]The form $\hat{\omega}$ is also equivariant under the action of the residual symmetry of the vacuum over which vortices are considered. See (2.112).

parameter ε for the Ω-background theory induced at the poles is given in terms of the radius of the S^2 by

$$\varepsilon = \frac{1}{r} \,. \tag{2.111}$$

It is pleasing that the $\mathcal{N} = (2, 2)$ theory near the poles yields the Ω-deformed theory, since the integral (2.110) for the $\mathcal{N} = (2, 2)$ theory in flat space suffers from ambiguities, such as infrared divergences. Fortunately, a closer inspection of the $\mathcal{N} = (2, 2)$ gauge theory on S^2 near the poles cures this problem, yielding finite, unambiguous results. In fact, the Ω-deformation was first introduced to regularize otherwise infrared divergent volume integrals such as (2.110).

The vortex partition function of an $\mathcal{N} = (2, 2)$ gauge theory in the Ω-background can be computed from the knowledge of the symplectic quotient construction of the vortex moduli space $\mathcal{M}^{\{p_i\},k}_{\text{vortex}}$ given in [75, 76]. Some details of this construction are presented in appendix 2.D. The volume (2.110) is then given by the matrix integral of a supersymmetric matrix theory action with $U(k)$ gauge group. This matrix theory can be obtained by dimensionally reducing a certain two dimensional $\mathcal{N} = (0, 2)$ $U(k)$ gauge theory to zero dimensions. This supersymmetric matrix theory inherits the supercharge $\mathcal{Q}$ of the $\mathcal{N} = (2, 2)$ theory in the Ω-background as well as an equivariant

$$U(1)_\varepsilon \times S[U(N)_{\text{diag}} \times U(N_F - N)] \times U(1) \times SU(\widetilde{N}_F) \tag{2.112}$$

symmetry. The first factor $U(1)_\varepsilon$ is the rotational symmetry of the Ω-background while the rest is the residual symmetry of the vacuum over which vortices are studied. The integral (2.110) receives contributions from isolated points in the vortex moduli space $\mathcal{M}^{\{p_i\},k}_{\text{vortex}}$, corresponding to the $\mathcal{Q}$-invariant configurations. These are labeled by a partition of k into N non-negative integers

$$k = \sum_{i=1}^{N} k_i \,. \tag{2.113}$$

To each such partition we associate an N-component vector $\vec{k} = (k_1, \ldots, k_N)$, describing how the total vortex number k is distributed among the N Cartan generators in $U(N)$ at this point.

For the choice of Higgs branch component of the $\mathcal{N} = (2, 2)$ gauge theory labelled by integers $\{p_i\} \subseteq \{1, \ldots, N_F\}$, the partition function of k-vortices admits the following contour integral representation [68, 77] (see appendix 2.D for details),

$$Z_k(\{p_i\}, M^\Omega, \widetilde{M}^\Omega) = \oint_{\Gamma_{\{p_i\},k}} \prod_{I=1}^{k} \frac{d\varphi_I}{2\pi i} \; \mathcal{Z}_{\text{vec}}(\varphi) \cdot \mathcal{Z}_{\text{fund}}(M^\Omega, \varphi) \cdot \mathcal{Z}_{\text{anti-fund}}(\widetilde{M}^\Omega, \varphi) \tag{2.114}$$

with

$$\mathcal{Z}_{\text{vec}}(\varphi) = \frac{1}{k! \; \varepsilon^k} \prod_{I \neq J}^{k} \frac{\varphi_I - \varphi_J}{\varphi_I - \varphi_J - \varepsilon} \tag{2.115}$$

$$\mathcal{Z}_{\text{fund}}(M^\Omega, \varphi) = \prod_{I=1}^{k} \prod_{s=1}^{N_F} \frac{1}{\varphi_I - M_s^\Omega} \tag{2.116}$$

$$\mathcal{Z}_{\text{anti-fund}}(\widetilde{M}^\Omega, \varphi) = \prod_{I=1}^{k} \prod_{t=1}^{\widetilde{N}_F} \left(\varphi_I + \widetilde{M}_t^\Omega\right) . \tag{2.117}$$

For each Higgs vacuum $\{p_i\}$ and vorticity $\vec{k}$, the integrand in (2.114) admits a pole at

$$\varphi_{(i,l)} = M_{p_i}^\Omega + (l-1)\varepsilon \qquad l = 1, 2, .., k_i \qquad i = 1, \ldots, N \,, \tag{2.118}$$

and the contour of integration $\Gamma_{\{p_i\},k}$ is carefully chosen to enclose all such poles for $\sum_{i=1}^{N} k_i = k$, and no other. The poles of (2.114) can be understood as the location of the fixed points under the action of $\mathcal{Q}$. Each factor in (2.114) reflects the contribution of the vortex collective coordinates associated to each of the $\mathcal{N} = (2,2)$ multiplets: the vector multiplet and fundamental and anti-fundamental chiral multiplets. Note here that the mass parameters in the Ω-background theory can be identified with the mass parameters of the theory on S^2,

$$M_{p_i}^\Omega = -i m_{p_i} \;, \qquad M_s^\Omega = -\varepsilon - i m_s \; (s \notin \{p_i\}) \;, \qquad \widetilde{M}_s^\Omega = -i \widetilde{m}_s \,. \tag{2.119}$$

We observe the same shift in masses as for $\mathcal{N} = 2$ gauge theories on S^4 found in [78]. Performing the contour integral and summing over all vortex charges $\vec{k}$, the vortex partition function for SQCD takes the following form

$$Z_{\text{vortex}}(\{p_i\}, m, \widetilde{m}, z) = \sum_{k_1 + \cdots + k_N = k} z^{|\vec{k}|} Z_{\vec{k}}(\{p_i\}, m, \widetilde{m}) \,, \tag{2.120}$$

with

$$Z_{\vec{k}}(\{p_i\}, m, \widetilde{m}) = \frac{1}{\prod_i k_i!} \frac{\prod_{s=1}^{\widetilde{N}_F} \prod_{i=1}^{N} (-irm_{p_i} - ir\widetilde{m}_s)_{k_i}}{\prod_{i \neq j} (irm_{p_j} - irm_{p_i} - k_j)_{k_i} \prod_{s \notin \{p\}}^{N_F} \prod_{i=1}^{N} (1 + irm_s - irm_{p_i})_{k_i}} \,. \tag{2.121}$$

This expression exactly agrees[31] with the expression (2.94) arising from factorization of the Coulomb branch representation of the partition function on S^2. Anti-vortices localized at the south pole provide an identical contribution, expanded in terms of the anti-vortex fugacity $\bar{z}$. The one loop determinant must be evaluated at the location of the Higgs branches, where there is a zero mode. Removing the zero mode amounts to taking the residue of the one-loop determinant. Summing over Higgs branch components finally leads to the Higgs branch representation of the partition function of $\mathcal{N}=(2,2)$ gauge theories on S^2

$$Z_{\text{Higgs}}(m,\tau)=\sum_{\substack{v_i=-m_{p_i}\\ \{p_i\}\subseteq\{1,\dots,N_F\}}} Z_{\text{cl}}(v,0,\tau)\ \underset{a=v}{\text{res}}\left[Z_{\text{one-loop}}(a,0,m)\right]\left|Z_{\text{vortex}}(\{p_i\},m,(-1)^{N_F}z)\right|^2_* \,, \quad (2.122)$$

where

$$\left|Z_{\text{vortex}}(\{p_i\},m,(-1)^{N_F}z)\right|^2_*=Z_{\text{vortex}}(\{p_i\},m,(-1)^{N_F}z)Z_{\text{vortex}}(\{p_i\},m,(-1)^{\widetilde{N}_F}\bar{z})\,. \quad (2.123)$$

This matches with the Coulomb branch representation of the partition function computed earlier.

[31]One must analytically continue the twisted masses $m\to\mathrm{M}$ and $\widetilde{m}\to\widetilde{\mathrm{M}}$ to restore non-zero R-charges.

Appendix

2.A Supersymmetric Configurations

In this appendix we present the derivation of the choice of SUSY parameters and the corresponding supersymmetric configurations.

2.A.1 Choice of Supercharge

The conformal Killing spinor equations on S^2 are

$$\nabla_\mu \epsilon = +\frac{1}{2r}\gamma_\mu \gamma^{\hat{3}}\epsilon \ , \tag{2.124}$$

$$\nabla_\mu \bar{\epsilon} = -\frac{1}{2r}\gamma_\mu \gamma^{\hat{3}}\bar{\epsilon} \ , \tag{2.125}$$

with the general solutions of the form

$$\epsilon = \exp\left(-\frac{i\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right)\epsilon_\circ \ , \tag{2.126}$$

$$\bar{\epsilon} = \exp\left(+\frac{i\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right)\bar{\epsilon}_\circ \ . \tag{2.127}$$

Here, the hatted γ indices denote the tangent space (flat) indices[32]. The corresponding bilinear $v^\mu = i\bar{\epsilon}\gamma^\mu\epsilon$ is given by

$$v^1 = \cos\varphi\left(i\bar{\epsilon}_\circ\hat{\gamma}^1\epsilon_\circ\right) + \sin\varphi\left(i\bar{\epsilon}_\circ\hat{\gamma}^2\epsilon_\circ\right) \ , \tag{2.128}$$

$$v^2 = \bar{\epsilon}_\circ\epsilon_\circ - \cot\theta\sin\varphi\left(i\bar{\epsilon}_\circ\hat{\gamma}^1\epsilon_\circ\right) + \cot\theta\cos\varphi\left(i\bar{\epsilon}_\circ\hat{\gamma}^2\epsilon_\circ\right) \ . \tag{2.129}$$

[32] See appendix A.

We wish to find spinors such that v^1 vanishes while v^2 is a non-zero constant. The vanishing on v^1 for all angles φ requires $\bar{\epsilon}_\circ \gamma^1 \epsilon_\circ = \bar{\epsilon}_\circ \gamma^2 \epsilon_\circ = 0$. This can be achieved by choosing $\epsilon_\circ$ and $\bar{\epsilon}_\circ$ to be chiral spinors with opposite chirality. We choose the constant spinors such that

$$\gamma^{\hat{3}} \epsilon_\circ = +\epsilon_\circ \ , \tag{2.130}$$

$$\gamma^{\hat{3}} \bar{\epsilon}_\circ = -\bar{\epsilon}_\circ \ , \tag{2.131}$$

and the conformal Killing spinors reduce to

$$\epsilon = \exp\left(-\frac{i\theta}{2}\gamma_{\hat{2}} + \frac{i\varphi}{2}\right)\epsilon_\circ \ , \tag{2.132}$$

$$\bar{\epsilon} = \exp\left(+\frac{i\theta}{2}\gamma_{\hat{2}} - \frac{i\varphi}{2}\right)\bar{\epsilon}_\circ \ . \tag{2.133}$$

The spinor bilinears constructed out of these spinors take the form

$$\bar{\epsilon}\epsilon = \bar{\epsilon}_\circ \epsilon_\circ \cos\theta \ , \tag{2.134}$$

$$v = \frac{1}{r}\bar{\epsilon}_\circ \epsilon_\circ \frac{\partial}{\partial \varphi} \ , \tag{2.135}$$

$$\alpha = -\frac{1}{2r}\bar{\epsilon}_\circ \epsilon_\circ \ . \tag{2.136}$$

2.A.2 SUSY Saddle Point Equations

Since after localization, only supersymmetric configurations can contribute, we write $\mathcal{Q}f = 0$ for all fermionic fields, with $\mathcal{Q}$ parametrized by the particular choice of ϵ and $\bar{\epsilon}$ we just derived. Let us fix the relative normalization of $\epsilon_\circ$ and $\bar{\epsilon}_\circ$ such that

$$\bar{\epsilon}_\circ = -i\gamma^{\hat{2}}\epsilon_\circ \tag{2.137}$$

We thus obtain the explicit expressions

$$\epsilon = e^{i\varphi/2}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \qquad \bar{\epsilon} = e^{-i\varphi/2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \tag{2.138}$$

$$\gamma^{\hat{1}}\epsilon = e^{i\varphi/2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \qquad \gamma^{\hat{1}}\bar{\epsilon} = e^{-i\varphi/2}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \tag{2.139}$$

$$\gamma^{\hat{2}}\epsilon = e^{i\varphi/2}\left(-i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \qquad \gamma^{\hat{2}}\bar{\epsilon} = e^{-i\varphi/2}\left(-i\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \tag{2.140}$$

$$\gamma^{\hat{3}}\epsilon = e^{i\varphi/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \qquad \gamma^{\hat{3}}\bar{\epsilon} = e^{-i\varphi/2}\left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \tag{2.141}$$

Thanks to those expressions for various gamma matrices acting on our conformal Killing spinors, $\delta\lambda = 0$ and $\delta\bar{\lambda} = 0$ may be written as

$$\begin{aligned} 0 = \delta\lambda = &\left[\sin\frac{\theta}{2}\left(iV_{\hat{1}} + V_{\hat{2}}\right) + i\cos\frac{\theta}{2}\left(V_{\hat{3}} + i\mathrm{D}\right)\right] e^{i\frac{\varphi}{2}}\epsilon_0 \\ &+ \left[\cos\frac{\theta}{2}\left(V_{\hat{1}} + iV_{\hat{2}}\right) - \sin\frac{\theta}{2}\left(V_{\hat{3}} - i\mathrm{D}\right)\right] e^{i\frac{\varphi}{2}}\gamma^{\hat{2}}\epsilon_0 \end{aligned} \tag{2.142}$$

$$\begin{aligned} 0 = \delta\bar{\lambda} = &\left[\cos\frac{\theta}{2}\left(i\bar{V}_{\hat{1}} + \bar{V}_{\hat{2}}\right) + i\sin\frac{\theta}{2}\left(\bar{V}_{\hat{3}} - i\mathrm{D}\right)\right] e^{-i\frac{\varphi}{2}}\epsilon_0 \\ &+ \left[\sin\frac{\theta}{2}\left(\bar{V}_{\hat{1}} + i\bar{V}_{\hat{2}}\right) - \cos\frac{\theta}{2}\left(\bar{V}_{\hat{3}} + i\mathrm{D}\right)\right] e^{-i\frac{\varphi}{2}}\gamma^{\hat{2}}\epsilon_0. \end{aligned} \tag{2.143}$$

while $\delta\psi = 0$ and $\delta\bar{\psi} = 0$ yields

$$\begin{aligned} 0 = \delta\psi = i&\left[\sin\frac{\theta}{2}\left(D_-\phi - ie^{-i\varphi}F\right) + \cos\frac{\theta}{2}\left(\sigma_1 - i\sigma_2 + \frac{q}{2r}\right)\phi\right] e^{i\frac{\varphi}{2}}\epsilon_\circ \\ &+ \left[\cos\frac{\theta}{2}\left(D_+\phi - ie^{-i\varphi}F\right) + \sin\frac{\theta}{2}\left(\sigma_1 + i\sigma_2 - \frac{q}{2r}\right)\phi\right] e^{i\frac{\varphi}{2}}\gamma^{\hat{2}}\epsilon_\circ, \end{aligned} \tag{2.144}$$

$$\begin{aligned} 0 = \delta\bar{\psi} = i&\left[\cos\frac{\theta}{2}\left(D_-\bar{\phi} - ie^{i\varphi}\bar{F}\right) + \sin\frac{\theta}{2}\,\bar{\phi}\left(\sigma_1 + i\sigma_2 + \frac{q}{2r}\right)\right] e^{-i\frac{\varphi}{2}}\epsilon_\circ \\ &+ \left[\sin\frac{\theta}{2}\left(D_+\bar{\phi} - ie^{i\varphi}\bar{F}\right) + \cos\frac{\theta}{2}\,\bar{\phi}\left(\sigma_1 - i\sigma_2 + \frac{q}{2r}\right)\right] e^{-i\frac{\varphi}{2}}\gamma^{\hat{2}}\epsilon_\circ\,. \end{aligned} \tag{2.145}$$

Here $D_\pm = D_{\hat{1}} \pm iD_{\hat{2}}$ and for future reference, we define $\sigma_\pm = \sigma_1 \pm i\sigma_2$. Since $\epsilon_\circ$ and $\gamma^{\hat{2}}\epsilon_\circ$ are linearly independent, each square bracket must vanish separately. Using the reality conditions

$$\begin{aligned} A_\mu^\dagger &= A_\mu & \qquad \bar{\phi}^\dagger &= \phi \\ \sigma_\pm^\dagger &= \sigma_\mp & \qquad \bar{F}^\dagger &= F \end{aligned} \tag{2.146}$$

we can write the equations as

$$\begin{aligned} \sin\frac{\theta}{2} D_\pm \sigma_+ + \cos\frac{\theta}{2}\left(F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} + i\mathrm{D} \mp i\left[\sigma_1, \sigma_2\right]\right) &= 0 \\ \cos\frac{\theta}{2} D_\pm \sigma_- - \sin\frac{\theta}{2}\left(F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} - i\mathrm{D} \pm i\left[\sigma_1, \sigma_2\right]\right) &= 0 \end{aligned} \tag{2.147}$$

$$\begin{aligned} \sin\frac{\theta}{2}\left(D_-\phi \pm i e^{-i\varphi} F\right) + \cos\frac{\theta}{2}\left(\sigma_\mp + \frac{q}{2r}\right)\phi &= 0 \\ \cos\frac{\theta}{2}\left(D_+\phi \pm i e^{-i\varphi} F\right) + \sin\frac{\theta}{2}\left(\sigma_\pm - \frac{q}{2r}\right)\phi &= 0\,. \end{aligned} \tag{2.148}$$

Taking linear combinations of each set of these equations and using the reality conditions, we obtain the desired SUSY equations

$$\begin{aligned} D_{\hat{2}}\sigma_1 = D_{\hat{2}}\sigma_2 = D_{\hat{1}}\sigma_2 &= 0 & \qquad \mathrm{Re}\,\mathrm{D} = [\sigma_1, \sigma_2] &= 0 \\ D_{\hat{1}}\sigma_1 - \mathrm{Im}\,\mathrm{D}\sin\theta &= 0 & \qquad F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} - \mathrm{Im}\,\mathrm{D}\cos\theta &= 0\,, \end{aligned} \tag{2.149}$$

$$\begin{aligned} \cos\frac{\theta}{2} D_+\phi + \sin\frac{\theta}{2}\left(\sigma_1 - \frac{q}{2r}\right)\phi &= 0 & \qquad \sigma_2\phi &= 0 \\ \sin\frac{\theta}{2} D_-\phi + \cos\frac{\theta}{2}\left(\sigma_1 + \frac{q}{2r}\right)\phi &= 0 & \qquad F &= 0\,. \end{aligned} \tag{2.150}$$

2.A.3 $\mathcal{Q}$-Supersymmetric Field Configurations

To compute the path integral using localization on supersymmetric configurations, we need to find the space of solutions of equations (2.149) and (2.150).

Let us first analyze the vector multiplet field equations.

For concreteness, we choose the coordinate patch $0 < \theta < \pi$, where we can gauge away the $\mathrm{d}\theta$-component of the gauge field[33]. The general solution to (2.149) takes the form

$$A = r\sigma_1 \cos\theta\ \mathrm{d}\varphi, \qquad \sigma_1 = \sigma_1(\theta), \qquad \sigma_2 = \sigma_2(\varphi)\,. \tag{2.151}$$

[33]Every 1-form $w = w_\theta \mathrm{d}\theta$ on S^2 is, up to $\mathrm{d}\varphi$ terms, closed and therefore exact – since the $H^1(S^2) = 0$.

Imposing the chiral multiplet supersymmetry equations (2.150) and plugging in the above form for the vector multiplet fields we obtain

$$\begin{aligned} \left(\sin\theta\ \partial_\theta + \frac{q}{2}\cos\theta + \sigma_1\right)\phi &= 0 \qquad\qquad & F &= 0 \\ \left(\partial_\varphi + i\frac{q}{2}\right)\phi &= 0 & (\sigma_2 + m)\phi &= 0 \end{aligned} \tag{2.152}$$

where we have also included the mass term which, as explained in section 2.1 is just a shift in σ_2 by a diagonal matrix valued in the flavor symmetry group. For generic values of R-charges q, the only solution of the above equations which is periodic in φ is

$$\phi = 0\,. \tag{2.153}$$

Consequently, in the absence of *effective* Fayet-Iliopoulos parameters[34], the reality conditions necessary for having a convergent path integral constrain the vector multiplet auxiliary field to vanish, *i.e.*

$$\Im \mathrm{D} = -g^2\phi\bar{\phi} = 0\,. \tag{2.154}$$

The vanishing of the auxiliary field in turn forces σ_1 to be a constant and the general solution to the supersymmetry equations (2.149) and (2.150) takes the form

$$\begin{aligned} A &= \frac{B}{2}(\kappa - \cos\theta)\mathrm{d}\varphi \qquad\qquad & \sigma_1 &= -\frac{B}{2r} \\ \sigma_2 &= a & \mathrm{D} &= 0 \\ \phi &= \bar{\phi}^\dagger = 0 & F &= \bar{F}^\dagger = 0 \end{aligned} \tag{2.155}$$

where $\delta A = \frac{\kappa B}{2}\mathrm{d}\varphi$ is the appropriate gauge transformation to extend the solution to the coordinate patches including the north pole (with $\kappa = 1$) or the south pole (where $\kappa = -1$). We conclude that for general R-charge assignments, $\mathcal{F}_0$ – the space of smooth solutions to the supersymmetry fixed point equations – is parametrized by two constant matrices, a & B, where B is further constrained by the first Chern class quantization to take integer values.

[34]To localize the path integral, we need to add to the action a $\mathcal{Q}$-exact deformation term with an arbitrary parameter t which we then take to ∞. The effective FI parameters are then ξ/t which vanish in the $t \to \infty$ limit.

We note in passing that for special values of the R-charges, there exist non-trivial solutions to the chiral multiplet supersymmetry equations which take the form

$$\phi = e^{\frac{i}{2}(\kappa B-q)\varphi}\frac{(\sin\frac{\theta}{2})^{\frac{B-q}{2}}}{(\cos\frac{\theta}{2})^{\frac{B+q}{2}}}\phi_\circ, \qquad \text{subject to} \qquad (a+m)\phi_\circ = 0\,. \tag{2.156}$$

2.B One-Loop Determinants

Here we present the computation of the one-loop determinants in the localization computation of the partition function. Our starting point is the quadratic part of the vector and chiral multiplet actions (2.4) and (2.10) in the background (2.64) with the addition of the gauge fixing ghosts $\bar{c}, c$ and the Lagrange multiplier b. The various terms are

$$\begin{aligned} S_{\mathrm{b}}^{\text{v.m.}} = \int \mathrm{d}^2x\sqrt{h}\,\mathrm{Tr}\Big\{ & A^\mu\left(\mathrm{M}^2+\frac{1}{r^2}\right)A_\mu + \frac{i}{2r^2}\varepsilon_{\mu\nu}A^\mu\big[B,A^\nu\big] + \frac{2}{r}\sigma_1\varepsilon^{\mu\nu}D_\mu A_\nu \\ & + \sigma_1\left(\mathrm{M}^2+\frac{1}{r^2}\right)\sigma_1 + \sigma_2\mathrm{M}^2\sigma_2 + \mathrm{D}^2 - \mathcal{G}^2\Big\}, \end{aligned} \tag{2.157}$$

$$S_{\mathrm{f}}^{\text{v.m.}} = \int \mathrm{d}^2x\sqrt{h}\,\mathrm{Tr}\left\{\bar\lambda\left(i\not{D} - \frac{i}{2r}\big[B,\cdot\,\big] + \gamma^{\hat{3}}\big[a,\cdot\,\big]\right)\lambda\right\}, \tag{2.158}$$

$$S_{\text{ghost}} = \int \mathrm{d}^2x\sqrt{h}\,\mathrm{Tr}\left\{\bar{c}\mathrm{M}^2c - b\,\mathcal{G}(A_i,\sigma_1,\sigma_2)\right\}, \tag{2.159}$$

$$S_{\mathrm{b}}^{\text{c.m.}} = \int \mathrm{d}^2x\,\sqrt{h}\quad\left\{\bar\phi\left(\mathrm{M}^2 + i\frac{q-1}{r}a - \frac{q^2-2q}{4r^2}\right)\phi + \bar{F}F\right\}, \tag{2.160}$$

$$S_{\mathrm{f}}^{\text{c.m.}} = \int \mathrm{d}^2x\,\sqrt{h}\quad\left\{\bar\psi\left(-i\not{D} - \frac{i}{2r}B - \left(a+\frac{iq}{2r}\right)\gamma^{\hat{3}}\right)\psi\right\}, \tag{2.161}$$

where $\mathcal{G}$ is the gauge fixing condition corresponding to the choice of gauge

$$\mathcal{G}(A_i,\sigma_1,\sigma_2) = D_\mu A^\mu + \frac{i}{2r}\big[B,\sigma_1\big] - i\big[a,\sigma_2\big] = 0, \tag{2.162}$$

and M^2 is given by

$$\mathrm{M}^2 = -D_\mu^2 + \frac{1}{4r^2}B^2 + a^2\,, \tag{2.163}$$

where a and B act in the appropriate representations. We note that (2.162) is the background gauge field choice $D_M A^M = 0$ in four dimensions dimensionally reduced to two dimensions. This choice simplifies computations considerably.

The integral over b imposes the background field gauge (2.162) while integrating out the auxiliary fields D and F yields a trivial factor. We now analyze the rest.

2.B.1 Dirac Operator in Monopole Background

Before computing the one-loop determinant contribution of fermionic fields, let us first derive the spectrum of the Dirac operator in the background (2.64). Since the index of the Dirac operator, acting in the representation $\mathbf{R}$ of the gauge algebra, is given by

$$\mathrm{ind}(\not{D}) = \frac{1}{2\pi}\int_{S^2} \mathrm{Tr}\, F = \mathrm{Tr}\, B\,, \tag{2.164}$$

we anticipate $|\,\mathrm{Tr}\, B|$ zero-modes. Excluding these modes, we may diagonalize the Dirac operator using spinor monopole harmonics. For each weight w of the representation $\mathbf{R}$ and each mode (J, m) such that $J > |B_w|/2$ and $-J \leq m \leq J$ we have

$$(i\not{D})_{J,m} = \begin{pmatrix} \lambda_{J,m} & 0 \\ 0 & -\lambda_{J,m} \end{pmatrix} \tag{2.165}$$

since $i\not{D}$ is traceless. The spectrum of $i\not{D}$ can easily be derived from the spectrum of $-\not{D}^2$ when expressed in terms of the scalar Laplacian

$$(i\not{D})^2 = \begin{pmatrix} -(D_\mu^-)^2 + \frac{1-B_w}{2r^2} & 0 \\ 0 & -(D_\mu^+)^2 + \frac{1+B_w}{2r^2} \end{pmatrix}. \tag{2.166}$$

Here $(D_\mu^\pm)^2 \equiv (\partial_\mu - i\frac{B_w \pm 1}{2}\omega_\mu)^2$ denotes the scalar Laplacian in the monopole background with monopole charge $\frac{B_w \pm 1}{2}$. The connection ω_i is expressed in terms of the spin connection (B.5) as $\omega_\mu = \omega_\mu^{\hat{1}\hat{2}}$. In the rest of this subsection, we drop the subscript in B_w to avoid cluttering the notation.

The eigen-value of the scalar Laplacian in the (J, m) mode is given by

$$-(D_\mu^\pm)^2_{J,m} = \frac{J(J+1)}{r^2} - \frac{(B \pm 1)^2}{4r^2}\,, \tag{2.167}$$

where J runs from $\frac{|B\pm1|}{2}$ to ∞ in integer steps and the multiplicity in each mode is $2J+1$. Using this expression for the eigenvalues and the relation between the eigenvalues of the scalar Laplacian, which can be easily read off from (2.165) and (2.166), we conclude that the spectrum of the Dirac operator consists of

$$0, \qquad \text{with multiplicity } |B|, \tag{2.168}$$

$$+\sqrt{\frac{(J+\frac{1}{2})^2-(\frac{B}{2})^2}{r^2}}, \qquad J=\frac{|B|+1}{2},\ldots \qquad \text{with multiplicity } 2J+1, \tag{2.169}$$

$$-\sqrt{\frac{(J+\frac{1}{2})^2-(\frac{B}{2})^2}{r^2}}, \qquad J=\frac{|B|+1}{2},\ldots \qquad \text{with multiplicity } 2J+1. \tag{2.170}$$

We also note that the fermonic zero-modes are spinors of a definite chirality, which depends on the sign of B.

2.B.2 Chiral Multiplet Determinant

Using the spectrum of the Dirac operator we just derived, we can easily compute the fermionic determinant of the chiral multiplet. First, note that $\gamma^{\hat{3}}$ anticommutes with $\not{D}$, hence, a shift in $\not{D}$ by $\gamma^{\hat{3}}$ results in a shift in the square of the eigenvalues. Therefore, we have

$$\begin{aligned}
\det \Delta_{\mathrm{f}}^{\text{c.m.}} &= \det\left[-i\not{D}-\frac{iB}{2r}-\left(a+\frac{iq}{2r}\right)\gamma^{\hat{3}}\right] \\
&= \prod_w (-i)^{B_w}\left(\frac{q+|B_w|}{2r}-ia_w\right)^{|B_w|} \\
&\quad\times \prod_{J=\frac{|B_w|+1}{2}}^{\infty}\left[-\left(\frac{B_w}{2r}\right)^2-\left(\frac{(J+\frac{1}{2})^2-(\frac{B_w}{2})^2}{r^2}+\left(a_w+\frac{iq}{2r}\right)^2\right)\right]^{2J+1} \\
&= \prod_w (-i)^{B_w}\prod_{J=0}^{\infty}\left[\left(\frac{J}{r}+\frac{|B_w|+q}{2r}-ia_w\right)^{2J+|B_w|}\right. \\
&\qquad\qquad \left.\times(-1)^{|B_w|}\left(\frac{J+1}{r}+\frac{|B_w|-q}{2r}+ia_w\right)^{2J+|B_w|+2}\right].
\end{aligned} \tag{2.171}$$

Here we have used the notation $x_w \equiv x \cdot w$, where w are the weights of the representation $\mathbf{R}$ under which the chiral multiplet transforms.

The bosonic determinant may be written as

$$
\begin{aligned}
(\det \Delta_{\mathrm{b}}^{\mathrm{c.m.}})^{\frac{1}{2}} &= \prod_w \prod_{J=\frac{|B_w|}{2}}^{\infty} \left[\left(\frac{J+\frac{1}{2}}{r}\right)^2 + \left(a_w + i\frac{q-1}{2r}\right)^2 \right]^{2J+1} \\
&= \prod_w \prod_{J=0}^{\infty} \left[\left(\frac{J}{r} + \frac{|B_w|+q}{2r} - ia_w\right) \cdot \left(\frac{J+1}{r} + \frac{|B_w|-q}{2r} + ia_w\right) \right]^{2J+|B_w|+1} .
\end{aligned} \tag{2.172}
$$

Putting the two together we have the one-loop contribution from the chiral multiplet fields:

$$
Z^{\mathrm{c.m.}}_{\text{one-loop}}(a,B,m) = \prod_{w\in\mathbf{R}} (-i)^{B_w} \prod_{J=0}^{\infty} (-1)^{|B_w|} \left[\frac{J+1+\frac{|B_w|-q}{2} + ira_w}{J+\frac{|B_w|+q}{2} - ira_w} \right] \tag{2.173}
$$

These infinite products can be regularized using Euler's gamma function

$$
\frac{1}{\Gamma(z)} = \left[\prod_{J=0}^{\infty} (z+J) \right]_{\text{reg}} \tag{2.174}
$$

to yield, in the presence of a twisted mass m introduced by shifting $a \to a+m$

$$
Z^{\mathrm{c.m.}}_{\text{one-loop}}(a,B,m) = \prod_{w\in\mathbf{R}} (-i)^{B_w} (-1)^{|B_w|/2} \frac{\Gamma\left(\frac{q}{2} - ir(a_w+m) + \frac{|B_w|}{2}\right)}{\Gamma\left(1-\frac{q}{2} + ir(a_w+m) + \frac{|B_w|}{2}\right)} . \tag{2.175}
$$

The chiral multiplet determinant has a pole when $a+m$ has a zero and q is a non-positive integer. More precisely, there is a pole whenever $|B| \leq -q$ with $B-q$ even when acting on ϕ. These poles are due to the zero modes found in (2.66), which exist precisely under these conditions. In evaluating the determinant for these tuned values of q, the zero modes must be excluded, thus yielding a finite result.

2.B.3 Vector Multiplet Determinant

The fermion contribution to the vector multiplet one-loop determinant is the same as that of a chiral multiplet in the adjoint representation with R-charge $q = 0$. It is given by

$$\begin{aligned}
\det\Delta_{\mathrm{f}}^{\mathrm{v.m.}} &= \prod_{\alpha\in\Delta}(-i)^{B_\alpha}\prod_{J=0}^{\infty}(-1)^{|B_\alpha|}\left[\left(\frac{J}{r}+\frac{|B_\alpha|}{2r}-ia_\alpha\right)^{2J+|B_\alpha|}\left(\frac{J+1}{r}+\frac{|B_\alpha|}{2r}+ia_\alpha\right)^{2J+|B_\alpha|+2}\right] \\
&= \prod_{\alpha\in\Delta_+}\prod_{J=0}^{\infty}\left\{\left[\left(\frac{J}{r}+\frac{|B_\alpha|}{2r}\right)^2+a_\alpha^2\right]^{2J+|B_\alpha|}\left[\left(\frac{J+1}{r}+\frac{|B_\alpha|}{2r}\right)^2+a_\alpha^2\right]^{2J+|B_\alpha|+2}\right\}.
\end{aligned} \tag{2.176}$$

where $\alpha \in \Delta_+$ are the positive roots of the Lie algebra of G.

In order to compute the contribution from the bosonic fields, we need to write down the mode expansion of the fields. For the scalars fields σ_1 and σ_2, we may use the expansion in the standard scalar monopole harmonics

$$\sigma_s^\alpha = \sum_{J=\frac{|B_\alpha|}{2}}^{\infty}\sum_{m=-J}^{J}\frac{1}{r}\sigma_{s,J,m}^\alpha Y_{J,m}^{\frac{|B.\alpha|}{2}} \tag{2.177}$$

where we have introduced a factor of $\frac{1}{r}$ for normalization and $s = 1, 2$. As for the gauge field, the mode expansion is much more subtle. A basis of monopole vector spherical harmonics is given in [79]. Expanding the gauge field in this basis we find

$$A_\mu^\alpha = \sum_{\lambda=\pm}\sum_{J=J_0^\lambda}^{\infty}\sum_{m=-J}^{J}A_{J,m}^{\alpha,\lambda}\left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)_\mu, \tag{2.178}$$

where $J_0^\pm = \frac{|B_\alpha|}{2} \mp 1$ for $\frac{|B_\alpha|}{2} \geq 1$ and $J_0^\pm = \frac{|B_\alpha|+1}{2} \mp \frac{1}{2}$ otherwise. The reality condition on the gauge field then implies $A_{-\alpha} = A_\alpha^*$ and for scalars $\sigma_{s,-\alpha} = \sigma_{s,\alpha}^*$. The explicit form of $\left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)_\mu$ is not necessary for our computation and will be omitted here. All we need are some basic properties of the basis elements which are

$$\delta_{\lambda}^{\lambda'}\delta_{J}^{J'}\delta_{m}^{m'} = \int \mathrm{d}^2x\sqrt{h}\left(C_{J',m'}^{\lambda',\frac{B_\alpha}{2}}\right)^*_{\mu}\left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)^{\mu} , \tag{2.179}$$

$$-D_\nu^2\left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)^{\mu} = \frac{1}{r^2}\left[J(J+1) - \left(\frac{|B_\alpha|}{2} - \lambda\right)^2\right]\left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)^{\mu} , \tag{2.180}$$

$$D_\mu\left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)^{\mu} = -\frac{1}{\sqrt{2}r^2}\sqrt{J(J+1) - \frac{|B_\alpha|}{2}\left(\frac{|B_\alpha|}{2} - \lambda\right)}Y_{J,m}^{\frac{|B_\alpha|}{2}} , \tag{2.181}$$

$$i\varepsilon_{\mu\nu}\left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)^{\nu} = -\lambda\left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)_{\mu} . \tag{2.182}$$

Using the above expansion for the gauge field and the scalars and performing the integral over S^2, the bosonic part of the vector multiplet action in (2.157) can be written as

$$\begin{aligned} S_{\mathrm{b}}^{\text{v.m.}} \simeq & \sum_{\lambda=\pm}\sum_{J=J_0^\lambda}^{\infty}\sum_{m=-J}^{J} A_{J,m}^{-\alpha,\lambda}\left[\frac{J(J+1)}{r^2} + a_\alpha^2 + \lambda\frac{B_\alpha}{2r^2}\right]A_{J,m}^{\alpha,\lambda} \\ & - \sum_{\lambda=\pm}\sum_{J=\frac{|B_\alpha|}{2}}^{\infty}\sum_{m=-J}^{J}\sigma_{1,J,m}^{-\alpha}i\lambda\sqrt{2}\,\frac{\sqrt{J(J+1) - \frac{|B_\alpha|}{2}\left(\frac{|B_\alpha|}{2} - \lambda\right)}}{r^2}A_{J,m}^{\alpha,\lambda} \\ & + \sum_{s=1,2}\sum_{J=\frac{|B_\alpha|}{2}}^{\infty}\sum_{m=-J}^{J}\sigma_{s,J,m}^{-\alpha}\left[\frac{J(J+1)}{r^2} + a_\alpha^2 + \frac{2-s}{r^2}\right]\sigma_{s,J,m}^{\alpha} , \end{aligned}$$

where there is an implicit summation over all roots $\alpha \in \Delta$.

In order to compute the determinant, it is best to break it down into three factors. The first one isolates the $J = \frac{|B_\alpha|}{2} - 1$ contribution, which is only non-trivial when $\frac{|B_\alpha|}{2} - 1$ is non-negative. In this case we have

$$\det(\Delta_{\mathrm{b},1}^{\text{v.m.}}) = \prod_{\alpha\in\Delta,|B_\alpha|\geq 2}\left[\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2\right]^{|B_\alpha|-1} . \tag{2.183}$$

The second factor is

$$\det(\Delta_{\mathrm{b},2}^{\text{v.m.}}) = \frac{\det(\mathrm{M}^2)}{\prod_{\alpha\in\Delta}\left[\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2\right]^{|B_\alpha|+1}} \tag{2.184}$$

where the numerator is just the contribution of σ_2 and the denominator is a factor that we have included to shift the lowest mode of A^- (which has $J = |B_\alpha|/2 + 1$). With this shift, the rest of the determinant is given by

$$
\begin{aligned}
&\det(\Delta^{\text{v.m.}}_{\text{b},3}) \\
&= \prod_\alpha \prod_{J=\frac{|B_\alpha|}{2}}^{\infty} \begin{vmatrix} \frac{J(J+1)-\frac{|B_\alpha|}{2}}{r^2} + a_\alpha^2 & 0 & -\frac{i}{r^2}\sqrt{\frac{J(J+1)-\frac{|B_\alpha|}{2}\left[\frac{|B_\alpha|}{2}+1\right]}{2}} \\ 0 & \frac{J(J+1)+\frac{|B_\alpha|}{2}}{r^2} + a_\alpha^2 & \frac{i}{r^2}\sqrt{\frac{J(J+1)-\frac{|B_\alpha|}{2}\left[\frac{|B_\alpha|}{2}-1\right]}{2}} \\ \frac{i}{r^2}\sqrt{\frac{J(J+1)-\frac{|B_\alpha|}{2}\left[\frac{|B_\alpha|}{2}+1\right]}{2}} & -\frac{i}{r^2}\sqrt{\frac{J(J+1)-\frac{|B_\alpha|}{2}\left[\frac{|B_\alpha|}{2}-1\right]}{2}} & \frac{J(J+1)+1}{r^2} + a_\alpha^2 \end{vmatrix}^{2J+1} \\
&= \prod_{\alpha\in\Delta} \prod_{J=\frac{|B_\alpha|}{2}}^{\infty} \left[\left(\frac{J(J+1)}{r^2} + a_\alpha^2\right)\left(\frac{J^2}{r^2} + a_\alpha^2\right)\left(\left(\frac{J+1}{r}\right)^2 + a_\alpha^2\right)\right]^{2J+1} \\
&= \det(\mathrm{M}^2) \prod_{\alpha\in\Delta} \prod_{J=0}^{\infty} \left[\left(\left(\frac{J}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2\right)\left(\left(\frac{J+1}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2\right)\right]^{2J+|B_\alpha|+1} ,
\end{aligned}
$$

where

$$
\det(\mathrm{M}^2) = \prod_{\alpha\in\Delta} \prod_{J=\frac{|B_\alpha|}{2}}^{\infty} \left[\frac{J(J+1)}{r^2} + a_\alpha^2\right]^{2J+1} . \tag{2.185}
$$

Note the shift in the lowest mode of A^- at the top left component in the matrix. As we mentioned earlier, this a factor that we multiply and divide by hand to avoid isolating the $J = \frac{|B_\alpha|}{2}$ mode. Note also that in this case the off-diagonal terms $(1,3)$ and $(3,1)$ vanish. Including the contribution from the ghosts – which is $\det(\mathrm{M}^2)$ – the one-loop partition function of the vector-multiplet becomes

$$
\begin{aligned}
\frac{\det(\Delta^{\text{v.m.}}_{\text{b}})^{\frac{1}{2}}}{\det(\mathrm{M}^2)} &= \frac{\prod_{\alpha\in\Delta_+}\prod_{J=0}^{\infty}\left[\left(\left(\frac{J}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2\right)\left(\left(\frac{J+1}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2\right)\right]^{2J+|B_\alpha|+1}}{\prod_{\alpha\in\Delta_+}\left[\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2\right]^{|B_\alpha|+1}\prod_{\alpha\in\Delta_+,|B_\alpha|\geq 2}\left[\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2\right]^{-|B_\alpha|+1}} \\
&= \det(\Delta^{\text{v.m.}}_{\text{f}}) \cdot \prod_{\alpha\in\Delta_+}\left[\frac{1}{\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2}\right]^{|B_\alpha|} \prod_{\alpha\in\Delta_+,|B_\alpha|\geq 2}\left[\frac{1}{\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2}\right]^{1-|B_\alpha|} .
\end{aligned}
\tag{2.186}
$$

Therefore, we find that

$$Z^{\text{v.m.}}_{\text{one-loop}}(a,B) = \prod_{\substack{\alpha\in\Delta^+ \\ B_\alpha\neq 0}} \left[\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2 \right] . \tag{2.187}$$

2.C One-Loop Running of FI Parameter

Consider a two dimensional $\mathcal{N}=(2,2)$ gauge theory with a $U(1)$ gauge group factor in the presence of an FI parameter ξ. When the sum of the $U(1)$ charges of the chiral multiplets $Q=\sum_i Q_i$ is non-vanishing, the FI parameter gets renormalized according to

$$\xi(\mu) = \xi + \frac{1}{2\pi}\sum_j Q_j \ln\left(\frac{\mu}{M_{\text{UV}}}\right) . \tag{2.188}$$

In our localization computation, some care has been taken to regularize the theory in a $\mathcal{Q}$-invariant way. We accomplish this by introducing an "expectator" chiral multiplet of charge $-Q$, mass M, and R-charge $q=0$. In this enriched theory the FI parameter does not run. However, we recover the original theory by decoupling the expectator chiral multiplet by taking its mass M to be large. We now demonstrate by analyzing the one-loop determinant of the expectator chiral multiplet that this yields the running of the FI parameter with $M_{\text{UV}}=M$ and $\mu = 1/r$.

The relevant one-loop determinant of the expectator chiral multiplet is

$$\ln Z^{\text{c.m.}}_{\text{one-loop}}(a,B,M) = \ln\left[\frac{\Gamma\left(\frac{QB+q}{2} + irQa - irM\right)}{\Gamma\left(1+\frac{QB-q}{2} - irQa + irM\right)}\right] + O(1) . \tag{2.189}$$

The asymptotic expansion of $\Gamma(z)$ with large imaginary argument is given by

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + O(1) \tag{2.190}$$

where the terms of order 1 depend on the sign of $\Im z$ but are irrelevant for renormalization of ξ. Using this asymptotic form for large mass M in (2.189) yields

$$\begin{aligned} \ln Z^{\text{c.m.}}_{\text{one-loop}}(a,B,M) &\underset{rM\gg 1}{\simeq} 2irM\left(1-\ln rM\right) + (q-1)\ln rM + 2irQa\ln rM \\ &= 2irM\left(1-\ln rM\right) + (q-1)\ln rM + 4\pi i r a\frac{1}{2\pi}Q\ln\left(\frac{M}{\varepsilon}\right) , \end{aligned} \tag{2.191}$$

where $\varepsilon = \frac{1}{r}$. Note that the first two terms do not have any physical effect since they just rescale the partition function by an a-independent factor. The last term, however, combines with the on-shell classical piece of the action

$$\ln Z_0 \simeq -4\pi i r a \xi \tag{2.192}$$

to account for the running of the FI parameter

$$\ln Z_0 \cdot Z^{\text{c.m.}}_{\text{one-loop}}(a, B, M) \simeq -4\pi i r a \xi_{\text{ren}} \,, \tag{2.193}$$

with

$$\xi_{\text{ren}} = \xi + \frac{1}{2\pi} \sum_i Q_i \ln\left(\frac{\varepsilon}{M}\right) \,. \tag{2.194}$$

2.D Vortex Partition Function

We describe in this appendix the procedure used to evaluate the contribution from vortex (and anti-vortex) configurations. For simplicity, we only consider the case of SQCD, the two-dimensional $\mathcal{N} = (2,2)$ $U(N)$ supersymmetric gauge theory with $N_F \geq N$ fundamental chiral multiplets of masses $(M_1, \ldots, M_{N_F})$ and $\widetilde{N}_F \leq N_F$ anti-fundamental chiral multiplets of masses $(\widetilde{M}_1, \ldots, \widetilde{M}_{N_F})$. The flavour group is $U(1)_{\text{anti-diag}} \times SU(N_F) \times SU(\widetilde{N}_F)$, hence $\sum_{s=1}^{N_F} M_s = \sum_{s=1}^{\widetilde{N}_F} \widetilde{M}_s$.

As we show in section 2.4, the presence of vortex/anti-vortex solutions requires the scalar field σ_2 to take specific values, labelled by a choice of N masses $M_{p_1}, \ldots, M_{p_N}$. For such a choice of Higgs vacuum, the moduli space of solutions to the vortex equations (2.97) splits into discrete components $\mathcal{M}^{\{p_i\},k}_{\text{vortex}}$, where the vorticity k is defined by

$$k = \frac{1}{2\pi} \int_{\mathbb{R}^2} \operatorname{Tr} F \,. \tag{2.195}$$

The equivariant volume of the moduli space $\mathcal{M}_{\text{vortex}}$ can be expressed as a finite dimensional integral [68]. We denote by $\hat{\mathbf{M}}$ the diagonal $N \times N$ matrix with eigenvalues M_{p_i}, by $\check{\mathbf{M}}$ the diagonal matrix whose eigenvalues are masses of the other $N_F - N$ (non-excited) fundamental chiral multiplets, and by $\widetilde{\mathbf{M}}$ the matrix of anti-fundamental masses.

2.D.1 Vortex Matrix Model

The moduli space $\mathcal{M}^{\{p_i\},k}_{\text{vortex}}$ of configurations with k vortices admits an ADHM-like construction, which can be understood as the supersymmetric vacua of a certain gauged matrix model preserving two supercharges [70, 74, 75]. The relevant representations of the supersymmetry algebra can be obtained from the dimensional reduction of $N = (2, 0)$ supersymmetry in two dimensions. This gauged matrix model involves one $U(k)$ vector multiplet $\Phi = (\varphi, l, \bar{l}, D)$, and is coupled to one adjoint chiral multiplet $\mathcal{X} = (X, \chi)$, N fundamental chiral multiplets $\mathcal{I} = (I, \mu)$, $N_F - N$ anti-fundamental chiral multiplet $\mathcal{J} = (J, \nu)$ and $\widetilde{N}_F$ fundamental fermi multiplets $\Xi = (\xi, G)$. The matrix model preserves three global symmetry groups $U(1)_R$, $U(1)_J$ and $U(1)_A$, which can be identified as the R-symmetry group, the rotational symmetry group J and the axial R-symmetry group of the given two-dimensional theory, respectively. As mentioned before, $U(1)_A$ may suffers from an axial anomaly. Under these three $U(1)$ symmetry groups, the supercharges Q and $\bar{Q}$ have charges $(-1, +1, -1)$ and $(+1, -1, -1)$. For later convenience, we summarize global and gauge charges of the matrix model variables in the table below.

	X	χ	I	μ	J	ν	ξ	$\bar{\varphi}$	l	$\bar{l}$
$U(1)_R$	0	-1	0	-1	0	-1	-1	0	-1	$+1$
$U(1)_{2J}$	-2	-1	0	$+1$	0	$+1$	$+1$	0	$+1$	-1
$U(1)_A$	0	-1	0	-1	0	-1	$+1$	$+2$	$+1$	$+1$
$U(1)_\varepsilon$	-2	-2	0	0	0	0	0	0	0	0
$U(k)$	**adj**		**k**		$\bar{\mathbf{k}}$		**k**	**adj**		

Here the $U(1)_\varepsilon$ symmetry group can be identified as a twisted rotational symmetry group $J + R/2$ of the two-dimensional theory. Note that the complex scalar field X represents the position of the k vortices while I and J represent orientation modes. The supersymmetric vacuum equation with a positive FI parameter $r \sim 1/g^2 > 0$ is given by

$$\begin{aligned}
&[X, X^\dagger] + II^\dagger - J^\dagger J = r\mathbb{1}_k \\
&\qquad \varphi I - I\hat{\mathbf{M}} = 0 \qquad [\varphi, \bar{\varphi}] = 0 \\
&\qquad J\varphi - \check{\mathbf{M}}J = 0 \qquad [\varphi, X] = 0\,,
\end{aligned} \tag{2.196}$$

where X, I and J denote $k \times k$, $k \times N$ and $(N_F - N) \times k$ matrices. The choice of Higgs vacuum in the original two-dimensional gauge theory is encoded in the matrices $\hat{\mathbf{M}}$ and $\check{\mathbf{M}}$.

The solutions of (2.196) describe the moduli space $\mathcal{M}_{\text{vortex}}^{\{p_i\},k}$ of k vortices, and the volume of the moduli space can be identified as the partition function of this matrix model.

2.D.2 Vortex Partition Function

Since the matrix model describing moduli space of vortices in $\mathbb{R}^2$ has an infinite volume, it must be modified by turning on a chemical potential associated to the twisted rotational symmetry group $U(1)_\varepsilon$. The chemical potential ε can be understood as the Omega deformation parameter in the given two-dimensional theory, which is the inverse radius of the sphere S^2.

In the context of the matrix model, the chemical potential can be introduced by weakly gauging $U(1)_\varepsilon$, hence modifying (2.196) to the deformed supersymmetry vacuum equation

$$\begin{aligned} &[X, X^\dagger] + II^\dagger - J^\dagger J = r\mathbb{1}_k \\ &\varphi I - I\hat{\mathbf{M}} = 0 \qquad [\varphi, \bar{\varphi}] = 0 \\ &J\varphi - \check{\mathbf{M}}J = 0 \qquad [\varphi, X] = \varepsilon X \ , \end{aligned} \tag{2.197}$$

and adding a new (deformed) fermion equation

$$\varphi\xi + \xi\widetilde{\mathbf{M}} = 0 \ . \tag{2.198}$$

Due to the chemical potential ε, the space of vacua is reduced to isolated points, fixed points of supersymmetry.

We explain how to characterize such fixed points. Suppose without loss of generality that ε is positive definite. One can show from the deformed supersymmetry vacuum equations that $J = 0$ and the N chiral multiplets I are each an eigenvector of the operator φ. More specifically, denoting by $|\alpha\rangle$ an eigenvector of the operator φ with eigenvalue α,

$$I = |M_{p_1}\rangle \oplus \cdots \oplus |M_{p_N}\rangle \ . \tag{2.199}$$

Then, the vector space of dimension k on which φ acts can be spanned by generators constructed by successive actions of X on $|M_{p_i}\rangle$

$$|M_{p_i} + l\varepsilon\rangle \overset{\text{def}}{\propto} X^l |M_{p_i}\rangle \qquad (l = 0, 1, .., k_i - 1) \ , \tag{2.200}$$

with $\sum_{i=1}^{N} k_i = k$. As a consequence, the fixed points are characterized by N one-dimensional Young diagrams. The number of boxes k_i of the i-th 1-d Young diagram determines the vorticity of the i-th $U(1)$ factor in the Cartan subalgebra of $U(N)$. The matrix components of X are then determined using the first relation of (2.197).

The partition function of the matrix model can be reduced to a Gaussian integral around such fixed points. The results are nicely expressed as the following contour-integral expression [68, 77]

$$Z_{\vec{k}}(\{p_i\}, M, \widetilde{M}) = \oint_{\Gamma_{\{p_i\},k}} \prod_{I=1}^{k} \frac{d\varphi_I}{2\pi i} \; \mathcal{Z}_{\text{vec}}(\varphi) \cdot \mathcal{Z}_{\text{fund}}(M, \varphi) \cdot \mathcal{Z}_{\text{anti-fund}}(\widetilde{M}, \varphi) \tag{2.201}$$

with

$$\mathcal{Z}_{\text{vec}}(\varphi) = \frac{1}{k!\; \varepsilon^k} \prod_{I \neq J}^{k} \frac{\varphi_I - \varphi_J}{\varphi_I - \varphi_J - \varepsilon} \tag{2.202}$$

$$\mathcal{Z}_{\text{fund}}(M, \varphi) = \prod_{I=1}^{k} \prod_{s=1}^{N_F} \frac{1}{\varphi_I - M_s} \tag{2.203}$$

$$\mathcal{Z}_{\text{anti-fund}}(\widetilde{M}, \varphi) = \prod_{I=1}^{k} \prod_{s=1}^{\widetilde{N}_F} \left(\varphi_I + \widetilde{M}_s\right) \;, \tag{2.204}$$

where the contour $\Gamma_{\{p_i\},k}$ is chosen such that it encircles poles at

$$\varphi_I = \varphi_{(i,l)} = M_{p_i} + (l-1)\varepsilon \qquad (l = 1, 2, .., k_i) \;, \tag{2.205}$$

which can be understood as the fixed points (2.200). The vortex partition function of the two-dimensional gauge theory in a specific choice of Higgs branch component $\{p_i\}$ thus takes the form

$$Z_{\text{vortex}}(\{p_i\}, M, \widetilde{M}, z) = \sum_{k_1 + \cdots + k_N = k} z^{|\vec{k}|} Z_{\vec{k}}(\{p_i\}, M, \widetilde{M}) \,. \tag{2.206}$$

The residues of (2.201) can be expressed as Pochhammer raising factorials $(x)_n = x(x+1)\cdots(x+n-1)$ and the full vortex partition function of SQCD in the Higgs vacuum labelled by $\{p_i\}$ is

$$Z^{\text{SQCD}}_{\text{vortex}} = \sum_{\vec{k}} \frac{z^{|\vec{k}|}}{\vec{k}!} \frac{\prod_{i=1}^{N} \prod_{s=1}^{\widetilde{N}_F} \left(\frac{1}{\varepsilon}(M_{p_i} + \widetilde{M}_s)\right)_{k_i}}{\prod_{i \neq j}^{N} \left(\frac{1}{\varepsilon}(M_{p_i} - M_{p_j}) - k_j\right)_{k_j} \prod_{i=1}^{N} \prod_{s \notin \{p_j\}}^{N_F} \left(\frac{1}{\varepsilon}(M_{p_i} - M_s)\right)_{k_i}} \,, \tag{2.207}$$

where $\vec{k}! = k_1! \cdots k_N!$.

Chapter 3

$SU(2|1)_B$ Invariant Gauge Theories

The goal of this chapter is to construct $SU(2|1)_B$ invariant two-dimensional $\mathcal{N} = (2,2)$ gauge theories. By definition, the $SU(2|1)_B$ invariant GLSMs with vector and chiral multiplets are equivalent – up to field redefinitions – to $SU(2|1)_A$ invariant GLSMs with twisted chiral and twisted vector multiplets. In flat space the Lagrangian of a vector coupled to a chiral multiplet is identical to the Lagrangian of a twisted vector coupled to a twisted chiral multiplet. This is no longer the case when the theory is placed on the two-sphere. The background fields [63] and curvature couplings needed to place the theory on a two-sphere in a supersymmetric way are different, and thus the resulting Lagrangians are different. We now proceed to construct the supersymmetry transformations and invariant couplings for the twisted vector and twisted chiral multiplets.

3.1 Twisted Vector Multiplet

An $\mathcal{N} = (2,2)$ twisted vector multiplet consists of a real vector, two complex scalars related by complex conjugation, two complex spinors and a real auxiliary scalar $(A_\mu, \sigma, \bar{\sigma}, \eta, \bar{\eta}, \mathrm{D})$, all of which are valued in the Lie algebra of the gauge group G. While a twisted vector multiplet and a vector multiplet with the same gauge group G have exactly the same field content, the supersymmetry transformations on the two multiplets are realized differently.

The $SU(2|1)_A$ supersymmetry transformations on the twisted vector multiplet fields are

$$
\begin{aligned}
\delta\eta &= i\not{D}(\sigma\epsilon) + \bar{\epsilon}(\mathrm{D} + iF) - \frac{i}{2}\gamma^{\hat{3}}\bar{\epsilon}[\sigma, \bar{\sigma}] \\
\delta\bar{\eta} &= i\not{D}(\bar{\sigma}\bar{\epsilon}) + \epsilon(\mathrm{D} - iF) - \frac{i}{2}\gamma^{\hat{3}}\epsilon[\sigma, \bar{\sigma}] \\
\delta A_\mu &= \frac{i}{2}(\epsilon\gamma^{\hat{3}}\gamma_\mu\eta - \bar{\epsilon}\gamma^{\hat{3}}\gamma_\mu\bar{\eta}) \\
\delta\sigma &= \bar{\epsilon}\eta \\
\delta\bar{\sigma} &= \epsilon\bar{\eta} \\
\delta\mathrm{D} &= \frac{i}{2}\left\{D_\mu(\epsilon\gamma^\mu\eta) - [\sigma, \epsilon\gamma^{\hat{3}}\bar{\eta}]\right\} + \frac{i}{2}\left\{D_\mu(\bar{\epsilon}\gamma^\mu\bar{\eta}) + [\bar{\sigma}, \bar{\epsilon}\gamma^{\hat{3}}\eta]\right\}\ .
\end{aligned}
\tag{3.1}
$$

They are parametrized by conformal Killing spinors ϵ and $\bar{\epsilon}$ obeying

$$
\nabla_\mu\epsilon = \frac{1}{2r}\gamma_\mu\gamma^{\hat{3}}\epsilon \qquad \nabla_\mu\bar{\epsilon} = -\frac{1}{2r}\gamma_\mu\gamma^{\hat{3}}\bar{\epsilon}\,,
\tag{3.2}
$$

where r is the radius of the two-sphere. These transformations realize the $SU(2|1)_A$ algebra off-shell up to gauge transformations. Concretely, the resulting algebra is

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] &= \delta_G(\Lambda) \\
[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] &= \delta_G(\bar{\Lambda}) \\
[\,\delta_\epsilon,\ \delta_{\bar{\epsilon}}\,] &= \delta_{SU(2)}(v) + \delta_R(\alpha) + \delta_G(\Omega)
\end{aligned}
\tag{3.3}
$$

where the $SU(2)$ isometry transformation is constructed from the S^2 Killing vector

$$
v = i\bar{\epsilon}\gamma^\mu\epsilon\,\partial_\mu\,,
\tag{3.4}
$$

and the $U(1)_R$ transformation is parametrized by the scalar

$$
\alpha = -\frac{1}{2r}\bar{\epsilon}\gamma^{\hat{3}}\epsilon\,.
\tag{3.5}
$$

The R-charges of the various fields are:[1]

σ	η_+	η_-	A_μ	D	$\bar{\eta}_+$	$\bar{\eta}_-$	$\bar{\sigma}$
-2	-1	-1	0	0	$+1$	$+1$	$+2$

[1] These are the $U(1)_{\mathcal{A}}$ charges of the vector multiplet fields for a vector multiplet of vanishing $U(1)_{\mathcal{A}}$ charge.

Finally, the field dependent gauge transformation parameters generated in the closure of the algebra are

$$\Lambda = -\epsilon_2\gamma^{\hat{3}}\epsilon_1\sigma \qquad \bar{\Lambda} = \bar{\epsilon}_2\gamma^{\hat{3}}\bar{\epsilon}_1\bar{\sigma} \qquad \Omega = -v^\mu A_\mu\,. \tag{3.6}$$

3.2 Twisted Chiral Multiplet

The field content of a twisted chiral multiplet is the same as the standard chiral multiplet but also has different supersymmetry transformations. A twisted chiral multiplet can be minimally coupled in a supersymmetric way to a twisted vector multiplet. It transforms in a representation $\mathbf{R}$ of the gauge group G. The $SU(2|1)_A$ supersymmetry transformations, invariant action and partition function of uncharged twisted chiral multiplets on S^2 appeared in [39].

The $SU(2|1)_A$ supersymmetry transformations of charged twisted chiral multiplet fields $(Y, \bar{Y}, \zeta, \bar{\zeta}, G, \bar{G})$ are

$$\begin{aligned}
\delta Y &= (\bar{\epsilon}\gamma_- - \epsilon\gamma_+)\zeta\\
\delta \bar{Y} &= (\bar{\epsilon}\gamma_+ - \epsilon\gamma_-)\bar{\zeta}\\
\delta\zeta_+ &= -\gamma_+(i\not{D}Y - G)\bar{\epsilon} + i\gamma_+\epsilon\sigma Y\\
\delta\zeta_- &= +\gamma_-(i\not{D}Y - G)\epsilon - i\gamma_-\bar{\epsilon}\bar{\sigma}Y\\
\delta\bar{\zeta}_+ &= +\gamma_+(i\not{D}\bar{Y} - \bar{G})\epsilon - i\gamma_+\bar{\epsilon}\,\bar{Y}\bar{\sigma}\\
\delta\bar{\zeta}_- &= -\gamma_-(i\not{D}\bar{Y} - \bar{G})\bar{\epsilon} + i\gamma_-\epsilon\,\bar{Y}\sigma\\
\delta G &= +i\epsilon\gamma_-\left(\not{D}\zeta - \eta Y - \sigma\zeta\right) - i\bar{\epsilon}\gamma_+\left(\not{D}\zeta + \bar{\eta}Y - \bar{\sigma}\zeta\right)\\
\delta\bar{G} &= +i\epsilon\gamma_+\left(\not{D}\bar{\zeta} - \bar{Y}\eta - \bar{\zeta}\sigma\right) - i\bar{\epsilon}\gamma_-\left(\not{D}\bar{\zeta} + \bar{Y}\bar{\eta} - \bar{\zeta}\bar{\sigma}\right)\,.
\end{aligned} \tag{3.7}$$

These supersymmetry transformations realize the off-shell $SU(2|1)_A$ algebra (3.3) with the same parameters and with the following R-charge assignments:[2]

[2]These are the same as the $U(1)_\mathcal{A}$ charges of the components of a chiral superfield with vanishing $U(1)_\mathcal{A}$ charge.

$\bar{G}$	$\bar{Y}$	$\bar{\zeta}_-$	$\bar{\zeta}_+$	ζ_-	ζ_+	Y	G
0	0	-1	$+1$	$+1$	-1	0	0

The supersymmetry transformations of a twisted chiral multiplet of $U(1)_{\mathcal{A}}$ charge Δ can be obtained from (3.7) by the field redefinition [39]

$$G \rightarrow G + \frac{\Delta}{2r} Y \,. \tag{3.8}$$

Since correlators do not depend on Δ, we take it to vanish.

The $U(1)_R$ transformation acts chirally on the twisted chiral multiplet fermions ζ and $\bar{\zeta}$. Since the $U(1)$ R-symmetry charge R appears explicitly in the anticommutator of supercharges in $SU(2|1)_A$, anomaly cancellation of R is required to write down an $SU(2|1)_A$ supersymmetric theory of twisted vectors and twisted chirals on the two-sphere. The R-current is quantum mechanically conserved whenever the sum of the gauge charges of all charged twisted chiral multiplets vanish for each abelian gauge group factor in G. This guarantees that if the flat space gauge theory is also invariant under the R-symmetry $\mathcal{A}$, that the gauge theory flows in the infrared to an $\mathcal{N} = (2,2)$ SCFT, and if it has a geometrical phase, to a Calabi-Yau NLSM.

3.3 Supersymmetric Lagrangian

We now write down the $SU(2|1)_A$-invariant action for a twisted vector multiplet coupled to a charged twisted chiral multiplet. The action has several couplings that are separately supersymmetric

$$S = S_{\text{t.v.m.}} + S_{\text{FI}} + S_{\text{top}} + S_{\text{t.c.m.}} + S_W + S_{\overline{W}} \,. \tag{3.9}$$

The supersymmetrized kinetic terms for the twisted vector multiplets fields are

$$\begin{aligned} \mathcal{L}_{\text{t.v.m.}} = \frac{1}{2g_{\text{YM}}^2} \operatorname{Tr} \Bigg\{ & F^2 + D^\mu \bar{\sigma} D_\mu \sigma + \frac{1}{4}[\sigma, \bar{\sigma}]^2 + \mathrm{D}^2 - i\bar{\eta} \left(\not{D} + \frac{1}{r}\gamma^{\hat{3}} \right) \eta \\ & + i\bar{\sigma}(\eta\gamma^{\hat{3}}\eta) - i\sigma(\bar{\eta}\gamma^{\hat{3}}\bar{\eta}) \Bigg\} \,, \end{aligned} \tag{3.10}$$

where $F \equiv \frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu}$. The supersymmetric Lagrangian for the charged twisted chiral multiplet fields is

$$\begin{aligned}\mathcal{L}_{\text{t.c.m.}} = \bar{Y}\left(-D_\mu^2 + i\mathrm{D} + \frac{\{\sigma,\bar{\sigma}\}}{2}\right)Y + \bar{G}G + i\bar{Y}(\bar{\eta}_- - \eta_+)\zeta + i\bar{\zeta}(\bar{\eta}_+ - \eta_-)Y \\ + i\bar{\zeta}(\not{D} - \bar{\sigma}\gamma_+ - \sigma\gamma_-)\zeta\,.\end{aligned} \tag{3.11}$$

Twisted chiral multiplets couple via a twisted superpotential[3] W

$$\mathcal{L}_W = \frac{i}{4\pi}\left(W''(Y)\zeta_+\zeta_- - W'(Y)G + \frac{i}{r}W(Y)\right)\,. \tag{3.12}$$

Each $U(1)$ factor in the gauge group admits a supersymmetric Fayet-Iliopoulos (FI) and topological term

$$\mathcal{L}_{\text{top}} + \mathcal{L}_{\text{FI}} = -i\,\mathrm{Tr}\left(\frac{\vartheta}{2\pi}F + \xi\mathrm{D}\right)\,. \tag{3.13}$$

For each abelian factor, the associated field strength multiplet Σ is a chiral superfield, and the FI and topological term can be encoded in a linear superpotential $\mathcal{W}$

$$\mathcal{W} = \frac{i\tau}{2}\Sigma\,, \tag{3.14}$$

where

$$\mathcal{L}_{\mathcal{W}} = \frac{\partial\mathcal{W}}{\partial\Sigma}F_\Sigma - \frac{\partial^2\mathcal{W}}{\partial\Sigma^2}\eta_+\eta_-\,. \tag{3.15}$$

Superpotential couplings are $SU(2|1)_A$ invariant if the superpotential $\mathcal{W}$ carries R-charge -2, which is the charge of Σ. For twisted vector multiplets on S^2, $SU(2|1)_A$-invariance implies that the complexified FI parameter

$$\tau = \frac{\vartheta}{2\pi} + i\xi \tag{3.16}$$

is an exactly marginal coupling.

The action in flat space, obtained by sending $r \to \infty$ in our expressions, has an additional $U(1)_\mathcal{A}$ R-symmetry if the charge of the twisted superpotential W is -2. On the

[3] We use here a convenient normalization.

two-sphere, however, the non-minimal $1/r$ couplings in the action required by supersymmetry break this $U(1)_A$ R-symmetry. This breaking can be understood as arising due to the non-trivial background fields in the supergravity multiplet required to couple the gauge theory to a supersymmetric supergravity background [63].

The parameters of the ultraviolet GLSM are the gauge couplings for each gauge group factor, the complex parameters appearing in the twisted superpotential and the complexified FI parameters appearing in the superpotential. We note that unlike $SU(2|1)_A$-invariant GLSM's based on vector and chiral multiplets, the twisted chiral multiplets have vanishing twisted masses, since the scalars in the twisted vector multiplet are charged under the $U(1)_R$ symmetry. For a Calabi-Yau GLSM, the complexified FI parameters are the Kähler moduli of the Calabi-Yau while the complex parameters in the twisted superpotential correspond to the complex structure moduli.

3.4 Localization of the Path Integral

In this section we perform the exact computation of the partition function of the gauge theories constructed in the previous section. This requires choosing a supercharge $\mathcal{Q}$ in $SU(2|1)_A$ and a suitable deformation of the Lagrangian

$$\mathcal{L} \to \mathcal{L} + t\mathcal{Q}V\,. \tag{3.17}$$

By the familiar t-independence of the path integral (in favorable situations), the path integral reduces to a one-loop integral over the space of saddle points $\mathcal{M}$ of $\mathcal{Q}V$. The measure of integration is determined by classical action evaluated on the saddle points and by the one-loop determinants $Z_{\text{1-loop}}$ of twisted vector and twisted chiral produced by the deformation term $\mathcal{Q}V$. The contribution of the gauge fixing multiplet must also be included.

In formulas, for a collection of $\mathcal{Q}$-invariant operators collectively denoted by $\mathcal{O}$, we have that

$$\langle\mathcal{O}\rangle = \int_{\mathcal{M}} e^{-S|_{\mathcal{M}}}\,\mathcal{O}|_{\mathcal{M}}\,Z_{\text{1-loop}}\,. \tag{3.18}$$

In this chapter, $\mathcal{O}$ is the two point function of a chiral operator $\mathcal{O}_a$ at the north pole and an anti-chiral operator operator $\mathcal{O}_{\bar{a}}$ at the south pole of the two-sphere.

3.4.1 Choice of Supercharge and Decoupling Theorems

We choose the following supercharge[4] $\mathcal{Q}$ in $SU(2|1)_A$

$$\mathcal{Q} = S_1 + Q_2\,. \tag{3.19}$$

This is the same supercharge (2.41) that we used in our localization computation in chapter 2. The $SU(1|1)$ subalgebra that $\mathcal{Q}$ generates is

$$\mathcal{Q}^2 = J_3 + \frac{R}{2} \qquad\qquad \left[J_3 + \frac{R}{2}, \mathcal{Q}\right] = 0\,, \tag{3.20}$$

where J_3 is a $U(1)$ isometry generator of S^2, and has two antipodal fixed points which we call the north and south poles of the two-sphere. R is the $U(1)$ R-symmetry generator in $SU(2|1)_A$.

The (Grassmann even) Killing spinors (3.2) parameterizing the transformations generated by $\mathcal{Q}$ are

$$\begin{aligned} \epsilon &= \exp\left(-\frac{i}{2}\theta\gamma^{\hat{2}} + \frac{i}{2}\varphi\right)\epsilon_\circ, \qquad \gamma^{\hat{3}}\epsilon_\circ = \epsilon_\circ \\ \bar{\epsilon} &= \exp\left(+\frac{i}{2}\theta\gamma^{\hat{2}} - \frac{i}{2}\varphi\right)\bar{\epsilon}_\circ, \qquad \gamma^{\hat{1}}\bar{\epsilon}_\circ = \epsilon_\circ\,, \end{aligned} \tag{3.21}$$

where (θ, φ) are the canonical coordinates on S^2.

At the north pole of the two-sphere, gauge invariant operators $O_a(Y)$ constructed from the lowest component of twisted chiral multiplets are $\mathcal{Q}$-invariant. Likewise, at the south pole, operators constructed from the lowest component of twisted anti-chiral multiplets $O_{\bar{a}}(\bar{Y})$ are also $\mathcal{Q}$-invariant. This follows from the supersymmetry transformation (3.7) generated by the spinors (3.21). Therefore the two-point function

$$\langle \mathcal{O}_a(Y)\, \mathcal{O}_{\bar{b}}(\bar{Y})\rangle \tag{3.22}$$

is $\mathcal{Q}$-invariant and can be computed by supersymmetric localization.

We now prove that the two-sphere partition function and two-point functions (3.22) are independent of some of the parameters of the Lagrangian. First, we note that the twisted

[4] We drop the index A, to avoid cluttering.

vector multiplet Lagrangian (3.10) as well as the FI and topological terms (3.13) are all $\mathcal{Q}$-exact. Explicitly

$$\mathcal{L}_{\text{t.v.m.}} = \frac{1}{4g_{\text{YM}}^2}\mathcal{Q}\widetilde{\mathcal{Q}}\,\text{Tr}\left(\eta\gamma^{\hat{3}}\bar{\eta} + \frac{i}{r}\sigma\bar{\sigma}\right) - \nabla_\mu J^\mu_{\text{t.v.m.}}\,, \tag{3.23}$$

where $\widetilde{\mathcal{Q}} = S_1 - Q_2$, a supercharge in $SU(2|1)_A$ parametrized by Killing spinors (3.21) $-\epsilon$ and $\bar{\epsilon}$.[5] Likewise

$$\begin{aligned}\mathcal{L}_{\text{FI}} &= \frac{\xi}{2i}\mathcal{Q}\,\text{Tr}\left(\bar{\epsilon}\gamma^{\hat{3}}\bar{\eta} + \epsilon\gamma^{\hat{3}}\eta\right) - \nabla_\mu J^\mu_{\text{FI}}\\ \mathcal{L}_{\text{top}} &= \frac{\vartheta}{4\pi}\mathcal{Q}\,\text{Tr}\left(\bar{\epsilon}\gamma^{\hat{3}}\bar{\eta} - \epsilon\gamma^{\hat{3}}\eta\right) - \nabla_\mu J^\mu_{\text{top}}\,.\end{aligned} \tag{3.24}$$

By virtue of equation (3.14), this follows from the more general result that the superpotential $\mathcal{W}$ couplings (3.15) are $\mathcal{Q}$-exact [37, 40]. The twisted chiral Lagrangian (3.11) is also $\mathcal{Q}$-exact

$$\mathcal{L}_{\text{t.c.m.}} = \frac{1}{2}\mathcal{Q}\widetilde{\mathcal{Q}}\left(\bar{G}Y - \bar{Y}G + \frac{i}{r}\bar{Y}Y\right) - \nabla_\mu J^\mu_{\text{t.c.m.}}\,. \tag{3.25}$$

We note, however, that twisted superpotential couplings (3.12) are not $\mathcal{Q}$-exact.

This shows that the gauge theory two-sphere partition function and two-point functions (3.22) are independent of the gauge couplings g_{YM}^2 and of the complexified FI parameters τ, but depend on the complex parameters in the twisted superpotential. Gauge coupling independence implies that the two-sphere partition function of a gauge theory is a renormalization group invariant observable. In particular, it coincides with the partition function of a SCFT theory in the extreme infrared, where $g_{\text{YM}}^2 \to \infty$. This is none other than the sought-after Calabi-Yau NLSM when the gauge theory has a geometric phase. Moreover, the Zamolodchikov metric (1.2) of operators in the chiral ring of the $\mathcal{N} = (2, 2)$ SCFT can be exactly computed in the ultraviolet GLSM, as these correlators have images in the ultraviolet GLSM through (3.22).[6] In conclusion, a gauge theory on the two-sphere computes the Kähler potential and associated Zamolodchikov metric of the infrared SCFT. When the GLSM has a geometric phase, the gauge theory computes these quantities for the complex structure moduli space of the Calabi-Yau.

[5] The total derivative terms are written down in appendix 3.B.

[6] In our choice of coordinates, where the infrared NLSM is described by twisted chiral multiplets, a chiral ring element in the infrared SCFT is the lowest component of a twisted chiral superfield while an operator in the conjugate ring is the lowest component of a twisted anti-chiral superfield.

3.4.2 $\mathcal{Q}$-Exact Deformation Term

We proceed by deforming the gauge theory action by a $\mathcal{Q}$-exact term

$$\mathcal{L} \to \mathcal{L} + t\mathcal{Q}\mathcal{V}\,. \tag{3.26}$$

Following our discussion in the previous subsection, we can take

$$\mathcal{V} = \frac{1}{4g^2}\widetilde{\mathcal{Q}}\,\mathrm{Tr}\left(\bar{\eta}\gamma^{\hat{3}}\eta + \frac{i}{r}\sigma\bar{\sigma}\right) - \frac{i}{2}\chi\,\mathrm{Tr}(\bar{\epsilon}\gamma^{\hat{3}}\bar{\eta} + \epsilon\gamma^{\hat{3}}\eta) + \frac{1}{2}\widetilde{\mathcal{Q}}\left(\bar{G}Y - \bar{Y}G + \frac{i}{r}\bar{Y}Y\right)\,. \tag{3.27}$$

The bosonic part of $\mathcal{QV}$ can be recast into the positive definite form

$$\frac{1}{2g^2}\,\mathrm{Tr}\left(|D_\mu\sigma|^2 + \frac{1}{4}[\sigma,\bar{\sigma}]^2 + F^2 + \widetilde{\mathrm{D}}^2\right) + |D_\mu Y|^2 + |G|^2 + \frac{1}{2}\left(|\sigma Y|^2 + |\bar{\sigma}Y|^2\right) + \frac{g^2}{2}\left(Y\bar{Y} - \chi\right)^2\,, \tag{3.28}$$

where $\widetilde{\mathrm{D}} = \mathrm{D} + ig^2(Y\bar{Y} - \chi)$. Positive definiteness follows from the reality conditions

$$\begin{aligned} &\sigma^\dagger = \bar{\sigma} && \widetilde{\mathrm{D}}^\dagger = \widetilde{\mathrm{D}} && F^\dagger = F \\ &Y^\dagger = \bar{Y} && G^\dagger = \bar{G} && A_\mu^\dagger = A_\mu\,. \end{aligned} \tag{3.29}$$

By adding this deformation term to the action and taking the limit $t \to \infty$, we are able to apply the saddle point method, which is exact, and localize the path integral to the extrema of $\mathcal{QV}$. Since the bosonic part of the deformation term is positive definite, all the paths that contribute to the path integral lie at the global minimum surface $\mathcal{QV} = 0$ in the space of fields. The space of saddle points that we must integrate over in the path integral is therefore[7]

$$\mathcal{M} = \left\{Y | Y = Y_\circ,\ \bar{Y}_\circ T_a Y_\circ - \chi_a = 0\right\}/G_{\mathrm{global}}\,, \tag{3.30}$$

where T_a are the $U(1)$ generators of the gauge group, with all the other fields vanishing. $Y = Y_\circ$ is constant on the two-sphere.[8] Field configurations related by the residual gauge transformation G_{global} (the global part of the gauge group G) must be identified

$$Y_\circ \simeq e^{i\alpha}Y_\circ\,, \tag{3.31}$$

[7]For $\chi = 0$, σ can be non-zero, but then at least one Y must vanish. The fermionic superpartner of this field, however, has a fermionic zero mode, and this saddle point does not contribute.

[8]Even though the parameter χ enters in the definition of $\mathcal{M}$, we shall prove that the partition function is independent of χ, as it should, since it is the coefficient of a $\mathcal{Q}$-exact term in (3.27).

where α acts on Y in the corresponding representation $\mathbf{R}$ of the gauge group. $\mathcal{M}$ is therefore the Kähler quotient space

$$\mathcal{M} = \mathbb{C}^{|\mathbf{R}|}//G_{\text{global}}\,. \tag{3.32}$$

Localization has to be performed for the gauge fixed functional integral (see appendix 3.A for details). For the background field configurations (3.30), we fix the Lorenz gauge which is compatible with $A_\mu = 0$. For the field fluctuations in the computation of the one-loop determinant however, it is much more convenient to fix an R_ξ-like gauge adapted to the Higgs phase of the theory. This requires introducing gauge fixing terms and a fermionic generator $\mathcal{Q}_{\text{BRST}}$. We localize the path integral with respect to the BRST deformed supercharge $\hat{\mathcal{Q}} = (\mathcal{Q} + \mathcal{Q}_{\text{BRST}})$ using as the deformation term $\hat{\mathcal{Q}}\mathcal{V}'$, where $\mathcal{V}' = \mathcal{V} + \mathcal{V}_{\text{G.F.}}$. The space of saddle points of the gauge fixed theory remains unaffected by the inclusion of the gauge fixing terms, however, the gauge fixing terms play an important role in the computation of the measure factor $Z_{\text{1-loop}}$.

3.4.3 Partition Function and Zamolodchikov Metric

Calculation of the measure of integration in the space of saddle points $\mathcal{M}$ requires computing the one-loop determinant $Z_{\text{1-loop}}$ of twisted vector, twisted chiral and ghost multiplets around the saddle point configurations $\mathcal{M}$. This is achieved by integrating out to quadratic order in the fluctuations the deformation and gauge fixing terms $\hat{\mathcal{Q}}\mathcal{V}'$.

Consider a gauge theory with gauge group $G = U(1)^{N_c}$ coupled to N_f twisted chiral multiplets with charges Q_I^a under $U(1)^{N_c}$, where $a = 1, \ldots, N_c$ and $I = 1, \ldots, N_f$. Supersymmetry on the two-sphere requires anomaly cancelation for the $U(1)_R$ R-symmetry, which yields the constraints

$$\sum_I Q_I^a = 0 \qquad a = 1, \ldots, N_c\,. \tag{3.33}$$

The one-loop determinant around the saddle points (3.30) is given by the determinant of an $N_c \times N_c$ matrix (see appendix 3.C for details)

$$Z_{\text{1-loop}} = \det(M^\dagger M)\,. \tag{3.34}$$

Here M is an the $N_f \times N_c$ mass matrix and $M^\dagger$ is its hermitian conjugate. They are given by[9]

$$M_I{}^a = Q_I^a Y_I \,, \qquad M_a^{\dagger\, I} = Q_I^a \bar{Y}_I \,. \tag{3.35}$$

We note that $N_f \geq N_c$ is a necessary condition for the matrix $M^\dagger M$ to be non-degenerate. For $N_f < N_c$, there is a linear combination of the $U(1)$ generators under which all the twisted chiral fields are neutral, and the associated gaugino has a fermionic zero mode,[10] and therefore the path integral vanishes.

Evaluating the classical action and operator insertions on the saddle points we obtain[11]

$$\langle \mathcal{O}_a(N)\mathcal{O}_{\bar{b}}(S)\rangle = \int \mathrm{vol}_{\mathcal{M}} \mathcal{O}_a(Y)\mathcal{O}_{\bar{b}}(\bar{Y})\, Z_{\text{1-loop}}\, e^{rW(Y)-r\overline{W}(\bar{Y})} \,, \tag{3.36}$$

where $\mathrm{vol}_{\mathcal{M}}$ is the volume form on the space of saddle points $\mathcal{M}$ (3.30). The volume form on $\mathcal{M}$, which is the quotient space (3.32), can be written in terms of the volume form of the ambient flat space $\mathbb{C}^{N_f}$ by inserting appropriately normalized Dirac delta distributions and dividing by the volume of the $U(1)^{N_c}$ gauge orbits:

$$\mathrm{vol}_{\mathcal{M}} = \frac{\mathrm{d}^{N_f} Y \wedge \mathrm{d}^{N_f}\bar{Y}}{\mathrm{vol}(G_{\text{global}})}\ \det\left[\frac{\partial F_a}{\partial Y_b}\right] \prod_a \delta\left(F_a\right)\,, \tag{3.37}$$

where

$$F_a = \sum_I Q_I^a |Y^I|^2 - \chi_a \,. \tag{3.38}$$

On the ambient space $\mathbb{C}^{N_f}$, we can define the Hamiltonian action of the complexification $U(1)^{N_c}_{\mathbb{C}}$ of the gauge group. The vector fields that generate the real gauge transformations are

$$\rho_a = i \sum_I Q_I^a \left(Y_I \partial_I - \bar{Y}_I \bar{\partial}_I\right) \qquad a = 1, \ldots, N_c \,, \tag{3.39}$$

while

$$v_a = -\sum_I Q_I^a \left(Y_I \partial_I + \bar{Y}_I \bar{\partial}_I\right) \qquad a = 1, \ldots, N_c \,, \tag{3.40}$$

[9] We drop the subindex of $Y_\circ$ in order to avoid cluttering.

[10] Given by $\lambda = \bar{\epsilon}$, where $\bar{\epsilon}$ is the conformal Killing spinor (3.21).

[11] As explained in [37], the partition function is also proportional to $r^{c/3}$, due to the usual conformal anomaly, where c is the central charge.

generate imaginary gauge transformations. They act, respectively, as

$$Y_I \to e^{i\sum_a Q_I^a \tau_1^a} Y_I\,, \qquad Y_I \to e^{-\sum_a Q_I^a \tau_2^a} Y_I \qquad \tau_1, \tau_2 \in \mathbb{R}\,. \tag{3.41}$$

The moment map associated with the imaginary transformation generated by the a-th $U(1)$ factor in the gauge group is given by

$$\mu_a = -\frac{1}{2}\sum_I Q_I^a |Y^I|^2 \qquad a = 1, \ldots, N_c\,, \tag{3.42}$$

as it obeys

$$\mathrm{d}\mu_a = \imath_{v_a}\omega\,, \tag{3.43}$$

where ω is the Kähler form in $\mathbb{C}^{N_f}$. Therefore, the D-term equations entering in the definition of $\mathcal{M}$ in (3.30)

$$\left\{\sum_I Q_I^a |Y^I|^2 = \chi \longleftrightarrow F_a = 0\,;\, a = 1, \ldots, N_c\right\}\,, \tag{3.44}$$

can be interpreted as the moments maps for the imaginary gauge transformations

$$2\mu_a + \chi = 0 \qquad a = 1, \ldots, N_c\,. \tag{3.45}$$

We note that these moment maps obey the equations

$$\mathrm{d}\mu_a \cdot \mathrm{d}\mu_b = \left(M^\dagger M\right)_{ab}\,, \tag{3.46}$$

where d denotes the exterior derivative and the inner product $\mathrm{d}\mu_a \cdot \mathrm{d}\mu_b$ is the $\mathbb{C}^{N_f}$ inner product. As a direct consequence of the anomaly cancellation conditions (3.33), the holomorphic and anti-holomorphic factors in the the measure

$$\mathrm{d}^{N_f}Y \wedge \mathrm{d}^{N_f}\bar{Y} \tag{3.47}$$

are each invariant under the complexified gauge transformations $U(1)_{\mathbb{C}}^{N_c}$. Furthermore, the twisted superpotential $W(Y)$ and $\overline{W}(\bar{Y})$ are also invariant under complex gauge transformations, whereas $Z_{\text{1-loop}}$ is only invariant under real gauge transformations. This observation suggests a change of coordinates $\{Y\} \to \{X, \tau\}$, to some gauge invariant coordinates X and the (complex) gauge orbit coordinates τ, where the integration over the complex gauge orbits is localized to the real gauge orbits due to the δ-distributions arising from the D-term equations.

In computing the volume form $\mathrm{vol}_{\mathcal{M}}$ we must quotient by the volume of the orbit of $U(1)^{N_c}$ real gauge transformations. It follows from (3.39) that it is given by

$$\mathrm{vol}(G_{\mathrm{global}}) = (2\pi)^{N_c} \det\left(\rho_a \cdot \rho_b\right)^{1/2} . \tag{3.48}$$

By virtue of (3.39) we have that

$$\rho_a \cdot \rho_b = 4\left(M^\dagger M\right)_{ab} , \tag{3.49}$$

which combined with (3.46) implies[12] that the Jacobian appearing with the delta functions in (3.37) precisely cancels with the volume of the gauge orbit.

Altogether, the correlator (3.36) can be written as

$$\langle \mathcal{O}_a(N)\mathcal{O}_{\bar{b}}(S)\rangle = \int \frac{\mathrm{d}^{N_f}Y \wedge \mathrm{d}^{N_f}\bar{Y}}{(2\pi)^{N_c}} \mathcal{O}_a(Y)\, \mathcal{O}_{\bar{b}}(\bar{Y})\, \det\left(M^\dagger M\right) \prod_a \delta\left(2\mu_a + \chi_a\right)\, e^{rW(Y)-r\overline{W}(\bar{Y})} , \tag{3.50}$$

with M and $M^\dagger$ defined in (3.35) and μ_a in (3.42). The partition function is obtained by placing the identity operator at the north and south poles of the two-sphere, yielding

$$\mathcal{Z}_B = \int \frac{\mathrm{d}^{N_f}Y \wedge \mathrm{d}^{N_f}\bar{Y}}{(2\pi)^{N_c}} \det\left(M^\dagger M\right) \prod_a \delta\left(2\mu_a + \chi_a\right)\, e^{rW(Y)-r\overline{W}(\bar{Y})} . \tag{3.51}$$

3.5 Calabi-Yau Geometries

The two-sphere partition function (3.51) of a Calabi-Yau GLSM is expected to compute the Kähler potential $\mathcal{K}_C$ for the complex structure moduli of the corresponding Calabi-Yau manifold. Concretely, we expect

$$\mathcal{Z}_B = e^{-\mathcal{K}_C} = i^{\,\dim M} \int_M \Omega \wedge \overline{\Omega} , \tag{3.52}$$

where Ω is the nowhere vanishing holomorphic top form of the corresponding Calabi-Yau. We now turn to explicitly demonstrating this for various families of Calabi-Yau geometries.

[12]The Jacobian factor $J_{\{b\}} = \det\left(\partial F_a/\partial Y_b\right)$ in (3.37) assumes that one carries out the integration over the $Y_{\{b\}}$ planes first, treating Y_I as constant for $I \neq b$. More covariantly, one may write $J = \sqrt{\det\left(\mathrm{d}F_a \cdot \mathrm{d}F_b\right)}$ which takes the order of integration into account.

3.5.1 Quintic Hypersurfaces in $\mathbb{CP}^4_{[Q_1,\ldots,Q_5]}$

Consider the partition function (3.51) in the case of a $U(1)$ gauge theory coupled to five twisted chiral multiplets Y_I with charges Q_I and a twisted chiral multiplet P with charge $-q$. The anomaly cancellation condition requires the sum of the charges of all of the twisted chiral multiplets vanish, *i.e.*

$$q = \sum_I Q_I \,. \tag{3.53}$$

The twisted superpotential for GLSMs corresponding quintic hypersurfaces in $\mathbb{CP}^4_{[Q_1,\ldots,Q_5]}$ has the general form

$$W = PG_5(Y)\,, \tag{3.54}$$

where $G_5(Y)$ is a transverse polynomial satisfying

$$G_5(\lambda^{Q_I}Y_I) = \lambda^q G_5(Y) \qquad \lambda \in \mathbb{C}^* \,. \tag{3.55}$$

The two-sphere partition function takes the form[13]

$$\mathcal{Z} = \frac{1}{2\pi}\int \mathrm{d}^5Y \wedge \mathrm{d}^5\bar{Y} \wedge \mathrm{d}P \wedge \mathrm{d}\bar{P}\ M^\dagger M\ \delta\left(2\mu + \chi\right) e^{W-\overline{W}}\,, \tag{3.56}$$

where the moment map and the mass matrix are given by

$$\begin{aligned} -2\mu &= \sum_I Q_I|Y_I|^2 - q|P|^2\,, \\ M^\dagger M &= \sum_I Q_I^2|Y_I|^2 + q^2|P|^2\,. \end{aligned} \tag{3.57}$$

We remark the the anomaly cancellation condition (3.53) guarantees that the flat measure and the twisted superpotential factor in (3.56) are invariant under global *complex* gauge transformation. It is therefore natural to consider the change of variables

$$\begin{aligned} Y_I &= e^{iQ_I\tau}x_I\,, \\ P &= e^{-iq\tau}p\,, \end{aligned} \tag{3.58}$$

with $x_5 =$ constant. In these coordinates, complex gauge transformations act only as a shift of the τ coordinate and therefore τ is the (complex) gauge orbit coordinate. The invariance

[13] We set $r = 1$ from now on.

of the ambient space volume form and the twisted superpotential under complex gauge transformations generated by ∂_τ becomes manifest in the new coordinates. The volume form of $\mathbb{C}^6$ in the new coordinates is

$$\mathrm{d}^5Y \wedge \mathrm{d}^5\bar{Y} \wedge \mathrm{d}P \wedge \mathrm{d}\bar{P} = Q_5^2\,|x_5|^2\; \mathrm{d}^4x \wedge \mathrm{d}^4\bar{x} \wedge \mathrm{d}p \wedge \mathrm{d}\bar{p} \wedge \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}\,. \tag{3.59}$$

while the twisted superpotential retains its original form

$$W = PG_5(Y) = p\,G_5(x)\,. \tag{3.60}$$

The moment map and the mass matrix (3.57), however, depend explicitly of the imaginary τ direction, denoted by τ_2, as they are only invariant under real gauge transformations, and may be rewritten as

$$\begin{aligned} -2\mu &= \sum_{I=1}^{5} Q_I\, e^{-2Q_I\tau_2}|x_I|^2 - q\, e^{2q\tau_2}|p|^2\,, \\ M^\dagger M &= \sum_{I=1}^{5} Q_I^2\, e^{-2Q_I\tau_2}|x_I|^2 + q^2 e^{2q\tau_2}|p|^2\,. \end{aligned} \tag{3.61}$$

The partition function (3.56) in the new coordinates is

$$\mathcal{Z} = -2iQ_5^2\,|x_5|^2 \int \mathrm{d}^4x \wedge \mathrm{d}^4\bar{x} \wedge \mathrm{d}p \wedge \mathrm{d}\bar{p}\left(e^{p\,G_5(x)-\bar{p}\,\bar{G}_5(\bar{x})}\int \mathrm{d}\tau_2\; M^\dagger M\;\delta\left(2\mu+\chi\right)\right)\,, \tag{3.62}$$

where we have carried out the integration over τ_1 which only contributes a factor of 2π. It is clear from (3.61) that $M^\dagger M$ is the Jacobian $\partial_{\tau_2}\mu$, that is

$$\mathrm{d}\tau_2 M^\dagger M = \mathrm{d}\tau_2\frac{\partial\mu}{\partial\tau_2} \doteq \mathrm{d}\mu \tag{3.63}$$

keeping x_I and p constant. This implies that the integration over τ_2 can be readily carried out yielding[14]

$$\int \mathrm{d}\tau_2\, M^\dagger M\delta(2\mu+\chi) \doteq \int \mathrm{d}\mu\,\delta(2\mu+\chi) = 1/2\,. \tag{3.64}$$

The partition function (3.62) can be put into the proposed form in terms of an integral over the holomorphic three-form by performing the integration over the complex variables

[14] In general, the equations $2\mu+\chi=0$ have multiple solutions for τ_2; this only introduces a multiplicative factor which we ignore.

p and $\bar{p}$ as well as the integration over one of the x planes, say x_4. Integrating over p imposes the embedding equation $G_5 = 0$ in $\mathbb{CP}^4_{[Q_1,\ldots,Q_5]}$ via δ distributions

$$\int \mathrm{d}p \wedge \mathrm{d}\bar{p}\; e^{p\,G_5 - \bar{p}\,\bar{G}_5} = -\frac{1}{4\pi^2}\delta(G_5)\delta(\bar{G}_5) \tag{3.65}$$

and finally, integrating over x_4 and $\bar{x}_4$ yields

$$\mathcal{Z} = i\frac{Q_5^2\,|x_5|^2}{4\pi^2} \sum_{\substack{\{x_4|G_5=0\}\\ \{\bar{x}_4|\bar{G}_5=0\}}} \int \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial_4 G_5(x)} \wedge \frac{\mathrm{d}\bar{x}_1 \wedge \mathrm{d}\bar{x}_2 \wedge \mathrm{d}\bar{x}_3}{\bar{\partial}_4 \bar{G}_5(\bar{x})}\,. \tag{3.66}$$

From (3.66) we can read off the holomorphic three-form to be

$$\Omega = \frac{Q_5}{2\pi}\;\frac{x_5\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial_4 G_5(x)}\,, \tag{3.67}$$

which matches the well known formulae for the holomorphic three-form presented in [80,81] of quintic hypersurfaces in $\mathbb{CP}^4_{[Q_1,\ldots,Q_5]}$. We remark that although (3.67) appears to have singularities whenever $\partial_4 G_5 = 0$, via a simple change of coordinates, corresponding to integrating (3.65) with respect to x_1 instead of x_4, it may be written as

$$\Omega = -\frac{Q_5}{2\pi}\;\frac{x_5\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4}{\partial_1 G_5(x)}\,. \tag{3.68}$$

Since the polynomial $G_5(x)$ is transversal and $x_5 \neq 0$, it follows that the holomorphic three-form Ω is non-singular and nowhere vanishing.

Mirror Quintic Complex Structure Kähler Potential

In [39] the $SU(2|1)_A$-invariant partition function for the familiar quintic three-fold in $\mathbb{CP}^4$ was shown to coincide with the $SU(2|1)_A$-invariant partition function of the Hori and Vafa mirror theory [32]. This is a $U(1)$ vector multiplet coupled to twisted chiral multiplets $(Y_1,\ldots,Y_5,Y_P)$ with a twisted superpotential

$$W = \left[i\,\Sigma\left(\sum_{a=1}^{5} Y^a - 5Y_P + 2\pi i\tau\right) - \left(\sum_{a=1}^{5} e^{-Y^a} + e^{-Y_P}\right)\right], \tag{3.69}$$

where Σ is the field strength multiplet. As shown in [39], the relation to the Mellin-Barnes like formula for $SU(2|1)_A$ invariant gauge theories derived in [37,40] follows by integrating out the twisted chiral multiplet fields. Explicitly, decomposing the integral into contours[15]

$$\int_{Y^*=\bar{Y}} dY d\bar{Y} e^{-e^{-Y}+iQ\Sigma Y} e^{e^{-\bar{Y}}+iQ\bar{\Sigma}\bar{Y}} = \int_0^\infty dt\, e^{-t}\, t^{-iQ\Sigma-1} \int_C dt\, e^{\bar{t}}\, \bar{t}^{-iQ\bar{\Sigma}-1}\,, \tag{3.70}$$

and with the identities

$$\int_0^\infty dt\, e^{-t}\, t^{-iQ\Sigma-1} = \Gamma(-iQ\Sigma)\,, \qquad \int_C dt\, e^{\bar{t}}\, \bar{t}^{-iQ\bar{\Sigma}-1} = \frac{2\pi i}{r\Gamma(1+iQ\bar{\Sigma})}\,, \tag{3.71}$$

one arrives at the gauge theory result [39].[16]

The two-sphere partition function of the mirror theory can be reduced to an orbifold Landau-Ginzburg model by integrating out Σ. This yields [39]

$$\mathcal{Z}_{\text{L.G.}} = \int \prod_{a=1}^{5} d\widetilde{X}_a d\overline{\widetilde{X}}_a\; e^{-W_{\text{eff}}+\overline{W}_{\text{eff}}}\,, \tag{3.72}$$

where the effective twisted superpotential is

$$W_{\text{eff}} = \sum_a \widetilde{X}_a^5 + e^{-2\pi i\tau/5} \prod_a \widetilde{X}_a\;. \tag{3.73}$$

The canonical variables $\widetilde{X}_a$ are given by

$$\widetilde{X}_a = e^{-\frac{1}{5}Y_a}\,, \tag{3.74}$$

and therefore we must orbifold by

$$\widetilde{X}_a \simeq e^{2\pi i/5}\widetilde{X}_a\,. \tag{3.75}$$

This orbifold Landau-Ginzburg model realizes the mirror Calabi-Yau geometry: the mirror quintic W. Indeed, it is easy to show that the orbifold Landau-Ginzburg model partition function also computes the Kähler potential on the complex structure moduli space of the mirror quintic W

$$\mathcal{Z}_{\text{L.G.}} = i\int_W \Omega\wedge\overline{\Omega}\,. \tag{3.76}$$

[15] C is the Hankel contour, which starts at $-\infty - i\epsilon$, then goes around the branch cut along the negative real t axis, and ends up at $\infty + i\epsilon$.

[16] This is a streamlined version of the identity derived in [39].

3.5.2 Complete Intersection Surfaces in $\mathbb{CP}^n_{[Q_1,\ldots,Q_{n+1}]}$

The analysis of the last section can be easily generalized to intersection of multiple hypersurfaces in $\mathbb{CP}^n_{[Q_1,\ldots,Q_{n+1}]}$. As the analysis is quite parallel to that of the last section, some details are omitted here.

Consider the partition function (3.51) in the case of a $U(1)$ gauge theory, this time coupled to $n+1$ twisted chiral multiplets Y_I with charges Q_I and m twisted chiral multiplet P_α with charges $-q_\alpha$. Imposing the anomaly cancellation condition restricts the charges to satisfy

$$\sum_\alpha q_\alpha = \sum_I Q_I \,. \tag{3.77}$$

The partition function takes the form

$$\mathcal{Z} = \frac{i^{n+m+1}}{2\pi} \int \mathrm{d}^{n+1}Y \wedge \mathrm{d}^{n+1}\bar{Y} \wedge \mathrm{d}^m P \wedge \mathrm{d}^m \bar{P} \;\, M^\dagger M \;\, \delta\left(2\mu + \chi\right) e^{W - \overline{W}} \tag{3.78}$$

where the twisted superpotential is linear in P_α and is a polynomial in Y_I,

$$W = \sum_\alpha P_\alpha G_\alpha(Y)\,, \tag{3.79}$$

with the polynomials G_α satisfying

$$G_\alpha(\lambda^{Q_I} Y_I) = \lambda^{q_\alpha} G_\alpha(Y)\,. \tag{3.80}$$

We emphasize again that both the twisted superpotential term and the volume form for the ambient space $\mathbb{C}^{n+1+m}$ are invariant under complex gauge transformations. The change of variables

$$\begin{aligned} Y_I &= e^{iQ_I \tau} x_I\,, \\ P_\alpha &= e^{-iq_\alpha \tau} p_\alpha\,, \end{aligned} \tag{3.81}$$

with x_{n+1} = constant, makes this invariance manifest as the gauge transformations in the new coordinates act simply as a shift in τ. The twisted superpotential in the new coordinates assumes the τ-independent form

$$W = \sum_\alpha p_\alpha G_\alpha(x) \tag{3.82}$$

and the volume form is

$$2i^{n+m} Q_{n+1}^2 \left|x_{n+1}\right|^2 \mathrm{d}^n x \wedge \mathrm{d}^n \bar{x} \wedge \mathrm{d}^m p \wedge \mathrm{d}^m \bar{p} \wedge \mathrm{d}\tau_1 \wedge \mathrm{d}\tau_2\,. \tag{3.83}$$

Here τ_1 and τ_2 are the real and imaginary parts of the τ coordinate parameterizing the compact and non-compact directions of the gauge orbit surface. The moment map, which has an explicit τ_2 dependence takes the form

$$-2\mu = \sum_I e^{-2Q_I\tau_2} Q_I |x_I|^2 - \sum_\alpha q_\alpha e^{2q_\alpha \tau_2} p_\alpha \,, \tag{3.84}$$

while $M^\dagger M$ can be related to the moment map, as in the case of quintic hypersurfaces, by τ_2 differentiation of the latter,

$$M^\dagger M = \frac{\partial \mu}{\partial \tau_2} \,. \tag{3.85}$$

The integration over τ_1 and τ_2 may then be carried out as was done for quintic hypersurfaces (3.62), yielding

$$\begin{aligned} \mathcal{Z} &= \frac{i^{n+m}}{\pi} Q_{n+1}^2 \, |x_{n+1}|^2 \int \mathrm{d}^n x \wedge \mathrm{d}^n \bar{x} \wedge \mathrm{d}^m p \wedge \mathrm{d}^m \bar{p} \left(e^{\sum_\alpha (p_\alpha G_\alpha - \bar{p}_\alpha \bar{G}_\alpha)} \int \mathrm{d}^2 \tau \frac{\partial \mu}{\partial \tau_2} \, \delta\left(2\mu + \chi\right) \right) \\ &= i^{n+m} Q_{n+1}^2 \, |x_{n+1}|^2 \int \mathrm{d}^n x \wedge \mathrm{d}^n \bar{x} \wedge \mathrm{d}^m p \wedge \mathrm{d}^m \bar{p} \; e^{\sum_\alpha (p_\alpha G_\alpha - \bar{p}_\alpha \bar{G}_\alpha)} \,. \end{aligned} \tag{3.86}$$

This is a simple generalization of the case of a hypersurface defined by a single embedding equation studied in the last section, with multiple p fields, one for each constraint. Integration over the p planes then imposes all the constraints leading to

$$\mathcal{Z} = \frac{i^{n-m} Q_{n+1}^2 \, |x_{n+1}|^2}{(2\pi)^{2m}} \int \mathrm{d}^n x \wedge \mathrm{d}^n \bar{x} \prod_\alpha \delta(G_\alpha) \delta(\bar{G}_\alpha) \,. \tag{3.87}$$

Carrying out the integration over the m dimensional space $\{x_I | I = n-m+1, \ldots, n\}$ we arrive at the desired expression

$$\mathcal{Z} = i^{n-m} \frac{Q_{n+1}^2 \, |x_{n+1}|^2}{(2\pi)^{2m}} \sum_{\substack{\{x_{n-m+\beta} | G_\alpha = 0\} \\ \{\bar{x}_{n-m+\beta} | \bar{G}_\alpha = 0\}}} \int \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-m}}{\det\left(\partial_{n-m+\beta} G_\alpha(x)\right)} \wedge \frac{\mathrm{d}\bar{x}_1 \wedge \cdots \wedge \mathrm{d}\bar{x}_{n-m}}{\det\left(\bar{\partial}_{n-m+\beta} \bar{G}_\alpha(\bar{x})\right)} \,, \tag{3.88}$$

where each determinant in the denominator is computed over the α and β indices. This yields the holomorphic $n-m$ form

$$\Omega = \frac{Q_{n+1}}{(2\pi)^m} \, \frac{x_{n+1} \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-m}}{\det\left(\partial_{n-m+\beta} G_\alpha(x)\right)} \tag{3.89}$$

for the intersection of m hypersurfaces in $\mathbb{CP}^n_{[Q_1,\ldots,Q_{n+1}]}$ [80, 81]. That Ω appears to be singular whenever $\det\left(\partial_{n-m+\beta}G_\alpha(x)\right) = 0$ is an artifact of the choice of coordinates. For these points on the manifold, there is a different choice $\{x_{\sigma(\alpha)}, \alpha = 1,\ldots,m\}$ of coordinates to integrate the δ-distributions in (3.87), such that (3.89) is non-singular.

3.5.3 Complete Intersection of Hypersurfaces in Product of Weighted Projective Spaces

As a much more general class of complete intersections with abelian GLSM realization, we now consider consider the partition function (3.51) in the case of $U(1)^{N_c}$ gauge theory with $N_f = n + m + N_c$ twisted chiral multiplets Y_I with charge matrix $\{Q_I^a | a = 1,\ldots,N_c;\ I = 1,\ldots,N_f\}$. The anomaly cancellation conditions restricts the charge matrix to obey

$$\sum_I Q_I^a = 0 \quad \text{for all } a\,. \tag{3.90}$$

The partition function has the general form (3.51) where the superpotential is a polynomial in $\{X_I = Y_I | I = 1,\ldots,N_f - m\}$ and is linear in $\{P_\alpha = Y_\alpha | \alpha = N_f - m + 1,\ldots,N_f\}$,

$$W = \sum_\alpha P_\alpha G_\alpha(X)\,. \tag{3.91}$$

The polynomials G_α satisfy

$$G_\alpha(\lambda^{Q_I^a} X_I) = \lambda^{-Q_\alpha^a} G_\alpha(X)\,, \tag{3.92}$$

which guarantees the invariance of the twisted superpotential under $U(1)_{\mathbb{C}}^{N_c}$ gauge transformations. As before, we introduce the complex τ^a coordinates, one for each $U(1)$ factor in the gauge group, via

$$\begin{aligned} X_I &= e^{i\sum_a Q_I^a \tau^a} x_I\,, \\ P_\alpha &= e^{i\sum_a Q_\alpha^a \tau^a} p_\alpha\,, \end{aligned} \tag{3.93}$$

and with[17] $x_{n+1} = \cdots = x_{n+N_c} = 1$. This isolates the action of each $U(1)_a$ factor in the gauge group to a shift in τ_a and highlights the gauge invariance of the twisted superpotential

$$W = \sum_\alpha p_\alpha G_\alpha(x)\,. \tag{3.94}$$

[17] This amounts to choosing inhomogeneous coordinates on the Calabi-Yau.

To write the volume form of $\mathbb{C}^{N_f}$ in the new coordinates, first consider the volume form of the subspace $\mathbb{C}^{N_c}$ of constant x_I. The holomorphic part of this volume form may be written as

$$\mathrm{d}X_{n+1} \wedge \cdots \wedge \mathrm{d}X_{n+N_c} = i^{N_c} \sum_{a_1,\ldots,a_{N_c}} Q^{a_1}_{n+1} \ldots Q^{a_{N_c}}_{n+N_c} \mathrm{d}\tau^{a_1} \wedge \cdots \wedge \tau^{a_{N_c}} = \det\left(iQ^a_{n+b}\right) \mathrm{d}^{N_c}\tau \,, \tag{3.95}$$

where the determinant is over the a and b indices. The partition function may then be written as

$$\mathcal{Z} = i^{N_f} \frac{\det(Q^a_{n+b})^2}{(2\pi)^{N_c}} \int \mathrm{d}^n x \wedge \mathrm{d}^n \bar{x} \wedge \mathrm{d}^m p \wedge \mathrm{d}^m \bar{p}\, e^{W-\overline{W}} \int \mathrm{d}^{N_c}\tau \wedge \mathrm{d}^{N_c}\bar{\tau} \det(M^\dagger M) \prod_a \delta(2\mu_a + \chi_a) \tag{3.96}$$

where the moment map depends on the imaginary part of τ^a according to

$$-2\mu_a = \sum_{I=1}^{n} e^{-2\sum_b Q^b_I \tau^b_2} Q^a_I |x_I|^2 + \sum_{I=n+1}^{n+N_c} Q^a_I e^{-2\sum_b Q^b_I \tau^b_2} + \sum_{\alpha=N_f-m+1}^{N_f} Q^a_\alpha e^{-2\sum_b Q^b_\alpha \tau^b_2} |p_\alpha|^2 \,, \tag{3.97}$$

and the mass matrix $M^\dagger M$ can be expressed in terms of the moment maps via

$$(M^\dagger M)_{ab} = \frac{\partial \mu_a}{\partial \tau^b_2} \,. \tag{3.98}$$

This last relation implies that $\det(M^\dagger M)$ is precisely the inverse of the Jacobian factor produced by the coordinate transformation $\{\tau^a_2\} \to \{\mu_a\}$. Consequently, the integration over the space of complex gauge orbits can be carried out leading to the numerical factor

$$\int \mathrm{d}^{N_c}\tau \wedge \mathrm{d}^{N_c}\bar{\tau} \det(M^\dagger M) \prod_a \delta(2\mu_a + \chi_a) = (-2i\pi)^{N_c} \,. \tag{3.99}$$

With the space of gauge orbits integrated out, the partition function (3.96) assumes the simple form

$$\mathcal{Z} = i^{n+m} \det(Q^a_{n+b})^2 \int \mathrm{d}^n x \wedge \mathrm{d}^n \bar{x} \wedge \mathrm{d}^m p \wedge \mathrm{d}^m \bar{p}\; e^{\sum_\alpha (p_\alpha G_\alpha - \bar{p}_\alpha \bar{G}_\alpha)} \,. \tag{3.100}$$

As in the last two examples, the p integrals impose the embedding equation constraints $G_\alpha = 0$ which can be used to solve for m of the coordinates x_I. This leads to the partition function

$$\mathcal{Z} = i^{n-m} \frac{\det(Q^a_{n+b})^2}{(2\pi)^{2m}} \sum_{\substack{\{x_{n-m+\beta}|G_\alpha=0\} \\ \{\bar{x}_{n-m+\beta}|\bar{G}_\alpha=0\}}} \int \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-m}}{\det\left(\partial_{n-m+\beta} G_\alpha(x)\right)} \wedge \frac{\mathrm{d}\bar{x}_1 \wedge \cdots \wedge \mathrm{d}\bar{x}_{n-m}}{\det\left(\bar{\partial}_{n-m+\beta} \bar{G}_\alpha(\bar{x})\right)} \, . \tag{3.101}$$

The resulting nowhere vanishing holomorphic $n - m$ form Ω is given by

$$\Omega = \frac{\det(Q^a_{n+b})}{(2\pi)^m} \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-m}}{\det\left(\partial_{n-m+\beta} G_\alpha(x)\right)} \, , \tag{3.102}$$

where the determinant in the denominator is over the α and β indices, thus realizing from gauge theory the formulae for the holomorphic form on a Calabi-Yau in [80–82].

Appendix

3.A BRST Supercharge and Gauge Fixing

As in any gauge theory, the formalism we have used has built in it a large redundancy which we need to remove in order to proceed with our computation of the partition function. This is achieved by introducing the supercharge $\mathcal{Q}_{\text{BRST}}$ and the ghost and anti-ghost multiplets $\{c, a_\circ\}$ and $\{\bar{c}, b\}$, where c and $\bar{c}$ are Grassmann odd and $a_\circ$ and b are Grassmann even scalars and they all have vanishing R-charge.

In terms of the ghost multiplet fields, the BRST operator is defined as

$$\mathcal{Q}_{\text{BRST}} = \delta_G(c), \qquad \mathcal{Q}_{\text{BRST}}^2 = \delta_G(a_\circ)\,, \tag{3.103}$$

where $a_\circ$ is assumed to be supersymmetric *i.e.* $\mathcal{Q}a_\circ = 0$. By construction, adding the BRST supercharge to the supersymmetry algebra (3.20) leaves the algebra invariant up to gauge transformations. We therefore define the supercharge $\hat{\mathcal{Q}} = \mathcal{Q} + \mathcal{Q}_{\text{BRST}}$ and require that it realizes the $su(1|1)$ algebra (3.20) as

$$\hat{\mathcal{Q}}^2 = \mathcal{L}_v - \frac{i}{2r}R + \delta_G(a_\circ) \tag{3.104}$$

where $\mathcal{L}_v$ denotes the Lie(-Lorenz) derivative along $v = 1/r\partial_\varphi$ and R is the generator of the $U(1)_R$ symmetry. This fixes the supersymmetry transformation rule for the ghost and anti-ghost multiplet fields completely. The action of $\hat{\mathcal{Q}}$ on the ghost multiplet fields is found to be

$$\hat{\mathcal{Q}}c = a + i\,cc + v^\mu A_\mu + \frac{1}{2}\sin\theta\left(\sigma e^{+i\varphi} - \bar{\sigma}e^{-i\varphi}\right), \qquad \hat{\mathcal{Q}}a_\circ = i[c, a_\circ]\,, \tag{3.105}$$

while the anti-ghost multiplet fields transform as

$$\hat{\mathcal{Q}}\bar{c} = ib, \qquad \hat{\mathcal{Q}}b = -i(\mathcal{L}_v + i[a, \cdot\,])\bar{c}\,. \tag{3.106}$$

We remark that by construction the action of $\hat{\mathcal{Q}}$ and $\mathcal{Q}$ coincide on all gauge invariant objects. In particular the deformation term (3.27) satisfies $\hat{\mathcal{Q}}\mathcal{V} = \mathcal{Q}\mathcal{V}$.

For a choice of gauge fixing functional $\mathcal{G}[A, \Phi]$, the gauge fixing condition, $\mathcal{G} = 0$, can be imposed on the path integral in a supersymmetric way by adding the deformation term $\hat{\mathcal{Q}}\mathcal{V}_{\text{G.F.}}$ to the action where

$$\mathcal{V}_{\text{G.F.}} = \frac{1}{2}\int \mathrm{d}^2x\sqrt{h}\ \mathrm{Tr}\left\{\bar{c}\left(\mathcal{G} - \frac{i}{4}b\right)\right\}\,. \tag{3.107}$$

Being exact in $\hat{\mathcal{Q}}$, this choice of deformation term guarantees the independence of the path integral from the choice of gauge fixing functional $\mathcal{G}$, provided that the ghost kinetic term, $\bar{c}\mathcal{Q}_{\text{BRST}}\mathcal{G}$, is non-degenerate.

In the presence of a Higgs branch, such as the saddle points (3.30), a particularly convenient choice for the gauge fixing functional $\mathcal{G}$ is the so called R_ξ gauge (with $\xi = 1$)

$$\mathcal{G} = \nabla_\mu A^\mu + i\left(Y\bar{Y}_\circ - Y_\circ\bar{Y}\right) \tag{3.108}$$

We remark that the gauge fixing condition on the saddle points reduces to the usual Lorenz gauge $\nabla_\mu A^\mu = 0$ which is compatible with the choice $A_\mu = 0$ in (3.30).

3.B $\hat{\mathcal{Q}}$-Exact Deformation Term

Here we spell out the precise deformation term $\hat{\mathcal{Q}}\mathcal{V}'$, including all the total derivative terms, which we use for the localization computation. We break $\mathcal{V}'$ into four pieces corresponding to the twisted vector, twisted chiral, Fayet-Iliopoulos and gauge fixing terms

$$\mathcal{V}' = \mathcal{V}_{\text{t.v.m.}} + \mathcal{V}_{\text{G.F.}} + \mathcal{V}_{\text{F.I.}} + \sum_I \mathcal{V}^I_{\text{t.c.m.}}\,. \tag{3.109}$$

For concreteness, let $\{T_a,\, a = 1, \ldots, \dim\mathfrak{g}\}$ be the set of normalized generators of the gauge algebra $\mathfrak{g}$. The twisted vector multiplet as well as the ghost fields are valued in the adjoint representation of $\mathfrak{g}$ while the twisted chiral multiplet fields live in a representation $\mathbf{r}$ of $\mathfrak{g}$. Suppressing the integration over the sphere, the various terms in $\mathcal{V}'$ are given by

$$
\begin{aligned}
\mathcal{V}_{\text{t.v.m.}} &= \frac{1}{4}\,\mathrm{Tr}\left\{\bar{\epsilon}\gamma^{\hat{3}}\bar{\eta}\,(\mathrm{D}+iF) + \epsilon\gamma^{\hat{3}}\eta\,(\mathrm{D}-iF) + \frac{i}{2}\,(\bar{\epsilon}\eta+\epsilon\bar{\eta})\,[\sigma,\,\bar{\sigma}] - i\bar{\epsilon}\gamma^{\hat{3}}\not{D}\bar{\sigma}\,\eta - i\epsilon\gamma^{\hat{3}}\not{D}\sigma\,\bar{\eta}\right\}\,, \\
\mathcal{V}_{\text{G.F.}} &= \frac{1}{2}\,\mathrm{Tr}\left\{\bar{c}\left(\nabla_\mu A^\mu + ig^2\left(Y\bar{Y}_\circ - Y_\circ\bar{Y}\right) - \frac{i}{4}b\right)\right\}\,, \\
\mathcal{V}_{\text{F.I.}} &= \frac{\chi}{2i}\,\mathrm{Tr}\left(\bar{\epsilon}\gamma^{\hat{3}}\bar{\eta} + \epsilon\gamma^{\hat{3}}\eta\right)\,, \\
\mathcal{V}_{\text{t.c.m.}} &= \frac{1}{2}\Big[\bar{G}\,(\bar{\epsilon}\zeta_- + \epsilon\zeta_+) - \left(\bar{\epsilon}\bar{\zeta}_+ + \epsilon\bar{\zeta}_-\right)G + i\bar{Y}D_\mu\,(\bar{\epsilon}\gamma^\mu\zeta_- + \epsilon\gamma^\mu\zeta_+) - i\bar{Y}\,(\bar{\sigma}\bar{\epsilon}\zeta_+ + \sigma\epsilon\zeta_-) \\
&\qquad - iD_\mu\left(\bar{\epsilon}\gamma^\mu\bar{\zeta}_+ + \epsilon\gamma^\mu\bar{\zeta}_-\right)Y + i\left(\bar{\epsilon}\bar{\zeta}_-\bar{\sigma} + \epsilon\bar{\zeta}_+\sigma\right)Y + i\bar{Y}\left(\bar{\epsilon}\gamma^{\hat{3}}\bar{\eta} + \epsilon\gamma^{\hat{3}}\eta\right)Y\Big]\,.
\end{aligned}
\tag{3.110}
$$

where there is an independent Fayet-Iliopoulos parameter χ_a for each $U(1)$ factor in the gauge group.[18] As we alluded to in section 3.4, the twisted vector and twisted chiral terms may be written in a more compact form as

$$
\begin{aligned}
\mathcal{V}_{\text{t.v.m.}} &= \frac{1}{4}\widetilde{\mathcal{Q}}\,\mathrm{Tr}\left(\eta\gamma^{\hat{3}}\bar{\eta} + \frac{i}{r}\sigma\bar{\sigma}\right)\,, \\
\mathcal{V}_{\text{t.c.m.}} &= \frac{1}{2}\widetilde{\mathcal{Q}}\left(\bar{G}Y - \bar{Y}G + \frac{i}{r}\bar{Y}Y\right)\,,
\end{aligned}
\tag{3.111}
$$

where $\widetilde{\mathcal{Q}} = S_1 - Q_2$. Using (3.110), the deformation term may be split into bosonic and fermionic pieces, up to a total derivative term, *i.e.*

$$
\hat{\mathcal{Q}}\mathcal{V} = \hat{\mathcal{Q}}\mathcal{V}\big|_{\text{bos.}} + \hat{\mathcal{Q}}\mathcal{V}\big|_{\text{fer.}} + \nabla_\mu J^\mu \tag{3.112}
$$

where the bosonic part is given by

$$
\begin{aligned}
t\hat{\mathcal{Q}}\mathcal{V}\big|_{\text{bos.}} &= \frac{t}{2}\sum_a\left\{F_a^2 + (D^\mu\sigma)_a(D_\mu\bar{\sigma})_a + \frac{1}{4}[\sigma,\,\bar{\sigma}]_a^2 + \widetilde{\mathrm{D}}_a^2 + \widetilde{b}_a^2 + \mathcal{G}_a^2 + \left(\bar{Y}T_aY - \chi\right)^2\right\} \\
&\qquad + t\left(\bar{G}G + D^\mu\bar{Y}D_\mu Y + \frac{1}{2}\bar{Y}\,\{\sigma,\,\bar{\sigma}\}\,Y\right)
\end{aligned}
\tag{3.113}
$$

[18]Without loss of generality, we have not chosen an independent parameter for all the different $\hat{\mathcal{Q}}$-exact pieces in the deformation term since $\hat{\mathcal{Q}}$-exactness guarantees that the final result will be independent of such parameters.

with $\widetilde{\mathrm{D}}_a = \mathrm{D}_a + i(\bar{Y}T_aY - \chi)$ and $\widetilde{b} = b/2 + i\mathcal{G}$. The fermionic part of $\hat{\mathcal{Q}}\mathcal{V}$ is given by

$$\begin{aligned} t\hat{\mathcal{Q}}\mathcal{V}\big|_{\text{fer.}} = -\frac{it}{2}\,\mathrm{Tr}\left\{\bar{\eta}\left(\not{D} + \frac{1}{r}\gamma^{\hat{3}}\right)\eta + \sigma\,\bar{\eta}\gamma^{\hat{3}}\bar{\eta} - \bar{\sigma}\,\eta\gamma^{\hat{3}}\eta - i\bar{c}\hat{\mathcal{Q}}\mathcal{G} + \frac{i}{4}\bar{c}\,(v^\mu\partial_\mu + i[a_\circ,\,\cdot\,])\,\bar{c}\right\} \\ + it\left(\bar{Y}(\bar{\eta}_- - \eta_+)\zeta + \bar{\zeta}(\bar{\eta}_+ - \eta_-)Y + \bar{\zeta}\left(\not{D} - \sigma\gamma_- - \bar{\sigma}\gamma_+\right)\zeta\right) \end{aligned} \tag{3.114}$$

and the total derivative term may be written as

$$\begin{aligned} J^\mu = \frac{i}{2}\,\mathrm{Tr}\left\{(\bar{\epsilon}\epsilon)\varepsilon^{\mu\nu}\bar{\sigma}D_\nu\sigma + \frac{1}{2r}v^\mu\bar{\sigma}\sigma + \frac{1}{2}\left(\bar{\epsilon}\gamma^\mu\bar{\eta}\right)\left(\epsilon\gamma^{\hat{3}}\eta\right) - \frac{1}{2}\left(\epsilon\bar{\eta}\right)\left(\bar{\epsilon}\gamma^{\hat{3}}\gamma^\mu\eta\right)\right\} \\ + \sum_I\left((\bar{\epsilon}\gamma_-\epsilon)D^\mu\bar{Y}_IY_I - (\bar{\epsilon}\gamma_+\epsilon)\bar{Y}_ID^\mu Y_I - \frac{i}{2}\left(\bar{\epsilon}\gamma^{\hat{3}}\gamma^\mu\epsilon\right)\left(\bar{G}_IY_I + \bar{Y}_IG_I\right) + i(\bar{\epsilon}\gamma_-\epsilon)\left(\bar{\zeta}_I\gamma^\mu\zeta_I\right)\right) \\ + \frac{\chi}{2}\,\mathrm{Tr}\left\{\left(\bar{\epsilon}\gamma^{\hat{3}}\gamma^\mu\bar{\epsilon}\right)\bar{\sigma} + \left(\epsilon\gamma^{\hat{3}}\gamma^\mu\epsilon\right)\sigma\right\}. \end{aligned} \tag{3.115}$$

3.C One-Loop Determinant

Consider the Abelian gauge theory with gauge group $U(1)^{N_c}$, minimally coupled to N_f twisted chiral multiplets with generic charges $\{Q_I^a | a = 1, \ldots, N_c; I = 1, \ldots, N_f\}$. We assume $N_f \geq N_c$ since, as will become clear, the one-loop determinant vanishes for $N_c > N_f$ due to fermionic zero modes.

Deforming the path integral by adding the deformation term $t\hat{\mathcal{Q}}\mathcal{V}$ to the action and taking the large t limit, the path integral localizes to the saddlepoints which are constant maps subject to the D-term constraints

$$\left\{Y = \text{constant}\,\middle|\,\sum_I Q_I^a|Y_I|^2 = \chi^a\right\}. \tag{3.116}$$

The measure of integration is defined by the one-loop – with respect to t – fluctuations of the fields around these saddle points. To extract this measure, we expand $\hat{\mathcal{Q}}\mathcal{V}$ to quadratic order around the saddle points (3.116), we therefore redefine the fields as

$$\Phi \to \frac{1}{\sqrt{t}}\Phi \tag{3.117}$$

for twisted vector fields and

$$Y_I \to Y_I + \frac{1}{\sqrt{t}} y_I\,, \quad G_I \to \frac{1}{\sqrt{t}} G_I\,, \quad \zeta_I \to \frac{1}{\sqrt{t}} \zeta \tag{3.118}$$

for twisted chiral fields. Imposing the R_ξ gauge on the gauge field fluctuations

$$\mathcal{G}_a = \frac{1}{\sqrt{t}} \left(\nabla_\mu A_a^\mu - i \sum_I Q_I^a \left(\bar{y}_I Y_I - \bar{Y}_I y_I \right) \right) \tag{3.119}$$

the quadratic part of the deformation term can be cast into the following form

$$\begin{aligned} t\hat{\mathcal{Q}}\mathcal{V}\big|_{\text{quad}} = & \frac{1}{2r^2} \sum_{a,b} \left[A_a^\mu \left(2M_{ab}^2 + \delta_{ab} - r^2 \delta_{ab} \nabla^2 \right) A_\mu^b + (\bar{\sigma}_a\,,\, \bar{c}_a) \left(2M_{ab}^2 - r^2 \delta_{ab} \nabla^2 \right) (\sigma_b\,,\, c_b)^T \right] \\ & + \frac{1}{r^2} \sum_{I,J} \bar{y}_I \left(2M_{IJ}^2 - r^2 \delta_{IJ} \nabla^2 \right) y_J + i \sum_{a,I} Q_I^a \left[\bar{\eta}_a \left(\bar{Y}_I \zeta_+^I + \bar{\zeta}_-^I Y_I \right) - \eta_a \left(\bar{Y}_I \zeta_-^I + \bar{\zeta}_+^I Y_I \right) \right] \\ & - \frac{i}{2r} \sum_a \bar{\eta}_a \left(r \not{\nabla} + \gamma^{\hat{3}} \right) \eta_a + i \sum_I \bar{\zeta}_I \not{\nabla} \zeta_I + \sum_I \bar{G}_I G_I + \frac{1}{2} \sum_a \left(\widetilde{\mathrm{D}}^2 + \widetilde{b}^2 \right) + \sum_a \bar{c}_a K_a \end{aligned} \tag{3.120}$$

where $\bar{c}K$ summerizes the fermionic terms that do not contribute to the one-loop determinant. Explicitly, K_a is given by

$$K_a = \frac{1}{8} v^\mu \partial_\mu \bar{c}_a - \frac{i}{4} \nabla_\mu \left(\epsilon \gamma^{\hat{3}} \gamma^\mu \eta_a - \bar{\epsilon} \gamma^{\hat{3}} \gamma^\mu \bar{\eta}_a \right) + \frac{i}{2} \sum_I Q_I^a \left((\bar{\epsilon} \bar{\zeta}_+^I - \epsilon \bar{\zeta}_-^I) Y_\circ^I - \bar{Y}_\circ^I (\bar{\epsilon} \zeta_-^I - \epsilon \zeta_+^I) \right) \,.$$

We define the $N_f \times N_c$ matrix M and it's hermitian conjugate $M^\dagger$ as

$$M_I{}^a = r Q_I^a Y_I\,, \qquad M_a^{\dagger\, I} = r Q_I^a \bar{Y}_I\,. \tag{3.121}$$

The mass matrices M_{ab}^2 and M_{IJ}^2 appearing in (3.120) are then given by

$$M_{ab}^2 = (M^\dagger M)_{ab}\,, \qquad M_{IJ}^2 = (M M^\dagger)_{IJ}\,. \tag{3.122}$$

For generic charges Q_I^a and with $N_f \geq N_c$, both of these matrices are of rank N_c. Furthermore, one can easily check that they have the same eigenvalues since for any eigenvector u of $MM^\dagger$, the vector $M^\dagger u$ is an eigenvector of $M^\dagger M$ with the same eigenvalue.

From (3.120), it is evident that the path integral over the auxiliary fields $\widetilde{\mathrm{D}}_a, \widetilde{b}_a$ and G_I is Gaussian and yields a trivial factor. It is also clear that the path integration over the twisted vector scalars $\{\sigma, \bar{\sigma}\}$ and the ghost and anti-ghost fields $\{c, \bar{c}\}$ yield canceling contributions.

As for the rest of the field, we begin our analysis by diagonalizing the Laplacian on the gauge field. Using the spectrum of the Laplacian operator on T^*S^2,

$$\text{spectrum}(-r^2\nabla^2\big|_{T^*S^2}) = \{(J^2 + J - 1)^{\times(4J+2)}; J = 1, 2, \dots\}\,, \tag{3.123}$$

we may compute the contribution of the gauge fields to the one-loop determinant to be

$$\Delta_A^{-1} = \prod_{J=1}^{\infty} \det\left(\frac{J(J+1)}{2}\mathbb{1} + 2M^\dagger M\right)^{2J+1}\,. \tag{3.124}$$

In order to compute the contribution to the one-loop determinant arising from the fluctuations of the twisted chiral scalar fields, we first need to isolate the zero modes satisfying

$$\nabla_\mu y_I = 0 \quad \text{and} \quad \sum_J M_{IJ}^2\, y_J = 0\,. \tag{3.125}$$

These are the longitudinal fluctuations that lie in the space of saddle points (3.116) and need to be excluded from the one-loop analysis. This amounts to removing the vanishing eigen values of $MM^\dagger$ from the $J = 0$ mode contribution. The contribution from the twisted chiral scalars is then

$$\begin{aligned}\Delta_y^{-1} &= \det{}'(2MM^\dagger)\prod_{J=1}^{\infty}\det\left(J(J+1) + 2MM^\dagger\right)^{2J+1}\\ &= \det(2M^\dagger M)\prod_{J=1}^{\infty}\left[J^{4J(N_f-N_c)}\det\left(J(J+1) + 2M^\dagger M\right)^{2J+1}\right]\,.\end{aligned} \tag{3.126}$$

Putting (3.124) and (3.126) together, the boson and ghost contributions to the one-loop determinant takes the form

$$\Delta_b^{-1} = \det(2M^\dagger M)\prod_{J=1}^{\infty}\left(\frac{1}{2}\right)^{(2J+1)N_c}\prod_{J=1}^{\infty}\left[J^{4J(N_f-N_c)}\det\left(J(J+1) + 2M^\dagger M\right)^{4J+2}\right]\,. \tag{3.127}$$

To compute the contribution to the one-loop determinant due to fermionic fields, consider the field redefinition

$$\psi = \bar{\zeta}_+ + \zeta_-\,, \qquad \bar{\psi} = \bar{\zeta}_- + \zeta_+\,. \tag{3.128}$$

In terms of ψ and $\bar{\psi}$, we may rewrite the quadratic fermion part of $t\hat{\mathcal{Q}}\mathcal{V}$ as

$$\bar{f}D_f f = \begin{pmatrix} \oplus_a \bar{\eta}_a \\ \oplus_I \psi_I \end{pmatrix}^T \begin{pmatrix} \mathbb{1}\otimes\left(-\frac{ir}{2}\nabla\!\!\!\!/ - \frac{i}{2}\gamma^{\hat{3}}\right) & +i(M^\dagger\otimes\gamma_+ + M^T\otimes\gamma_-) \\ -i(M\otimes\gamma_- + M^*\otimes\gamma_+) & \mathbb{1}\otimes ir\nabla\!\!\!\!/ \end{pmatrix} \begin{pmatrix} \oplus_a \eta_a \\ \oplus_I \bar{\psi}_I \end{pmatrix}, \tag{3.129}$$

where the operator D_f is block diagonal, *i.e.* it does not mix the eigenmodes of the Dirac operator. Exploiting this fact we may consider each block separately. In the Jth mode, the Dirac operator is diagonal while the chirality operator $\gamma^{\hat{3}}$ has only non-zero off-diagonal elements. Explicitly, we have

$$ir\nabla\!\!\!\!/\big|_J = (J+1/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^{\hat{3}}\big|_J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.130}$$

in the basis of eigenspinors of the Dirac operator. In this basis, the Jth block of the operator D_f in (3.129) takes the form

$$D_f[J] = \begin{pmatrix} \mathbb{1}\otimes\begin{pmatrix} -\frac{J+1/2}{2} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{J+1/2}{2} \end{pmatrix} & \frac{i}{2}\begin{pmatrix} M^\dagger + M^T & M^\dagger - M^T \\ M^\dagger - M^T & M^\dagger + M^T \end{pmatrix} \\ -\frac{i}{2}\begin{pmatrix} M^* + M & M^* - M \\ M^* - M & M^* + M \end{pmatrix} & \mathbb{1}\otimes\begin{pmatrix} J+\frac{1}{2} & 0 \\ 0 & -J-\frac{1}{2} \end{pmatrix} \end{pmatrix}, \tag{3.131}$$

and the fermion contribution to the one-loop determinant takes the form

$$\Delta_f = \prod_{J=1/2}^{\infty} \big|D_f[J]\big|^{2J+1} = \prod_{J=1}^{\infty} \big|D_f[J-1/2]\big|^{2J}. \tag{3.132}$$

The finite dimensional determinant $\big|D_f[J-1/2]\big|$ can easily be computed since the bottom right $N_f \times N_f$ block of (3.131) is diagonal which allows us to put the matrix $D_f[J]$ in a lower triangular form. This is achieved via the non-degenerate matrix

$$U[J-1/2] = \begin{pmatrix} \mathbb{1} & -\frac{i}{2}\begin{pmatrix} M^\dagger + M^T & M^\dagger - M^T \\ M^\dagger - M^T & M^\dagger + M^T \end{pmatrix} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1}\otimes\begin{pmatrix} J^{-1} & 0 \\ 0 & -J^{-1} \end{pmatrix} \end{pmatrix} \tag{3.133}$$

whose determinant is given by

$$\big|U[J-1/2]\big| = (-1)^{N_f} J^{-2N_f}. \tag{3.134}$$

Using this matrix, $|D_f[J]|$ in (3.132) decomposes as

$$\left|D_f[J-1/2]\right| = \frac{1}{|U|}\left|UD_f\right| = \frac{1}{|U|}\begin{vmatrix} D'_f & 0 \\ U' & \mathbb{1} \end{vmatrix} = (-1)^{N_f} J^{2N_f} |D'_f[J-1/2]| \tag{3.135}$$

where $D'_f[J-1/2]$ is given by

$$\begin{aligned} D'_f[J-1/2] &= \mathbb{1} \otimes \begin{pmatrix} -\frac{J}{2} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{J}{2} \end{pmatrix} - \frac{1}{4J}\begin{pmatrix} M^\dagger + M^T & M^\dagger - M^T \\ M^\dagger - M^T & M^\dagger + M^T \end{pmatrix}\begin{pmatrix} M^* + M & M^* - M \\ M - M^* & -M^* - M \end{pmatrix} \\ &= \begin{pmatrix} -\frac{J}{2}\mathbb{1} - \frac{r^2 M^\dagger M}{J} & -\frac{i}{2}\mathbb{1} \\ -\frac{i}{2}\mathbb{1} & \frac{J}{2}\mathbb{1} + \frac{r^2 M^\dagger M}{J} \end{pmatrix} \end{aligned} \tag{3.136}$$

and it's determinant is given by

$$\left|D'_f[J-1/2]\right| = \left(\frac{-1}{(2J)^2}\right)^{N_c} \det\left[\left(J(J+1) + 2M^\dagger M\right)\left(J(J-1) + 2M^\dagger M\right)\right] . \tag{3.137}$$

Using this result, substituting (3.135) in (3.132) yields

$$\Delta_f = \prod_{J=1}^{\infty} 2^{-4JN_c} \prod_{J=1}^{\infty} J^{4J(N_f - N_c)} \prod_{J=0}^{\infty} \det\left(J(J+1) + 2M^\dagger M\right)^{4J+2} . \tag{3.138}$$

Combining (3.127) and (3.138), the one-loop determinant is given by

$$\Delta = \det(M^\dagger M) \tag{3.139}$$

up to an irrelevant divergent factor which may be regularized via zeta function regularization to $2^{2N_c/3}$.

Chapter 4

Elliptic Genera

In this chapter we compute the elliptic genus of $\mathcal{N} = (2, 2)$ supersymmetric gauge theories with Stückelberg fields. We compute the elliptic genus, first for the case of GLSMs with a single Stückelberg field with one or many chiral multiplets. We then generalize the results to GLSMs with multiple Stückelberg fields.

4.1 Gauge Theories with Stückelberg Fields

In this section we review a class of gauged linear sigma models with one Stückelberg field [59,60], and its relation to non-linear sigma models [83]. Next, we recall gauged linear sigma models with multiple Stückelberg fields.

4.1.1 The Stückelberg Field

The superspace action for the gauged linear sigma model of interest is given by [59]

$$S = \frac{1}{2\pi} \int d^2x d^4\theta \left[\sum_{i=1}^{N} \bar{\Phi}_i e^V \Phi_i + \frac{k}{4}(P + \bar{P} + V)^2 - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right] . \tag{4.1}$$

The chiral multiplets Φ_i have unit charge under the $U(1)$ gauge group, and the superfield Σ is a twisted chiral superfield derived from the vector superfield V [28]. The superfield

P is also a chiral multiplet with the complex scalar $p = p_1 + ip_2$ as its lowest component. While the field p_1 is a real uncharged non-compact bosonic field, the field p_2 is compact with period $2\pi\sqrt{\alpha'}$ and we set $\alpha' = 1$ as in [59]. The field P is charged under the gauge group additively. It is a Stückelberg field.

With suitable linear dilaton boundary conditions [59], the theory flows in the infrared to a conformal field theory which has $\mathcal{N} = (2, 2)$ supersymmetry and central charge

$$c = 3N\left(1 + \frac{2N}{k}\right). \tag{4.2}$$

To lowest order in α' these conformal field theories are described by a non-linear sigma model on a $2N$-dimensional Kähler manifold which has $U(N)$ isometry and a linear dilaton along a non-compact direction:

$$\begin{aligned} ds^2 &= \frac{g_N(Y)}{2}dY^2 + \frac{2}{N^2 g_N(Y)}(d\psi + NA_{FS})^2 + 2Y ds^2_{FS}\,, \\ \Phi &= -\frac{NY}{k}\,. \end{aligned} \tag{4.3}$$

The explicit form of $g_N(Y)$ was found in [83].

4.1.2 Multiple Stückelberg Fields

More general gauged linear sigma models exist [59] in which one considers a $(U(1))^M$ gauge theory with N chiral fields Φ_i with charge R_{ia} under the ath gauge group, and M Stückelberg fields P_a. The superspace action is a simple generalization of the action in (4.1):

$$S = \frac{1}{2\pi}\int d^2x d^4\theta \left[\sum_{i=1}^{N} \bar{\Phi}_i e^{\sum_a R_{ia}V_a}\Phi_i + \sum_{a=1}^{M}\frac{k_a}{4}(P_a + \bar{P}_a + V_a)^2 - \sum_{a=1}^{M}\frac{1}{2e_a^2}|\Sigma_a|^2\right]. \tag{4.4}$$

The gauge transformations under the $U(1)^M$ are given by

$$\Phi_i \to e^{i\sum_{a=1}^{M} R_{ia}\Lambda_a}\Phi_i \quad \text{and} \quad P_a \to P_a + i\Lambda_a\,. \tag{4.5}$$

The central charge of the conformal field theory to which this theory flows is given by

$$c = 3\left(N + \sum_{a=1}^{M}\frac{2b_a^2}{k_a}\right). \tag{4.6}$$

Here, b_a is given by the sum over the charges of the chiral multiplets:

$$b_a = \sum_{i=1}^{N} R_{ia} \,. \tag{4.7}$$

4.2 Elliptic Genus Through Localization

In this section, we compute the elliptic genera of the class of models reviewed in section 4.1. In the Hamiltonian formalism the elliptic genus is given by

$$\chi = \mathrm{Tr}_{\mathcal{H}_R}(-1)^F \, q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}} z^{J_0} \,, \tag{4.8}$$

where L_0 and $\bar{L}_0$ are the right-moving and left-moving conformal dimensions in the CFT respectively and J_0 is the zero mode of the right-moving R-charge.

We will evaluate the trace (4.8) in the path integral formalism where the insertion of $(-1)^F$ amounts to imposing periodic boundary conditions for bosonic as well as fermionic fields. Furthermore, the insertion of z^{J_0} twists the periodic boundary conditions in a manner that depends explicitly on the R-charge of the fields.

Exploiting the invariance of the elliptic genus under the renormalization group flow, the computation can be carried out in the ultraviolet using the super-renormalizable gauged linear sigma model description [28,50]. The R-charges of the fields in the GLSM can be read off from the explicit expression for the right-moving R-current in the GLSM realization of the $\mathcal{N} = (2,2)$ superconformal algebra constructed in [59]. Consequently, we can compute the elliptic genus by evaluating the partition function of the ultraviolet gauged linear sigma model with twisted boundary conditions using supersymmetric localization, as has been done for various compact models in [49–51].

4.2.1 Preliminaries

In what follows we carry out the path integration of the GLSM described by the action (4.1) with twisted boundary conditions using supersymmetric localization. To avoid clutter, we present the computation for a single chiral multiplet Φ minimally coupled, with gauge charge $q_\Phi = 1$, to a $U(1)$ vector multiplet V which is rendered massive by a single

Stückelberg superfield P. The generalization to multiple chiral multiplets and multiple massive vector multiplets is straightforward.

After integrating over the Grassmann odd superspace coordinates, the action (4.1) takes the form[1]

$$S = \frac{i}{4\pi} \int \mathrm{d}^2 w \left(\mathcal{L}_{\text{c.m.}} + \frac{1}{2e^2} \mathcal{L}_{\text{v.m.}} + \frac{k}{2} \mathcal{L}_{\text{St.}} \right) , \tag{4.9}$$

where the chiral multiplet, vector multiplet and Stückelberg multiplet Lagrangians are given by

$$\mathcal{L}_{\text{c.m.}} = \bar{\phi} \left(-D_\mu^2 + \sigma\bar{\sigma} + i\mathrm{D} \right) \phi + \bar{F}F - i\bar{\psi} \left(\not{D} - \sigma\gamma_- - \bar{\sigma}\gamma_+ \right) \psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi \,, \tag{4.10}$$

$$\mathcal{L}_{\text{v.m.}} = \mathcal{F}^2 + \partial_\mu \sigma \partial^\mu \bar{\sigma} + \mathrm{D}^2 + i\bar{\lambda}\not{\partial}\lambda \,, \tag{4.11}$$

$$\mathcal{L}_{\text{St.}} = \bar{G}G + \bar{\sigma}\sigma + D_\mu \bar{p} D^\mu p + i\mathrm{D}(p + \bar{p}) - i\bar{\chi}\not{\partial}\chi - i\bar{\lambda}\chi + i\bar{\chi}\lambda \,. \tag{4.12}$$

By D_μ we denote the gauge covariant derivative which acts canonically on the chiral multiplet fields while its action on the on the Stückelberg scalar is given by

$$D_\mu p = \partial_\mu p - iA_\mu \,. \tag{4.13}$$

The action (4.9) is invariant under $\mathcal{N} = (2,2)$ super-Poincaré transformations generated by the Dirac spinor supercharges Q and $\bar{Q}$. The explicit realization of the supersymmetry algebra on the fields can be found in appendix 4.A.

Localization supercharge

To compute the elliptic genus via supersymmetric localization we choose the supercharge

$$\mathcal{Q} = -Q_1 - \bar{Q}_1 \,, \tag{4.14}$$

whose action on the fields is parametrized by the Grassmann even spinors

$$\epsilon = \bar{\epsilon} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{4.15}$$

This supercharge satisfies the algebra

$$\mathcal{Q}^2 = -2i\partial_{\bar{w}} + 2i\delta_G(A_{\bar{w}}) \tag{4.16}$$

[1] Note that the volume form in the complex coordinates $\{w, \bar{w}\}$ takes the form $\mathrm{d}^2 x = \frac{i}{2} \mathrm{d}^2 w$.

where δ_G denotes a gauge transformation. One can easily show that the vector multiplet and chiral multiplet Lagrangians are, up to total derivatives, $\mathcal{Q}$-exact, *i.e.*

$$\begin{aligned}\mathcal{L}_{\text{v.m.}} &= \mathcal{Q}\mathcal{V}_{\text{v.m.}} + \partial_\mu J^\mu_{\text{v.m.}}\,,\\ \mathcal{L}_{\text{c.m.}} &= \mathcal{Q}\mathcal{V}_{\text{c.m.}} + \partial_\mu J^\mu_{\text{c.m.}}\,.\end{aligned} \tag{4.17}$$

The explicit form of $\mathcal{V}_{\text{v.m.}}$ and $\mathcal{V}_{\text{c.m.}}$ can be found in appendix 4.B.

In contrast to the vector and chiral multiplets, the action governing the dynamics of the Stückelberg field P is not globally $\mathcal{Q}$-exact[2] [59]. This must be the case since the coefficient of the P-field action, k, appears explicitly in the expression for the central charge (4.2). Therefore to obtain the contribution from the Stückelberg multiplet to the path integral via supersymmetric localization, a non-degenerate and globally $\mathcal{Q}$-exact deformation term would need to be constructed. This, however, is not necessary since the Stückelberg Lagrangian (4.12) is quadratic, leading to a Gaussian path integral which can be explicitly carried out.

Consequently, exploiting the $\mathcal{Q}$-exactness of the vector multiplet and chiral multiplet Lagrangians, we may rescale them independently by positive real numbers leaving the path integral invariant. While rescaling the chiral multiplet amounts to the replacement $\mathcal{L}_{\text{c.m.}} \to t\mathcal{L}_{\text{c.m.}}$, rescaling the vector multiplet Lagrangian is equivalent to rescaling of the Yang-Mills coupling e. In particular, we may compute the path integral in the large t and $1/e^2$ limit, keeping the product te^2 finite. The saddle-point approximation is one-loop exact.

R-charges and twisted boundary conditions

In order to compute the path integral corresponding to the elliptic genus (4.8), we need to identify the charge assignments of the GLSM fields under the right moving R-symmetry. Using the explicit expression [59] for the corresponding current

$$\begin{aligned}j^R_w &= -i\left[\bar{\psi}_1\psi_1 + \frac{k}{2}\bar{\chi}_1\chi_1 + \frac{i}{e^2}\bar{\sigma}\partial\sigma - iD_w(p-\bar{p})\right]\,,\\ j^R_{\bar{w}} &= -i\left[\frac{1}{2e^2}\bar{\lambda}_2\lambda_2 + \frac{i}{e^2}\bar{\sigma}\bar{\partial}\sigma - iD_{\bar{w}}(p-\bar{p})\right]\,,\end{aligned} \tag{4.18}$$

[2] Locally, one can write the Stückelberg action as $\mathcal{Q}\Lambda$, however, one can check that Λ does not fall off fast enough near infinity to be in the Hilbert space of the theory.

yields the charge assignments

$$q^R_\sigma = q^R_{\lambda_2} = q^R_{\psi_1} = q^R_{\chi_1} = 1\,, \tag{4.19}$$

and the opposite charge for the barred fields. The zero mode of p_2 also carries R-charge, equal to $-\frac{1}{k}$. In addition to the dynamical fields, supersymmetry also fixes the R-charges of the auxiliary fields to be $q_F = q_G = 1$.

The R-charges above determine the boundary conditions that need to be imposed on the GLSM path integral[3]. Equivalently, the boundary conditions can be implemented via weakly gauging the right moving R-symmetry. This amounts to turning on a background gauge-field

$$a^R = \frac{v}{2i\tau_2}(\mathrm{d}w - \mathrm{d}\bar{w})\,, \tag{4.20}$$

for the R-symmetry with the constant parameter v satisfying $z = e^{2\pi i v}$. Note that only the boundary condition along one cycle of the torus is affected; this will also ensure a holomorphic dependence on the variable z. The background gauge field is incorporated into the theory via gauge covariantization

$$\partial_\mu \to \partial_\mu - \delta_R(a^R)\,. \tag{4.21}$$

Gauge fixing and supersymmetric Faddeev-Popov ghosts

To impose the Lorentz gauge $\partial_\mu A^\mu = 0$ in the path integral in a supersymmetric way, we introduce the Grassmann odd BRST operator $\mathcal{Q}_{\mathrm{BRST}}$, the gauge fixed localization supercharge $\hat{\mathcal{Q}} = \mathcal{Q} + \mathcal{Q}_{\mathrm{BRST}}$ and the ghost and anti-ghost doublets $\{c, a_\circ\}$ and $\{\bar{c}, b\}$ such that

$$\begin{aligned} \mathcal{Q}_{\mathrm{BRST}} &= \delta_G(c)\,, \\ \mathcal{Q}^2_{\mathrm{BRST}} &= \delta_G(a_\circ)\,, \\ \hat{\mathcal{Q}}^2 &= -2i\bar{\partial} + 2i\delta_R(a^R) + 2i\delta_G(a_\circ)\,. \end{aligned} \tag{4.22}$$

This fixes the supersymmetry transformations of the ghost and anti-ghost fields up to field redefinitions[4]. Note that the vector and chiral multiplet Lagrangians are also $\hat{\mathcal{Q}}$ exact by

[3] This is the method followed in [53] for the gauged Wess-Zumino-Witten model that describes the cigar conformal field theory.

[4] See appendix 4.B for details.

virtue of the gauge invariance of $\mathcal{V}_{\text{v.m.}}$ and $\mathcal{V}_{\text{c.m.}}$. We further add to the action the $\hat{\mathcal{Q}}$-exact gauge fixing term

$$\begin{aligned}\frac{1}{2e^2}\hat{\mathcal{Q}}\mathcal{V}_{\text{G.F.}} = \frac{1}{2e^2}\big[&(\partial_\mu A^\mu)^2 + (i\partial_\mu A^\mu + b/2)^2 - \bar{c}\partial_\mu^2 c - i\bar{c}\bar{\partial}\left(\bar{c} + 2\lambda_1 + 2\bar{\lambda}_1\right) \\ &- ib_\circ b - i\bar{c}_\circ c + i\bar{c}c_\circ - \bar{b}_\circ\left(a_\circ - 2iA_{\bar{w}}\right)\big]\,,\end{aligned} \tag{4.23}$$

where we have introduced the constant ghost doublets $\{b_\circ, c_\circ\}$ and $\{\bar{b}_\circ, \bar{c}_\circ\}$ in order to remove the ghost zero-mode[4].

4.2.2 Evaluation of the Path Integral

The path integral that we are interested in takes the form

$$\chi = \int \mathcal{D}[\Phi, V, C, P] e^{-S_{\text{St.}} - \frac{i}{4\pi}\int \mathrm{d}^2 w \hat{\mathcal{Q}}\mathcal{V}} \tag{4.24}$$

where

$$\mathcal{V} = t\mathcal{V}_{\text{c.m.}} + \frac{1}{2e^2}(\mathcal{V}_{\text{v.m.}} + \mathcal{V}_{\text{G.F.}}) \equiv t\mathcal{V}_{\text{c.m.}} + \frac{1}{2e^2}\mathcal{V}^{\text{G.F.}}_{\text{v.m.}}\,. \tag{4.25}$$

As explained in section 4.2.1, the $\hat{\mathcal{Q}}$-exactness of $\hat{\mathcal{Q}}\mathcal{V}$ ensures that the path integral is independent of the couplings t and e. We may therefore carry out the path integration in large t and $1/e^2$ limit, while keeping te^2 finite, where the saddle-point approximation is valid.

Consequently, we first have to extract the space of saddle points of $\hat{\mathcal{Q}}\mathcal{V}$ which we denote by $\mathcal{M}$. Explicitly, the chiral multiplet and the gauge fixed vector multiplet terms in $\hat{\mathcal{Q}}\mathcal{V}$ are given by

$$\begin{aligned}\hat{\mathcal{Q}}\mathcal{V}_{\text{c.m.}} &= \bar{F}F + D^\mu\bar{\phi}D_\mu\phi + \bar{\phi}(\bar{\sigma}\sigma + i\mathrm{D})\phi - 2i\bar{\psi}_2 D_w\psi_2 + 2i\bar{\psi}_1(D_{\bar{w}} - ia^R_{\bar{w}})\psi_1 \\ &\quad + i\bar{\psi}_2\bar{\sigma}\psi_1 - i\bar{\psi}_1\sigma\psi_2 + i\bar{\phi}(\bar{\lambda}_1\psi_2 - \bar{\lambda}_2\psi_1) - i(\bar{\psi}_1\lambda_2 - \bar{\psi}_2\lambda_1)\phi\,, \\ \hat{\mathcal{Q}}\mathcal{V}^{\text{G.F}}_{\text{v.m.}} &= \partial^\mu A^\nu\partial_\mu A_\nu + \mathrm{D}^2 + \tilde{b}^2 + (\partial^\mu + ia^\mu_R)\bar{\sigma}(\partial_\mu - ia^R_\mu)\sigma - ib_\circ b - \bar{b}_\circ\left(a_\circ - 2iA_{\bar{w}}\right) \\ &\quad - 2i\bar{\lambda}_1\bar{\partial}\lambda_1 + 2i\bar{\lambda}_2(\partial - ia^R_w)\lambda_2 + \partial^\mu\bar{c}\partial_\mu c - i\bar{c}\bar{\partial}\left(\bar{c} + 2\lambda_1 + 2\bar{\lambda}_1\right) - i\bar{c}_\circ c + i\bar{c}c_\circ\,,\end{aligned} \tag{4.26}$$

where $\tilde{b} = b/2 + i\partial_\mu A^\mu$. Before we look for the space of saddle points $\mathcal{M}$, note that the constant ghost multiplet fields $\{c_\circ, \bar{c}_\circ, b_\circ, \bar{b}_\circ\}$ appear as Lagrange multipliers and can be integrated out. This yields a delta function for the ghost zero-modes effectively removing them from the spectrum. The only remaining fermionic zero-mode is $\lambda_1 = \lambda_0$, whereas the space of bosonic zero modes can be identified with the first De Rham cohomology of the torus and can be parametrized by a constant parameter u as

$$A = \frac{\bar{u}}{2i\tau_2}\mathrm{d}w - \frac{u}{2i\tau_2}\mathrm{d}\bar{w}\,. \tag{4.27}$$

We remark that the bosonic superpartner of the fermionic zero-mode λ_0 is the constant mode of the vector multiplet auxiliary field, D_0, and has to be treated separately. The space of saddle-points is therefore parametrized by $\{\mathrm{D}_0, u, \bar{u}, \lambda_0, \bar{\lambda}_0\}$. We normalize all bosonic and fermionic zero modes to have unit norm when Gaussian wavefunctions are integrated over the torus worldsheet. With this in mind, the partition function (4.24) reduces to the Gaussian path integral

$$\chi = \int \frac{\mathrm{d}^2 u}{2i\tau_2} \int \mathrm{dD}_0 \int d^2\lambda_0 \int \mathcal{D}[P] \int \hat{\mathcal{D}}[eV, eC, t^{-1/2}\Phi] e^{-S_{\mathrm{St.}}|_{\mathcal{M}} - \frac{i}{4\pi}\int \mathrm{d}^2 w \hat{\mathcal{Q}}\mathcal{V}|_{\mathrm{quad}\ \mathcal{M}}} \tag{4.28}$$

where $\hat{\mathcal{D}}[eV, eC, t^{-1/2}\Phi]$ denotes the path integral measure with the zero-modes removed. Here $\hat{\mathcal{Q}}\mathcal{V}|_{\mathrm{quad}\ \mathcal{M}}$ is the quadratic action for the fluctuations of order e and order $t^{-1/2}$ for the vector multiplet and chiral multiplet fields respectively. The integral over u is performed over the whole of the complex plane. The origin of this plane is on the one hand the torus of holonomies of the gauge field, and on the other hand the winding modes of the compact boson p_2 (the imaginary part of the Stückelberg field) on the toroidal worldsheet. The latter can be soaked up into the holonomy variable u such that the integral indeed covers the complex plane once.

The Stückelberg Lagrangian evaluated on the saddle points $\mathcal{M}$ is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{St.}}\Big|_{\mathcal{M}} &= |G|^2 + 4|\partial p_1|^2 + 4\left(\partial p_2 - \frac{\bar{u} - v/k}{2i\tau_2}\right)\left(\bar{\partial} p_2 - \frac{u - v/k}{2i\tau_2}\right) \\ &\quad + 2i\bar{\chi}_1(\bar{\partial} + \frac{v}{2\tau_2})\chi_1 - 2i\bar{\chi}_2\partial\chi_2 + 2i\mathrm{D}p_1 + i\bar{\chi}_2\lambda_0 + i\bar{\lambda}_0\chi_2\,.\end{aligned} \tag{4.29}$$

Note that the kinetic term for the Stückelberg multiplet is not canonically normalized due to the factor of k out front in equation (4.9). To this end we rescale each field in the Stückelberg multiplet by $\sqrt{k}$. This allows us to define a canonical measure in the path integral. With this rescaling, a few things have to be kept in mind: firstly, the periodicity of the imaginary part of the Stückelberg field, p_2, becomes $2\pi\sqrt{k}$. Secondly, the quadratic terms involving the zero-modes of the vector multiplet fields acquire an overall factor $\sqrt{k}$.

The first integral to carry out is over the fermionic zero modes. To perform this integral, we isolate all the terms that depend on λ_0:

$$\int d^2\lambda_0 e^{\frac{1}{4\pi}\int d^2w\left(\bar{\phi}\bar{\lambda}_0\psi_2+\bar{\psi}_2\lambda_0\phi+\sqrt{k}\bar{\chi}_2\lambda_0+\sqrt{k}\bar{\lambda}_0\chi_2\right)}\,. \tag{4.30}$$

We pause here to point out an important difference with earlier calculations of the elliptic genera of gauged linear sigma models [49–51]. This involves the coupling of the gaugino zero modes with the fermionic partners χ_2 of the Stückelberg field p. In the path integral over the P multiplet, we also have to soak up the fermionic zero modes of χ_2, as can be seen from the Lagrangian in (4.29). Therefore, on expanding the zero mode part of the Lagrangian, the only term that contributes is the quartic term in the fermions and that leads to a factor of k.

In the models with only chiral and vector multiplets [51], one obtains rather a four-point correlator involving the chiral multiplet fields. The further coupling to the P-multiplet determines the fact that another correlator is to be evaluated in the chiral multiplet sector, which turns out to be just $\langle 1\rangle$. The only coupling between the Stückelberg multiplet and the vector multiplet that remains is the coupling to the zero mode of the auxiliary field D. Separating out this integral, the result of doing the λ_0 and χ_0 zero mode integrals we obtain

$$\chi = k\int\frac{\mathrm{d}^2u}{2i\tau_2}\int \mathrm{dD}_0\int\hat{\mathcal{D}}[P]e^{-\int d^2w\mathcal{L}_{\mathrm{St.}}|_{\lambda_0=\bar{\lambda}_0=0}}\,\langle 1\rangle_{\mathrm{free}}\,, \tag{4.31}$$

where the expectation value is in the chiral and vector multiplet sector and the hat indicates that the fermionic zero mode of the P-multiplet is excluded in the path integral. The free partition function is well known and is given by [50]

$$\langle 1\rangle_{\mathrm{free}} = \chi_{\mathrm{v.m.}}\,\chi_{\mathrm{c.m.}}\,, \tag{4.32}$$

where these are given by[5]

$$\chi_{\mathrm{v.m.}} = \frac{\hat{\det}(\bar{\partial})}{\det(\bar{\partial}+\frac{v}{2\tau_2})}\qquad\text{and}\qquad \chi_{\mathrm{c.m.}} = \frac{\det(\bar{\partial}+\frac{u+v}{2\tau_2})}{\det(\bar{\partial}+\frac{v}{2\tau_2})}\,. \tag{4.33}$$

[5]Strictly speaking we should write Pfaffians for the fermionic path integrals.

See Appendix 4.C for the explicit evaluation of the chiral multiplet contribution. The vector multiplet contribution will naturally combine with the Stückelberg fields. Turning to the latter, we have a product of functional determinants Δ_i for each of the component fields. For the field χ_2, it is given by

$$\hat{\Delta}_{\chi_2} = \hat{\det}(\partial)\,. \tag{4.34}$$

The hat over the χ_2 determinant denotes that the zero mode has been removed. The χ_1 fermion is charged under the R-current and leads to

$$\Delta_{\chi_1} = \det(\bar{\partial} + \frac{v}{2\tau_2})\,. \tag{4.35}$$

Let us consider the field p_1, the real part of p. It has a bosonic zero mode and has to be treated carefully. Taking care of the coupling of p_1 to the auxiliary field D_0, we find that

$$\begin{aligned} \int \mathrm{d}\mathrm{D}_0\ \Delta_{p_1} &= \int \mathrm{d}\mathrm{D}_0 \int \mathcal{D}[p_1] e^{\int \mathrm{d}^2 w\left[-\mathrm{D}_0^2 + 4p_1(\partial\bar{\partial})p_1 - 2i\sqrt{k}\mathrm{D}_0 p_1\right]} \\ &= \frac{1}{(\hat{\det}(\partial\bar{\partial}))^{\frac{1}{2}}} \int \mathrm{d}\mathrm{D}_0 \int \mathrm{d}p_{1,0} e^{-\int d^2 w(\mathrm{D}_0^2 + 2i\sqrt{k}\mathrm{D}_0 p_{1,0})} \\ &= \frac{1}{\sqrt{k}} \frac{1}{(\hat{\det}(\partial\bar{\partial}))^{\frac{1}{2}}} \end{aligned} \tag{4.36}$$

Therefore, up to constant factors up front we obtain just the square root of the inverse determinant. The last component field left is the imaginary part p_2 of the Stückelberg field. This is a periodic variable with period $2\pi\sqrt{k}$, on account of the earlier rescaling. It is only the zero mode of this field that is charged under the gauge field and the R-current while the non-zero modes are uncharged. The partition function for such a field has been reviewed in [53] and is given by

$$\Delta_{p_2} = \frac{\sqrt{k}}{(\hat{\det}(\partial\bar{\partial}))^{\frac{1}{2}}} \times e^{-\frac{\pi k}{\tau_2}(u-\frac{v}{k})(\bar{u}-\frac{v}{k})}\,. \tag{4.37}$$

The factor of $\sqrt{k}$ arises from the radius of the compact direction [84]. Note that this contribution is not holomorphic. The non-holomorphicity arises from the momentum and winding modes along the compact direction. The Stückelberg field therefore contributes a factor

$$\chi_{\mathrm{St.}} = \frac{\det(\bar{\partial} + \frac{v}{2\tau_2})}{\hat{\det}(\bar{\partial})}\, e^{-\frac{\pi k}{\tau_2}(u-\frac{v}{k})(\bar{u}-\frac{v}{k})}\,. \tag{4.38}$$

A crucial point to note is that non-zero modes of the P multiplet have combined to produce exactly the inverse of the contribution from the vector multiplet. This is as expected from the supersymmetric Higgs mechanism. Combining all of the above factors, we find that the path integral takes the form

$$\chi(\tau, v) = k \int \frac{d^2u}{2i\tau_2}\, \chi_{\text{c.m.}}(\tau, u, v)\, e^{-\frac{k\pi}{\tau_2}(u-\frac{v}{k})(\bar{u}-\frac{v}{k})}\,. \tag{4.39}$$

Using the results in Appendix 4.C, one can write this as

$$\chi(\tau, v) = k \int \frac{d^2u}{2i\tau_2} \frac{\theta_{11}(\tau, u+v)}{\theta_{11}(\tau, u)}\, e^{-\frac{k\pi}{\tau_2}(u-\frac{v}{k})(\bar{u}-\frac{v}{k})}\,. \tag{4.40}$$

Shifting the holonomy variable u by $\frac{v}{k}$ and using the rewriting the u-integral in terms of the variables (s_1, s_2) and momentum and winding numbers[6] (m, w), we obtain

$$\chi(\tau, v) = k \int_0^1 ds_1 \int_0^1 ds_2 \frac{\theta_{11}(\tau, s_1\tau + s_2 + v)}{\theta_{11}(\tau, s_1\tau + s_2)} \sum_{n,m} e^{2\pi i n v} e^{-\frac{k\pi}{\tau_2}|(n+s_1)\tau + s_2 + m + \frac{v}{k}|^2} e^{2\pi i v_2(m + s_2 + \frac{v}{k} + \tau(n+s_1))}\,. \tag{4.41}$$

This is the elliptic genus of the cigar conformal field theory [53], here exhibited in the form valid for complexified chemical potentials [85].

4.2.3 Elliptic Genera for GLSMs with Multiple Chiral Fields

From the discussion in the preceding section, and especially equation (4.39), it is clear how to obtain the elliptic genera of the models with more chiral multiplets. The interaction Lagrangian that couples the Stückelberg field to the vector multiplet remains the same; therefore the discussion regarding the fermionic zero modes also remains the same. Consequently the correlator to be calculated in the chiral multiplet path integral continues to be the identity. Therefore, we include the free partition function of a chiral multiplet in equation (4.70) for each of the N chiral multiplets. The only difference is in the R-charge of the Stückelberg field; from the discussion in [59], it is clear that the R-charge is given by $-\frac{N}{k}$.

[6] A note about ranges: in [53], the conventions are such that the gauge holonomy variables (s_1, s_2) take values between 0 and 1. It is possible to combine them along with the winding and momentum quantum numbers (n, m) to obtain a complex holonomy variable u which takes values on the complex plane.

Putting all this together the path integral therefore is given by

$$\chi(\tau, v) = k \int \frac{d^2u}{2i\tau_2} \left[\frac{\theta_{11}(\tau, u+v)}{\theta_{11}(\tau, u)}\right]^N e^{-\frac{k\pi}{\tau_2}(u-\frac{Nv}{k})(\bar{u}-\frac{Nv}{k})} . \tag{4.42}$$

This is precisely the elliptic genus that was proposed in [86], on the basis of its modular and elliptic properties as well as its coding of wound bound states [54] in the background spacetime in (4.3). All properties are consistent with it being the elliptic genus of a conformal field theory with central charge $c = 3N(1+2N/k)$. Indeed, we have now derived this fact from first principles, through localization. As shown in [53, 86], it is also possible to define a twisted elliptic genus by including chemical potentials for global symmetries; in this case these are the $U(1)^N$ phase rotations of each of the chiral multiplet fields Φ_i. The resulting twisted genera take the form

$$\chi(\tau, v, \beta_i) = k \int \frac{d^2u}{2i\tau_2} \prod_{i=1}^{N} \left[\frac{\theta_{11}(\tau, u+v+\beta_i)}{\theta_{11}(\tau, u+\beta_i)}\right] e^{-\frac{k\pi}{\tau_2}(u-\frac{Nv}{k})(\bar{u}-\frac{Nv}{k})} . \tag{4.43}$$

These twisted genera were decomposed in holomorphic and remainder contributions in [86]. We refer to [86] for the calculation of the shadow and an interpretation of the remainder term in terms of the asymptotic geometry.

4.2.4 Elliptic Genera for GLSMs with Multiple Stückelberg Fields

In subsection 4.1.2 we discussed gauged linear sigma models with gauge groups $U(1)^M$ and M Stückelberg fields. We specified the gauge charges R_{ia} of all the chiral fields. In order to write the formula for the elliptic genus, we need the R-charges of the component fields as well. These can be read off from the R-current recorded in [59]. The fermions have unit R-charge while the zero mode of the P_a field has charge $-\frac{b_a}{k_a}$, where b_a is given in equation (4.7). Using the same logic as before, one can write down the elliptic genus of such a theory as an integral over the M holonomies of the $U(1)^M$ gauge theory:

$$\chi(\tau, v) = \int \prod_{a=1}^{M} k_a \frac{d^2u_a}{2i\tau_2} \prod_{i=1}^{N} \left[\frac{\theta_{11}(\tau, \sum_{a=1}^{M} R_{ia}u_a + v)}{\theta_{11}(\tau, \sum_{a=1}^{M} R_{ia}u_a)}\right] e^{-\sum_{a=1}^{M} \frac{k_a\pi}{\tau_2}(u_a-\frac{b_a}{k_a}v)(\bar{u}_a-\frac{b_a}{k_a}v)} . \tag{4.44}$$

One can further generalize this result by including chemical potentials for global symmetries of the model. It would also be interesting to analyze the decomposition of this formula in terms of holomorphic contributions and non-holomorphic terms that modularly covariantize the contributions of right-moving ground states, following [53, 57, 58, 86]

Appendix

4.A Supersymmetry variations and Lagrangians

In this appendix we record Lagrangians and supersymmetry variations of the fields. We follow the notations and conventions of [37] regarding spinors and gamma matrices. We choose a basis such that the two-dimensional γ_μ matrices coincide with the Pauli matrices $\sigma^{1,2}$. The chirality matrix is given by

$$\gamma_3 = -i\gamma^1\gamma^2 = \sigma^3 \,. \tag{4.45}$$

This allows to define projection operators

$$\gamma_\pm = \frac{1}{2}(1 \pm \gamma_3) \,, \tag{4.46}$$

which we will use in the supersymmetry variations below. With this choice, if we consider a two component Dirac spinor λ, with

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \,, \tag{4.47}$$

then the components λ_1 and λ_2 have definite chirality ± 1 respectively.

4.A.1 Vector Multiplet

The vector multiplet supersymmetry transformations are given by

$$\begin{aligned} \delta\sigma &= \bar{\epsilon}\gamma_-\lambda - \epsilon\gamma_+\bar{\lambda} \\ \delta\bar{\sigma} &= \bar{\epsilon}\gamma_+\lambda - \epsilon\gamma_-\bar{\lambda} \\ \delta\lambda &= i\left(\partial\!\!\!/\sigma\gamma_+ + \partial\!\!\!/\bar{\sigma}\gamma_- + \gamma^3\mathcal{F} + i\mathrm{D}\right)\epsilon \\ \delta\bar{\lambda} &= -i\left(\partial\!\!\!/\sigma\gamma_- + \partial\!\!\!/\bar{\sigma}\gamma_+ - \gamma^3\mathcal{F} + i\mathrm{D}\right)\bar{\epsilon} \\ \delta A_\mu &= -\frac{i}{2}\left(\bar{\epsilon}\gamma_\mu\lambda + \epsilon\gamma_\mu\bar{\lambda}\right) \\ \delta\mathrm{D} &= -\frac{i}{2}\left(\bar{\epsilon}\partial\!\!\!/\lambda - \epsilon\partial\!\!\!/\bar{\lambda}\right) . \end{aligned} \tag{4.48}$$

The Lagrangian governing the dynamics of the vector multiplet fields may be written as

$$\mathcal{L}_{\text{v.m.}} = \frac{1}{2e^2}\left(\mathcal{F}^2 + \partial_\mu\sigma\partial^\mu\bar{\sigma} + \mathrm{D}^2 + i\lambda\partial\!\!\!/\bar{\lambda}\right) . \tag{4.49}$$

4.A.2 Chiral Multiplet with Minimal Coupling

The supersymmetry transformations for a chiral multiplet with minimal coupling to the vector multiplet are

$$\begin{aligned} \delta\phi &= \bar{\epsilon}\psi \\ \delta\bar{\phi} &= \epsilon\bar{\psi} \\ \delta\psi &= i\left(D\!\!\!\!/\,\phi + \sigma\phi\gamma_+ + \bar{\sigma}\phi\gamma_-\right)\epsilon + \bar{\epsilon}F \\ \delta\bar{\psi} &= i\left(D\!\!\!\!/\,\bar{\phi} + \bar{\phi}\sigma\gamma_- + \bar{\phi}\bar{\sigma}\gamma_+\right)\bar{\epsilon} + \epsilon\bar{F} \\ \delta F &= i\left(D_\mu\psi\gamma^\mu + \sigma\psi\gamma_+ + \bar{\sigma}\psi\gamma_- + \lambda\phi\right)\epsilon \\ \delta\bar{F} &= i\left(D_\mu\bar{\psi}\gamma^\mu + \bar{\psi}\sigma\gamma_- + \bar{\psi}\bar{\sigma}\gamma_+ - \bar{\phi}\bar{\lambda}\right)\bar{\epsilon}\,, \end{aligned} \tag{4.50}$$

and the corresponding Lagrangian is given by

$$\mathcal{L}_{\text{c.m.}} = \bar{\phi}\left(-D_\mu^2 + \sigma\bar{\sigma} + i\mathrm{D}\right)\phi + \bar{F}F - i\bar{\psi}\left(D\!\!\!\!/ - \sigma\gamma_- - \bar{\sigma}\gamma_+\right)\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi\,. \tag{4.51}$$

4.A.3 Chiral Multiplet with Stückelberg Coupling

The Stückelberg field is coupled to the gauge field via the covariant differentiation

$$D_\mu p = \partial_\mu p - iA_\mu \,. \tag{4.52}$$

The supersymmetry transformations then take the form

$$\begin{aligned}
\delta p &= \bar\epsilon\chi \\
\delta\bar p &= \epsilon\bar\chi \\
\delta\chi &= i\left(\not{D}p + \sigma\gamma_+ + \bar\sigma\gamma_-\right)\epsilon + \bar\epsilon G \\
\delta\bar\chi &= i\left(\not{D}\bar p + \sigma\gamma_- + \bar\sigma\gamma_+\right)\bar\epsilon + \epsilon\bar G \\
\delta G &= -i\left(\partial_\mu\psi\gamma^\mu + \lambda\right)\epsilon \\
\delta\bar G &= -i\left(\partial_\mu\bar\psi\gamma^\mu - \bar\lambda\right)\bar\epsilon \,,
\end{aligned} \tag{4.53}$$

and the Lagrangian is given by

$$\mathcal{L}_{\text{St.}} = k\left(\bar G G + \bar\sigma\sigma + D_\mu\bar p D^\mu p - i\bar\chi\not{\partial}\chi - i\bar\lambda\chi + i\bar\chi\lambda + i\mathrm{D}(p+\bar p)\right) \,. \tag{4.54}$$

4.B Deformation Lagrangian

In this appendix, we discuss the supersymmetry variations of the fields under the localization supercharge, the exactness of various Lagrangians, as well as the technical subtleties in the localization scheme due to the gauge invariance of the model.

4.B.1 Vector Multiplets and Chiral Multiplets

The supersymmetry transformation of the vector and chiral multiplet fields, including the background R-current, take the form

$$\begin{aligned}
\mathcal{Q}\sigma &= -\lambda_2 & \mathcal{Q}\lambda_2 &= 2i(\bar\partial - ia^R_{\bar w})\sigma \\
\mathcal{Q}\bar\sigma &= \bar\lambda_2 & \mathcal{Q}\bar\lambda_2 &= -2i(\bar\partial + ia^R_{\bar w})\bar\sigma \\
\mathcal{Q}A_w &= i(\lambda_1 + \bar\lambda_1)/2 & \mathcal{Q}\lambda_1 &= i\mathcal{F} - \mathrm{D} \\
\mathcal{Q}A_{\bar w} &= 0 & \mathcal{Q}\bar\lambda_1 &= i\mathcal{F} + \mathrm{D} \\
\mathcal{Q}\mathrm{D} &= i\bar\partial(\lambda_1 - \bar\lambda_1) & \mathcal{Q}\mathcal{F} &= -\bar\partial(\lambda_1 + \bar\lambda_1)
\end{aligned} \tag{4.55}$$

and

$$
\begin{aligned}
\mathcal{Q}\phi &= -\psi_2 & \mathcal{Q}\psi_2 &= 2iD_{\bar{w}}\phi \\
\mathcal{Q}\bar{\phi} &= -\bar{\psi}_2 & \mathcal{Q}\bar{\psi}_2 &= 2iD_{\bar{w}}\bar{\phi} \\
\mathcal{Q}\psi_1 &= F + i\sigma\phi & \mathcal{Q}F &= -2i(D_{\bar{w}} - ia^R_{\bar{w}})\psi_1 + i\sigma\psi_2 + i\lambda_2\phi \\
\mathcal{Q}\bar{\psi}_1 &= \bar{F} + i\bar{\phi}\bar{\sigma} & \mathcal{Q}\bar{F} &= -2i(D_{\bar{w}} + ia^R_{\bar{w}})\bar{\psi}_1 + i\bar{\psi}_2\bar{\sigma} - i\bar{\phi}\bar{\lambda}_2 \,.
\end{aligned}
\tag{4.56}
$$

It is straightforward to check that the Lagrangian of the vector and chiral multiplets, including the background R-current couplings, is Q-exact: if $\tilde{\mathcal{L}} = \mathcal{L}_{\text{v.m.}} + \mathcal{L}_{\text{c.m.}}$, then $\tilde{\mathcal{L}} = \mathcal{Q}\mathcal{V}$ where

$$
\mathcal{V} = \mathcal{V}_{\text{v.m.}} + \mathcal{V}_{\text{c.m.}} \,, \tag{4.57}
$$

and

$$
\mathcal{V}_{\text{v.m.}} = \frac{1}{4g^2}\left[\bar{\lambda}_1(\mathrm{D} - i\mathcal{F}) - \lambda_1(\mathrm{D} + i\mathcal{F}) + 2i\bar{\lambda}_2\,(\partial - ia^R_w)\sigma - 2i\lambda_2\,(\partial + ia^R_w)\bar{\sigma}\right] \,, \tag{4.58}
$$

$$
\mathcal{V}_{\text{c.m.}} = \frac{1}{2}\left[\bar{\psi}_1(F - i\sigma\phi) + (\bar{F} - i\bar{\phi}\bar{\sigma})\psi_1 - 2i\bar{\psi}_2\,D_w\phi - 2iD_w\bar{\phi}\,\psi_2 - i\bar{\phi}(\lambda_1 - \bar{\lambda}_1)\phi\right] \,. \tag{4.59}
$$

4.B.2 Gauge Fixing and Ghosts

To implement the gauge fixing condition we define the (Grassmann odd) BRST operator $\mathcal{Q}_{\text{BRST}}$ and the ghost multiplet $\{c, a\}$ such that

$$
\begin{aligned}
\mathcal{Q}_{\text{BRST}} &= iq_G c \\
\mathcal{Q}^2_{\text{BRST}} &= iq_G a \,.
\end{aligned}
\tag{4.60}
$$

To fix the supersymmetry transformation rules for the ghost multiplet, we require that the supercharge $\hat{\mathcal{Q}} = \mathcal{Q} + \mathcal{Q}_{\text{BRST}}$ satisfy the algebra

$$
\hat{\mathcal{Q}}^2 = -2i\bar{\partial} - 2q_R a^R_{\bar{w}} - 2q_G a \,. \tag{4.61}
$$

This requires the ghost field c to transform as

$$
\hat{\mathcal{Q}}c = a - 2iA_{\bar{w}} \,, \tag{4.62}
$$

while the bosonic superpartner of the ghost field, a, must be supersymmetric, *i.e.* $\hat{\mathcal{Q}}a = 0$. We next define the anti-ghost multiplet $\{\bar{c}, b\}$ and the constant (zero-mode) multiplets

$\{a_\circ, c_\circ\}$ and $\{\bar{c}_\circ, b_\circ\}$ and add to our deformation term the gauge fixing terms

$$\begin{aligned}\hat{\mathcal{Q}}\mathcal{V}_{\text{G.F.}} &= \frac{1}{2}\hat{\mathcal{Q}}\left(\bar{c}\mathcal{G} - \frac{i}{4}\bar{c}b - \bar{c}a_\circ + b_\circ c\right) \\ &= \frac{1}{2}\left(\mathcal{G}^2 + (i\mathcal{G} + b/2)^2 - \bar{c}\hat{\mathcal{Q}}\mathcal{G} - \frac{i}{2}\bar{c}\bar{\partial}\bar{c} + iba_\circ - i\bar{c}c_\circ + i\bar{c}_\circ c + b_\circ(a - 2iA_{\bar{w}})\right)\,,\end{aligned} \tag{4.63}$$

where we have used the supersymmetry transformations

$$\begin{aligned}&\hat{\mathcal{Q}}\bar{c} = ib \qquad && \hat{\mathcal{Q}}c_\circ = 0 \qquad && \hat{\mathcal{Q}}\bar{c}_\circ = 0 \\ &\hat{\mathcal{Q}}b = -2\bar{\partial}\bar{c} && \hat{\mathcal{Q}}a_\circ = ic_\circ && \hat{\mathcal{Q}}b_\circ = i\bar{c}_\circ\,.\end{aligned} \tag{4.64}$$

In Lorentz gauge, the ghost deformation term therefore has the form

$$\begin{aligned}\hat{\mathcal{Q}}\mathcal{V}_{\text{G.F.}} = \frac{1}{2}\Big(&(\partial_\mu A^\mu)^2 + (b/2 + i\partial_\mu A^\mu)^2 - 4\bar{c}\partial\bar{\partial}c - i\bar{c}\bar{\partial}\left(\bar{c} + 2\lambda_1 + 2\bar{\lambda}_1\right) \\ &+ iba_\circ - i\bar{c}c_\circ + i\bar{c}_\circ c + b_\circ(a - 2iA_{\bar{w}})\Big)\,.\end{aligned} \tag{4.65}$$

4.C Product representation of theta functions

In this appendix, we record some formulas for calculating functional determinants of free fields with twisted boundary conditions on the torus, and their representation in terms of θ functions. The free (twisted) path integral of the chiral multiplets which we encountered in the main text can be put in the form

$$\chi_{\text{c.m.}} = \frac{\det(\bar{\partial} + \frac{u+v}{2\tau_2})}{\det(\bar{\partial} + \frac{u}{2\tau_2})} \tag{4.66}$$

We will diagonalize these differential operators on the torus by using the following infinite set of functions:

$$f_{r,s}(w, \bar{w}) = \frac{1}{2i\tau_2}((r + s\tau)\bar{w} - (r + s\bar{\tau})w)\,, \tag{4.67}$$

where $r, s \in \mathbb{Z}$. One can check that $\Psi_{r,s} = e^{if_{r,s}}$ is single valued under the transformations

$$w \to w + 2\pi \qquad w \to w + 2\pi\tau\,. \tag{4.68}$$

Using this basis, it is clear that the ratio of determinants takes form of an infinite product

$$\chi_{\text{c.m.}} = \frac{u+v}{v} \prod_{\{r,s\} \neq \{0,0\}} \frac{((r+s\tau)+u+v)}{((r+s\tau)+u)} \,. \tag{4.69}$$

The factor out front can be absorbed by including the $(r,s) = (0,0)$ in the infinite product. One can check explicitly that this is a Jacobi form with a given weight and index. Using this knowledge, one can rewrite the expression as

$$\chi_{\text{c.m.}} = \prod_{\{r,s\}} \frac{((r+s\tau)+u+v)}{((r+s\tau)+u)} = \frac{\theta_{11}(\tau, u+v)}{\theta_{11}(\tau, u)} \,. \tag{4.70}$$

Similar formulae are also used in [50].

Chapter 5

Conclusion and Future Directions

In this dissertation we have performed exact computations in two dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories. We have computed the exact sphere partition function for GLSMs with $SU(2|1)_A$ symmetry and demonstrated that the partition function admits two equivalent representations. In the Coulomb branch of the sphere partition function for these theories, we integrate over the Cartan subgroup of the gauge group and sum over the overall Dirac monopole configurations. The integration may be carried out by closing the contour in a suitable way leading to a double sum representation of the partition function. Moreover, this expression can be shown to factorize into product of vortex and anti-vortex partition functions obtained by Shadchin [68]. We have shown that this expression can be obtained directly via localizing the path integral to the Higgs branch which consists of vortex and anti-vortex configurations localized at the north and the south poles of the sphere. We have also computed the exact sphere partition function for GLSMs with $SU(2|1)_B$ symmetry. The partition function localizes to the Higgs branch of the theory which is generically a Kähler quotient manifold.

In the light of the conjecture by Jockers *et. al.* [38] which was later proved by Gomis and Lee [39], our results provide a purely gauge theoretic derivation of the Kähler potentials on the Kähler moduli and the complex structure moduli of Calabi-Yau 3-folds. This derivation does not rely on mirror symmetry and was used in [38] to compute the exact Kähler potential for the Kähler moduli of Calabi-Yau manifolds which do not have known mirrors. Using the prescription in [38] one can extract the Gromov-Witten invariants from the Kähler potential on the Kähler moduli of any Calabi-Yau 3-fold whose GLSM description is known. These invariants encode all worldsheet instanton corrections in the infrared CFT. Our results also pave the road to search for new mirror manifolds. Mirror symmetry maps the Kähler potential on the Kähler moduli of a Calabi-Yau M to the Kähler potential on the complex structure moduli of the mirror Calabi-Yau W. The corresponding $SU(2|1)_A$ and $SU(2|1)_B$ invariant GLSM sphere partition functions obey

$$Z_A(M) = Z_B(W)\,, \quad Z_A(W) = Z_B(M)\,. \tag{5.1}$$

Our results have many applications to dualities among GLSMs including Seiberg duality, and correspondence with Toda CFTs. Some early results in these directions were presented in[1] [37] and are developed further in [69].

An interesting application of our results has been in [87] where the hemisphere partition function of the $SU(2|1)_A$ invariant GLSMs are computed. The hemisphere partition function is shown to compute the central charge of a D-brane. It would be quite interesting to extend this result to $SU(2|1)_B$ invariant GLSMs.

Another interesting direction to pursue is to study spherical surface defects of gauge theories in higher dimensions. It is then possible to couple the gauge theories constructed here in a supersymmetric fashion to supersymmetric gauge theories in the ambient space. For instance, inserting such surface defects in $\mathcal{N}=2$ gauge theories on a four dimensional sphere may help developing the gauge theory/Toda CFT correspondence further.

In the last chapter, we have shown that in the presence of Stückelberg superfields, we can still fruitfully apply the technique of localization. The dynamics determines the observable to be calculated by localization in the chiral and vector multiplet sectors. We have demonstrated that the appearance of extra fermionic zero modes simplifies the observable to be calculated. After applying localization to the chiral and vector multiplet sectors, we are left with a Gaussian integration in the Stückelberg sector. Performing this path integral, one finds that the non-zero modes of the Stückelberg multiplet cancel the contribution from the vector multiplet, as one would expect from the supersymmetric Higgs mechanism. We thereby have a derivation of the elliptic genera of gauged linear sigma models from first principles. The associated models are non-compact and the elliptic genera are real Jacobi forms.

We were thus able to prove, from first principles, a formula for elliptic genera of asymptotic linear dilaton spaces conjectured in [86]. Moreover, we have generalized this formula

[1] Some results on Seiberg duality were also presented in [40].

to abelian gauge theories in two dimensions with multiple Stückelberg fields.

These models appear in the context of mirror symmetry in two dimensions [27, 32] and in the worldsheet description of wrapped NS5 branes [60]. It will be interesting to verify mirror symmetry at the level of the elliptic genera. Verifications of mirror symmetry in tensor products and orbifolds of the cigar conformal field theory and minimal models were performed in [88]. In order to check the mirror duality for the genera computed in this here, one has to calculate elliptic genera of non-compact Landau-Ginzburg models and their orbifolds more generally then has been done hitherto.

Applying the calculation of these worldsheet indices to space-time string theory BPS state counting, along the lines of [89–92], would be a further worthwhile endeavour. It would also be interesting to find examples of non-holomorphic elliptic genera in higher dimensions, perhaps by the addition of Stückelberg fields. Since the phenomenon of non-holomorphic contributions to indices is generic for theories with continuous spectra, higher dimensional manifestations are likely to be found.

It would also be quite interesting to generalize the work on sphere partition functions to GLSMs with Stückelberg fields.

Appendix A

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Nima Doroud <ndoroud@perimeterinstitute.ca>

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2 messages

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Appendix B

Notations and Conventions

We use the following conventions for indices:

$\mu, \nu, \dots = 1, 2$	coordinate indices on S^2
$\hat{\mu}, \hat{\nu}, \dots = \hat{1}, \hat{2}$	tangent space indices
$\alpha, \beta, \gamma, \dots = 1, 2$	Dirac spinor indices
$m, n, p = 1, 2, 3$	indices for $SU(2)$ generators

B.1 S^2 Conventions

We work in polar coordinates $(x^1, x^2) = (\theta, \varphi)$ where the metric on S^2 can be written as

$$\mathrm{d}s^2 = r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) . \tag{B.1}$$

The canonical choice of orientation is

$$\varepsilon_{12} = \sqrt{h}\, \varepsilon_{\hat{1}\hat{2}} = r^2 \sin\theta \,, \tag{B.2}$$

with the corresponding volume-form

$$\mathrm{d}^2x\sqrt{h} \; = r^2 \sin\theta \; \mathrm{d}\theta \wedge \mathrm{d}\varphi \,. \tag{B.3}$$

The simplest choice of zweibein is

$$e^{\hat{1}} = r\mathrm{d}\theta \qquad \text{and} \qquad e^{\hat{2}} = r\sin\theta \; \mathrm{d}\varphi \,, \tag{B.4}$$

with the spin connection given by

$$\omega^{\hat{\mu}\hat{\nu}} = -\varepsilon^{\hat{\mu}\hat{\nu}} \cos\theta \, \mathrm{d}\varphi \,. \tag{B.5}$$

By D_i we denote the gauge-covariant derivative

$$D_\mu = \nabla_\mu - iA_\mu \,, \tag{B.6}$$

where ∇_i is the usual covariant derivative and A_i is the gauge field. The corresponding curvature is given by

$$F_{\mu\nu} = \varepsilon_{\mu\nu} F_{\hat{\mu}\hat{\nu}} = \nabla_\mu A_\nu - \nabla_\nu A_\mu - i[A_\mu, A_\nu] \,. \tag{B.7}$$

B.2 Spinors and the Clifford Algebra

Our conventions for spinors are the same as in [93] and are listed below. Let τ_m denote the standard Pauli matrices given by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.8}$$

We take our spinors to be anti-commuting Dirac spinors ϵ_α. These spinors are acted on by the γ-matrices defined by

$$(\gamma_{\hat{m}})_\alpha^{\ \beta} : \quad \gamma_{\hat{m}} = \tau_{\hat{m}} \ . \tag{B.9}$$

Evidently, the matrices $\gamma^{\hat{\mu}}$ satisfy the two dimensional Clifford algebra

$$\left\{\gamma^{\hat{\mu}}, \gamma^{\hat{\nu}}\right\} = 2\delta^{\hat{\mu}\hat{\nu}} \,, \tag{B.10}$$

and $\gamma^{\hat{3}} = -i\gamma^{\hat{1}}\gamma^{\hat{2}}$ is the two dimensional chirality matrix.[1]

The spinor indices are raised and lowered by the (anti-symmetric) charge conjugation matrix as

$$\epsilon^\alpha = C^{\alpha\beta}\epsilon_\beta \qquad \text{and} \qquad \epsilon_\alpha = C_{\alpha\beta}\epsilon^\beta \,, \tag{B.11}$$

[1] In terms of the σ and $\bar{\sigma}$ matrices introduced in [93], the γ-matrices are given by $\gamma^{m\ \beta}_{\ \alpha} = \frac{i}{2}\epsilon^{mnp}\sigma_{n\,\alpha\dot{\alpha}}\bar{\sigma}_p^{\dot{\alpha}\beta}$.

with the consistency condition

$$C_{\alpha\gamma}C^{\gamma\beta} = \delta_\alpha^\beta\,. \tag{B.12}$$

More explicitly, we take $C^{12} = C_{21} = 1$ and $C^{21} = C_{12} = -1$.

We adapt the Northwest-Southeast convention for the implicit contraction of the spinor indices, *i.e.* for two spinors ϵ and λ we define

$$\epsilon\lambda \equiv \epsilon^\alpha\lambda_\alpha = \lambda\epsilon \qquad \text{and} \qquad \epsilon\gamma^{\hat{m}}\lambda \equiv \epsilon^\alpha(\gamma^{\hat{m}})_\alpha{}^\beta\lambda_\beta = -\lambda\gamma^{\hat{m}}\epsilon\,. \tag{B.13}$$

Note that the γ-matrices with both spinor indices lowered

$$(\gamma^{\hat{m}})_{\alpha\beta} \equiv C_{\beta\delta}\gamma^{\hat{m}\,\delta}_{\ \alpha}\,, \tag{B.14}$$

are symmetric and are numerically equal to $(-\tau_3, -i, \tau_1)$ for $\hat{m} = (1, 2, 3)$ respectively.

B.3 Fierz Identities

Let $\bar{\epsilon}, \lambda$ and ϵ be anticommuting spinors. The following Fierz identities are used extensively in our calculations

$$(\bar{\epsilon}\lambda)\epsilon + (\lambda\epsilon)\bar{\epsilon} + (\bar{\epsilon}\epsilon)\lambda = 0\,, \tag{B.15}$$

$$(\bar{\epsilon}\gamma_{\hat{m}}\lambda)\gamma^{\hat{m}}\epsilon + (\bar{\epsilon}\lambda)\epsilon + 2(\bar{\epsilon}\epsilon)\lambda = 0\,. \tag{B.16}$$

Appendix C

$\mathcal{N} = (2,2)$ Supersymmetry on S^2

C.1 The Superconformal Algebra in the Standard Basis

The globally defined $\mathcal{N} = (2,2)$ superconformal group in two dimensions is generated by the bosonic symmetries $\{J_0, L_0, L_\pm; \bar{J}_0, \bar{L}_0, \bar{L}_\pm\}$ and the fermionic generators $\{G^\pm_\pm; \bar{G}^\pm_\pm\}$ satisfying the (anti-)commutation relations [9]

$$\begin{aligned}
\left[L_0, G^s_\pm\right] &= \mp\frac{1}{2}G^s_\pm & \left[\bar{L}_0, \bar{G}^s_\pm\right] &= \mp\frac{1}{2}\bar{G}^s_\pm \\
\left[L_\pm, G^s_\mp\right] &= \pm G^s_\pm & \left[\bar{L}_\pm, \bar{G}^s_\mp\right] &= \pm\bar{G}^s_\pm \\
\left[J_0, G^\pm_s\right] &= \pm G^\pm_s & \left[\bar{J}_0, \bar{G}^\pm_s\right] &= \pm\bar{G}^\pm_s \\
\left[L_m, L_n\right] &= (m-n)L_{m+n} & \left[\bar{L}_m, \bar{L}_n\right] &= (m-n)\bar{L}_{m+n} \\
\left\{G^+_\pm, G^-_\pm\right\} &= 2L_\pm & \left\{\bar{G}^+_\pm, \bar{G}^-_\pm\right\} &= 2\bar{L}_\pm \\
\left\{G^+_\pm, G^-_\mp\right\} &= 2L_0 \pm J_0 & \left\{\bar{G}^+_\pm, \bar{G}^-_\mp\right\} &= 2\bar{L}_0 \pm \bar{J}_0
\end{aligned} \tag{C.1}$$

with all the other (anti-)commutations vanishing. This algebra admits an automorphism σ whose action on the generators is given by

$$\sigma\left(G^\pm_\pm\right) = G^\mp_\pm, \quad \sigma(J_0) = -J_0, \quad \sigma = 1 \text{ otherwise.} \tag{C.2}$$

We shall see below that this is precisely the map between the $su(2|1)_A$ and the $su(2|1)_B$ subalgebras.

C.2 The Superconformal Algebra in the S^2 Basis

The $\mathcal{N} = (2,2)$ superconformal algebra in the S^2 basis is spanned by the bosonic generators

$$J_m, K_m, R, \mathcal{A}\,, \tag{C.3}$$

and the supercharges

$$Q_\alpha, S_\alpha, \bar{Q}_\alpha, \bar{S}_\alpha\,. \tag{C.4}$$

J_m generate the $SU(2)$ isometries of S^2 while K_m generate the conformal symmetries of S^2. R and $\mathcal{A}$ are each a $U(1)$ R-symmetry generator, the first being non-chiral and the latter being chiral. These generators are related to the standard generators introduced above via

$$\begin{aligned}
J_1 &= \frac{i}{2}\left(L_- + L_+ + \bar{L}_- + \bar{L}_+\right) & K_1 &= -\frac{1}{2}\left(L_- + L_+ - \bar{L}_- - \bar{L}_+\right) \\
J_2 &= \frac{1}{2}\left(L_- - L_+ - \bar{L}_- + \bar{L}_+\right) & K_2 &= \frac{i}{2}\left(L_- - L_+ + \bar{L}_- - \bar{L}_+\right) \\
J_3 &= L_0 - \bar{L}_0 & K_3 &= i\left(L_0 + \bar{L}_0\right) \\
R &= J_0 + \bar{J}_0 & \mathcal{A} &= -J_0 + \bar{J}_0 \\
S &= \frac{1}{\sqrt{2}}\begin{pmatrix} G^+_+ + i\bar{G}^+_- \\ iG^+_- + \bar{G}^+_+ \end{pmatrix} & \bar{S} &= \frac{1}{\sqrt{2}}\begin{pmatrix} G^-_+ + i\bar{G}^-_- \\ iG^-_- + \bar{G}^-_+ \end{pmatrix} \\
Q &= \frac{1}{\sqrt{2}}\begin{pmatrix} -iG^-_+ - \bar{G}^-_- \\ G^-_- + i\bar{G}^-_+ \end{pmatrix} & \bar{Q} &= \frac{1}{\sqrt{2}}\begin{pmatrix} iG^+_+ + \bar{G}^+_- \\ -G^+_- - i\bar{G}^+_+ \end{pmatrix}
\end{aligned} \tag{C.5}$$

and satisfy the algebra[1]

$$\begin{aligned}
&\{S_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R && [J_m, S^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} S_\beta && [R, S_\alpha] = +S_\alpha \\
&\{\bar{S}_\alpha, \bar{Q}_\beta\} = -\gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R && [J_m, Q^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} Q_\beta && [R, Q_\alpha] = -Q_\alpha \\
&\{Q_\alpha, \bar{Q}_\beta\} = \gamma^m_{\alpha\beta} K_m - \frac{i}{2} C_{\alpha\beta} \mathcal{A} && [J_m, \bar{Q}^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} \bar{Q}_\beta && [R, \bar{Q}_\alpha] = +\bar{Q}_\alpha \\
&\{S_\alpha, \bar{S}_\beta\} = \gamma^m_{\alpha\beta} K_m + \frac{i}{2} C_{\alpha\beta} \mathcal{A} && [J_m, \bar{S}^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} \bar{S}_\beta && [R, \bar{S}_\alpha] = -\bar{S}_\alpha \\
&[J_m, J_n] = i\epsilon_{mnp} J^p && [K_m, S^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} \bar{Q}_\beta && [\mathcal{A}, S_\alpha] = i\bar{Q}_\alpha \\
&[K_m, K_n] = -i\epsilon_{mnp} J^p && [K_m, Q^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} \bar{S}_\beta && [\mathcal{A}, Q_\alpha] = -i\bar{S}_\alpha \\
&[J_m, K_n] = i\epsilon_{mnp} K^p && [K_m, \bar{Q}^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} S_\beta && [\mathcal{A}, \bar{Q}_\alpha] = -iS_\alpha \\
& && [K_m, \bar{S}^\alpha] = -\frac{1}{2}\gamma_m{}^{\alpha\beta} Q_\beta && [\mathcal{A}, \bar{S}_\alpha] = iQ_\alpha \,.
\end{aligned} \tag{C.6}$$

This algebra admits a $\mathbb{Z}_2$ automorphism, under which

$$\begin{aligned}
J_m, R, Q_\alpha, S_\alpha &\to J_m, R, Q_\alpha, S_\alpha \\
K_m, \mathcal{A}, \bar{Q}_\alpha, \bar{S}_\alpha &\to -K_m, -\mathcal{A}, -\bar{Q}_\alpha, -\bar{S}_\alpha \,.
\end{aligned} \tag{C.7}$$

The generators $\{J_m, R, S, Q\}$ form a subalgebra which is the $SU(2|1)_A$ algebra

$$\begin{aligned}
&[J_m, J_n] = i\epsilon_{mnp} J_p && [J_m, Q_\alpha] = -\frac{1}{2}\gamma_m{}^\beta{}_\alpha Q_\beta && [J_m, S_\alpha] = -\frac{1}{2}\gamma_m{}^\beta{}_\alpha S_\beta \\
&\{S_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R && [R, Q_\alpha] = -Q_\alpha && [R, S_\alpha] = S_\alpha \,,
\end{aligned} \tag{C.8}$$

which was used in [37]. In addition to this automorphism, the algebra (C.6) inherits the automorphism σ defined in (C.2). This implies that $\{\sigma(J_m), \sigma(R), \sigma(S), \sigma(Q)\}$ given by

$$\begin{aligned}
\sigma(J_m) &= J_m & \sigma(S) &= \frac{S + \bar{S}}{2} + i\frac{Q + \bar{Q}}{2} \\
\sigma(R) &= \mathcal{A} & \sigma(Q) &= -i\frac{S - \bar{S}}{2} + \frac{Q - \bar{Q}}{2}
\end{aligned} \tag{C.9}$$

[1] The generator of the $U(1)$ axial symmetry $\mathcal{A}$ used here defers from the one used in [37] by a factor of i.

also form the $SU(2|1)_B$ subalgebra with the (anti-)commutation relations

$$\begin{aligned}
&[J_m, J_n] = i\epsilon_{mnp}J_p && [J_m, \sigma(Q)] = -\frac{1}{2}\gamma_m\sigma(Q) && [J_m, \sigma(S)] = -\frac{1}{2}\gamma_m\sigma(S) \\
&\{\sigma(S)_\alpha, \sigma(Q)_\beta\} = \gamma^m_{\alpha\beta}J_m - \frac{1}{2}C_{\alpha\beta}\mathcal{A} && [\mathcal{A}, \sigma(Q)] = -\sigma(Q) && [\mathcal{A}, \sigma(S)] = \sigma(S)\,,
\end{aligned} \tag{C.10}$$

which is precisely the $SU(2|1)_B$ algebra used in [41].

C.3 Weyl Covariantization

The superconformal transformations are easily obtained from the Poincaré supersymmetry transformations in flat space by demanding that once the flat metric is replaced by a curved metric, that the supersymmetry transformations are covariant under Weyl transformations. In this process, the constant supersymmetry parameters of flat space are replaced by conformal Killing spinors, which obey

$$\nabla_\mu\epsilon = \gamma_\mu\tilde{\epsilon} \qquad \nabla_\mu\bar{\epsilon} = \gamma_\mu\tilde{\bar{\epsilon}}\,. \tag{C.11}$$

Using that the fields and conformal Killing spinors transform with definite weight under a Weyl transformation

$$g_{\mu\nu} \to e^{2\Omega(x)}g_{\mu\nu} \tag{C.12}$$

we obtain the required superconformal transformations by imposing Weyl covariance. The terms that need to be modified in the vector and chiral multiplet flat space supersymmetry transformations (which can be obtained by dimensionally reducing the four dimensional $\mathcal{N} = 1$ supersymmetry transformations in [93] to two dimensions) to make them Weyl covariant are[2]

$$
\begin{aligned}
\bar\epsilon \not{D}\lambda &\longrightarrow \bar\epsilon\not{D}\lambda - \lambda\not{\nabla}\bar\epsilon \\
\epsilon\not{D}\bar\lambda &\longrightarrow \epsilon\not{D}\bar\lambda - \bar\lambda\not{\nabla}\epsilon \\
\not{D}\sigma_{1,2}\epsilon &\longrightarrow \not{D}\sigma_{1,2}\epsilon + \sigma_{1,2}\not{\nabla}\epsilon \\
\not{D}\sigma_{1,2}\bar\epsilon &\longrightarrow \not{D}\sigma_{1,2}\bar\epsilon + \sigma_{1,2}\not{\nabla}\bar\epsilon \\
\not{D}\phi\epsilon &\longrightarrow \not{D}\phi\epsilon + \frac{q}{2}\phi\not{\nabla}\epsilon \\
\not{D}\bar\phi\bar\epsilon &\longrightarrow \not{D}\bar\phi\bar\epsilon + \frac{q}{2}\phi\not{\nabla}\bar\epsilon \\
\not{D}\psi\epsilon &\longrightarrow \not{D}\psi\epsilon - \frac{q}{2}\psi\not{\nabla}\epsilon \\
\not{D}\bar\psi\bar\epsilon &\longrightarrow \not{D}\bar\psi\bar\epsilon - \frac{q}{2}\bar\psi\not{\nabla}\bar\epsilon\,,
\end{aligned}
\tag{C.13}
$$

where we have used the following Weyl weights w

SUSY		vector multiplet						chiral multiplet					
ϵ	$\bar\epsilon$	A_μ	σ_1	σ_2	λ	$\bar\lambda$	D	ϕ	ψ	F	$\bar\phi$	$\bar\psi$	$\bar F$
$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	1	$\frac{3}{2}$	$\frac{3}{2}$	2	$\frac{q}{2}$	$\frac{q+1}{2}$	$\frac{q+2}{2}$	$\frac{q}{2}$	$\frac{q+1}{2}$	$\frac{q+2}{2}$

where ω is the charge $\varphi \to e^{-w\Omega(x)}\varphi$ under the Weyl transformation (C.12).

[2]The coefficients of the extra terms are fixed by demanding that the combination transforms *covariantly* under Weyl transformations and, in general, depend on the Weyl weight of the fields as well as the dimension of space.

In this way, we obtain the two dimensional $\mathcal{N}=(2,2)$ superconformal transformations for the vector multiplet

$$\begin{aligned}
\delta A_\mu &= -\frac{i}{2}\left(\bar{\epsilon}\gamma_\mu\lambda + \epsilon\gamma_\mu\bar{\lambda}\right),\\
\delta\sigma_1 &= \frac{1}{2}\left(\bar{\epsilon}\lambda - \epsilon\bar{\lambda}\right),\\
\delta\sigma_2 &= -\frac{i}{2}\left(\bar{\epsilon}\gamma_{\hat{3}}\lambda + \epsilon\gamma_{\hat{3}}\bar{\lambda}\right),\\
\delta \mathrm{D} &= -\frac{i}{2}\bar{\epsilon}\left(\not{D}\lambda + [\sigma_1,\lambda] - i\left[\sigma_2,\gamma^{\hat{3}}\lambda\right]\right)\\
&\quad + \frac{i}{2}\lambda\not{\nabla}\bar{\epsilon} + \frac{i}{2}\epsilon\left(\not{D}\bar{\lambda} - \left[\sigma_1,\bar{\lambda}\right] - i\left[\sigma_2,\gamma^{\hat{3}}\bar{\lambda}\right]\right) - \frac{i}{2}\bar{\lambda}\not{\nabla}\epsilon,\\
\delta\lambda &= \left(i\gamma^{\hat{3}}F_{\hat{1}\hat{2}} - \gamma^{\hat{3}}\not{D}\sigma_2 + i\not{D}\sigma_1 - \gamma^{\hat{3}}[\sigma_1,\sigma_2] - \mathrm{D}\right)\epsilon + i\sigma_1\not{\nabla}\epsilon - \sigma_2\gamma^{\hat{3}}\not{\nabla}\epsilon,\\
\delta\bar{\lambda} &= \left(i\gamma^{\hat{3}}F_{\hat{1}\hat{2}} - \gamma^{\hat{3}}\not{D}\sigma_2 - i\not{D}\sigma_1 + \gamma^{\hat{3}}[\sigma_1,\sigma_2] + \mathrm{D}\right)\bar{\epsilon} - i\sigma_1\not{\nabla}\bar{\epsilon} - \sigma_2\gamma^{\hat{3}}\not{\nabla}\bar{\epsilon},
\end{aligned} \tag{C.14}$$

and chiral multiplet

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi\\
\delta\bar{\phi} &= \epsilon\bar{\psi}\\
\delta\psi &= i\left(\not{D}\phi + \sigma_1\phi - i\sigma_2\phi\gamma^{\hat{3}} + \frac{q}{2}\phi\not{\nabla}\right)\epsilon + \bar{\epsilon}F\\
\delta\bar{\psi} &= i\left(\not{D}\bar{\phi} + \bar{\phi}\sigma_1 + i\bar{\phi}\sigma_2\gamma^{\hat{3}} + \frac{q}{2}\bar{\phi}\not{\nabla}\right)\bar{\epsilon} + \epsilon\bar{F}\\
\delta F &= -i\left(D_\mu\psi\gamma^\mu + \sigma_1\psi - i\sigma_2\psi\gamma^{\hat{3}} + \lambda\phi + \frac{q}{2}\psi\not{\nabla}\right)\epsilon\\
\delta\bar{F} &= -i\left(D_\mu\bar{\psi}\gamma^\mu + \bar{\psi}\sigma_1 + i\bar{\psi}\sigma_2\gamma^{\hat{3}} - \bar{\phi}\bar{\lambda} + \frac{q}{2}\bar{\psi}\not{\nabla}\right)\bar{\epsilon}\,.
\end{aligned} \tag{C.15}$$

The spinors ϵ and $\bar{\epsilon}$ serve as the parameters of the superconformal transformations, such that each independent conformal Killing spinor is associated with one of the supercharges in the superconformal algebra. On S^2, we can take the conformal Killing spinors to satisfy

$$\nabla_\mu\epsilon_s = \frac{s}{2r}\gamma_\mu\gamma^{\hat{3}}\epsilon_s \qquad \text{and} \qquad \nabla_\mu\bar{\epsilon}_{\bar{s}} = \frac{\bar{s}}{2r}\gamma_\mu\gamma^{\hat{3}}\bar{\epsilon}_{\bar{s}} \tag{C.16}$$

with $s, \bar{s} = \pm$. There are four independent solutions to these equations

$$\epsilon_s = \exp\left(-s\frac{i\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right)\epsilon_\circ^s\,, \tag{C.17}$$

$$\bar{\epsilon}_{\bar{s}} = \exp\left(-\bar{s}\frac{i\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{i\varphi}{2}\gamma^{\hat{3}}\right)\bar{\epsilon}_\circ^{\bar{s}}\,. \tag{C.18}$$

parametrized by four independent constant spinors $\epsilon_\circ^\pm$ and $\bar{\epsilon}_\circ^\pm$. A general superconformal transformation is then generated by a linear combination of the supercharges parametrized as follows

$$\delta_{\epsilon_+} = \epsilon_\circ^+\tilde{\gamma}_+ Q, \qquad \delta_{\epsilon_-} = \epsilon_\circ^-\tilde{\gamma}_-\bar{S}, \qquad \bar{\delta}_{\bar{\epsilon}_+} = \bar{\epsilon}_\circ^+\tilde{\gamma}_+\bar{Q}, \qquad \bar{\delta}_{\bar{\epsilon}_-} = -\bar{\epsilon}_\circ^-\tilde{\gamma}_- S \tag{C.19}$$

where $\tilde{\gamma}_\pm$ satisfy

$$\begin{aligned}
\tilde{\gamma}_\pm &= \frac{1}{\sqrt{2}}\left(\mathbb{1} \pm i\gamma^3\right) = \pm i\gamma^3\tilde{\gamma}_\mp \\
\tilde{\gamma}_+^2 &= -\tilde{\gamma}_-^2 = i\gamma^3, \quad \tilde{\gamma}_+\tilde{\gamma}_- = \mathbb{1}\,.
\end{aligned} \tag{C.20}$$

Using the conformal Killing spinor equations above, the superconformal algebra is realized on the vector multiplet fields as

$$\begin{aligned}
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\lambda &= \mathcal{L}_v\lambda + i\,[\Lambda, \lambda] + i\frac{s-\bar{s}}{2}\alpha\lambda + i\frac{s+\bar{s}}{2}\Theta\gamma^{\hat{3}}\lambda - 3i\frac{s+\bar{s}}{2}\alpha\lambda\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\bar{\lambda} &= \mathcal{L}_v\bar{\lambda} + i\left[\Lambda, \bar{\lambda}\right] - i\frac{s-\bar{s}}{2}\alpha\bar{\lambda} - i\frac{s+\bar{s}}{2}\Theta\gamma^{\hat{3}}\bar{\lambda} - 3i\frac{s+\bar{s}}{2}\alpha\bar{\lambda}\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,A_\mu &= (\mathcal{L}_v A)_\mu + D_\mu\Lambda\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\sigma_1 &= \mathcal{L}_v\sigma_1 + i\,[\Lambda, \sigma_1] - (s+\bar{s})\Theta\sigma_2 - i(s+\bar{s})\alpha\sigma_1\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\sigma_2 &= \mathcal{L}_v\sigma_2 + i\,[\Lambda, \sigma_2] + (s+\bar{s})\Theta\sigma_1 - i(s+\bar{s})\alpha\sigma_2\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\mathrm{D} &= \mathcal{L}_v\mathrm{D} + i\,[\Lambda, \mathrm{D}] - 2i(s+\bar{s})\alpha\mathrm{D}\,,
\end{aligned} \tag{C.21}$$

and $[\delta_\epsilon, \delta_\epsilon] = [\delta_{\bar{\epsilon}}, \delta_{\bar{\epsilon}}] = 0$ on all the fields. Therefore $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ generates a space-time transformation as well as a gauge transformation, an R and $\mathcal{A}$ R-symmetry transformation and a Weyl transformation. The parameters of these transformations are given by

$$\begin{aligned}
v^\mu &= i\bar{\epsilon}\gamma^\mu\epsilon\,, \\
\Lambda &= (\bar{\epsilon}\epsilon)\sigma_1 - i(\bar{\epsilon}\gamma^{\hat{3}}\epsilon)\sigma_2 - v^\mu A_\mu\,, \\
\Theta &= \frac{1}{2r}\bar{\epsilon}\epsilon\,, \\
\alpha &= -\frac{1}{2r}\bar{\epsilon}\gamma^{\hat{3}}\epsilon\,,
\end{aligned} \tag{C.22}$$

where we have omitted the subscript s and $\bar{s}$ on the spinors. Note that the spacetime transformation is realized by the Lie derivative on bosonic fields and by the Lie-Lorentz derivative on the fermions. More explicitly, the Lie-Lorentz derivative along the vector field ξ is given by

$$\mathcal{L}_v \equiv v^\mu \nabla_\mu + \frac{1}{4} \nabla_\mu v_\nu \gamma^{\mu\nu} \, . \tag{C.23}$$

The superconformal algebra is realized on the chiral multiplet fields as

$$\begin{aligned}
[\delta_\epsilon, \delta_{\bar{\epsilon}}] \, \psi &= \mathcal{L}_v \psi + i\Lambda\psi + i\frac{s-\bar{s}}{2}(1-q)\alpha\psi - i\frac{s+\bar{s}}{2}\Theta\gamma^{\hat{3}}\psi - i\frac{s+\bar{s}}{2}(q+1)\alpha\psi \, , \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}] \, \bar{\psi} &= \mathcal{L}_v \bar{\psi} - i\bar{\psi}\Lambda + i\frac{s-\bar{s}}{2}(q-1)\alpha\bar{\psi} + i\frac{s+\bar{s}}{2}\Theta\gamma^{\hat{3}}\bar{\psi} - i\frac{s+\bar{s}}{2}(q+1)\alpha\bar{\psi} \, , \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}] \, \phi &= \mathcal{L}_v \phi + i\Lambda\phi - i\frac{s-\bar{s}}{2}q\alpha\phi - i\frac{s+\bar{s}}{2}q\alpha\phi \, , \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}] \, \bar{\phi} &= \mathcal{L}_v \bar{\phi} - i\bar{\phi}\Lambda + i\frac{s-\bar{s}}{2}q\alpha\bar{\phi} - i\frac{s+\bar{s}}{2}q\alpha\bar{\phi} \, , \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}] \, F &= \mathcal{L}_v F + i\Lambda F + i\frac{s-\bar{s}}{2}(2-q)\alpha F - i\frac{s+\bar{s}}{2}(q+2)\alpha F \, , \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}] \, \bar{F} &= \mathcal{L}_v \bar{F} - i\bar{F}\Lambda + i\frac{s-\bar{s}}{2}(q-2)\alpha\bar{F} - i\frac{s+\bar{s}}{2}(q+2)\alpha\bar{F} \, ,
\end{aligned} \tag{C.24}$$

where the parameters of the transformations are the same as those for the vector multiplet fields (C.22).

To obtain the $su(2|1)_A$ supersymmetry transformations, we restrict the superconformal transformations (C.14) and (C.15) we have constructed to those associated with Q_α and S_α, which are parametrized by ϵ_+ and $\bar{\epsilon}_-$. The corresponding realization of the algebra on the fields is given by (C.21) and (C.24) with $s = 1$ and $\bar{s} = -1$.

On the other hand, in order to obtain the $su(2|1)_B$ supersymmetry transformations we need to restrict the superconformal transformations (C.14) and (C.15) to those associated with $\sigma(Q)$ and $\sigma(S)$. This is equivalent – up to field redefinitions – to realizing the $su(2|1)_A$ algebra on the twisted chiral and twisted vector multiplets. The resulting transformations are presented in chapter 3.

In the rest of this dissertation, we find it convenient to perform the field redefinition $\mathrm{D} \to \mathrm{D} + \sigma_2/r$, after which we obtain the supersymmetry transformations presented in chapter 2.

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