Rayleigh Property of Lattice Path Matroids

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.
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Abstract

In this work, we studied the class of lattice path matroids $\mathcal{L}$, which was first introduced by J.E. Bonin. A.D. Mier, and M. Noy in [3]. Lattice path matroids are transversal, and $\mathcal{L}$ is closed under duals and minors, which in general the class of transversal matroids is not. We give a combinatorial proof of the fact that lattice path matroids are Rayleigh. In addition, this leads us to several research directions, such as which positroids are Rayleigh and which subclass of lattice path matroids are strongly Rayleigh.

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Chapter 1

Introduction

This thesis is motivated by finding classes of matroids that admit the Rayleigh property, which is an algebraic condition on bases enumerator of matroids. Consider the following graph G:

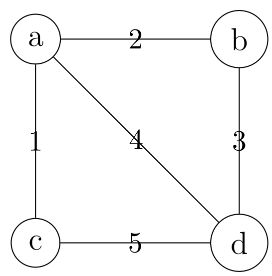

Figure 1.1. G

It is easy to verify that G has 8 spanning trees $\{124, 134, 345, 245, 123, 125, 235, 135\}$. Let those spanning be chosen equally likely. Then we choose a pair of them, and consider the following two cases: one of these two contains both edge 1 and 4 and the other does not contain either of them (case 1); one of these two contains edge 1 but not edge 4 and the other one contains edge 4 but not 1 (case 2). The probability of case 1 is $1/14$, which is less than or equal to the probability of case 2 that is $3/14$. When this happens we say edge 1 and 4 are negatively correlated. In addition, it is not hard to check not only edges 1 and 4 are negatively correlated. Every pair of edges for this graph is negatively correlated.

People who study matroid theory abstract this concept from graphs to matroids. Consider a cycle matroid of a connected graph, then the edges of the graph correspond to the elements in the ground set of the matroid and the spanning trees are the corresponding bases of the matroid . Let $\mathcal{M} = (E, \mathcal{B})$ be this matroid with ground set E and collections of bases $\mathcal{B}$. Define the bases enumerator as follows,

$$M(x) = \sum_{B \in \mathcal{B}} x^B, \quad \text{where } x^B = \prod_{i \in B} x_i$$

Consider a pair of elements e and f in E, and partition $\mathcal{B}$ into 4 sets

- $\mathcal{B}_e^f = \{B \in \mathcal{B} : e \in B, f \in E - B\}$.
- $\mathcal{B}_f^e = \{B \in \mathcal{B} : f \in B, e \in E - B\}$.
- $\mathcal{B}_{ef} = \{B \in \mathcal{B} : e, f \in B\}$.
- $\mathcal{B}^{ef} = \{B \in \mathcal{B} : e, f \in E - B\}$.

If we can partition $M(x)$ corresponding to these 4 sets, then the difference of probabilities that we mentioned above will be generalized as the following expression

$$\frac{\sum_{B\in\mathcal{B}_e^f} x^B \sum_{B\in\mathcal{B}_f^e} x^B - \sum_{B\in\mathcal{B}_{ef}} x^B \sum_{B\in\mathcal{B}^{ef}} x^B}{\sum_{B_1,B_2\in\mathcal{B}} x^{B_1}x^{B_2}}$$

If we set every $x_i = 1$, then we get the original probability. Since we consider the difference compared to 0, we can ignore the denominator. In this thesis, we study the the more general result, if we assign any value for x_i, what will happen? In fact, for some matroids if we give positive real assignment, then the result will always be greater than or equal to 0. Such matroids we call *Rayleigh matroids*. Not all matroids are Rayleigh, so we wonder which classes of matroids are Rayleigh.
Define M^e and M_f as follows:

$$M^e = M(x)\big|_{x_e=0}, \quad \text{and} \quad M_f = \frac{\partial M(x)}{\partial x_f}$$

Then we can write

$$M(x) = M^{ef} + x_e M_e^f + x_f M_f^e + x_e x_f M_{ef} \tag{1.1}$$

For elements $e, f \in E$, e and f are said to be *negatively correlated* if

$$\frac{M_e}{M(x)} \geq \frac{M_{ef}}{M_f}, \text{ when } x_i \geq 0 \text{ for all } i \in E$$

The concept of negative correlation was firstly introduced by Seymour and Welsh in [12](1975) for the case $x_i = 1$ for all i in the ground set of a matroid. After that Feder and Mihail defined *balance matroids* according to this in [9], a matroid is *negatively correlated* if for every $e, f \in E$

$$\frac{M_e}{M(x)} \geq \frac{M_{ef}}{M_f}$$

And a matroid M is *balanced* if every minor of M is negatively correlated.

It is not hard to show the above inequality is equivalent to to the following by several substitutions

$$M_f^e M_e^f - M_{ef}M^{ef} \geq 0$$

Define *Rayleigh difference* of a matroid M with respect to e, f as

$$\Delta M(e,f) = M_f^e M_e^f - M_{ef} M^{ef}$$

Then the problem will become to determine whether $\Delta M(e,f)$ is non-negative. A matroid is said to be *Rayleigh* if $\Delta M(e,f)$ is non-negative for any pair of e and f with nonnegative real assignment for x_i, where i is the element in ground set. Choe and Wagner gave this definition and proved several classes of matroids admits this property in [7], such as graphical matroids, sixth-root-of-unity matroids, uniform matroids and so on. I will give a more detailed review in 2.2.

People studying this desired to determine which classes of matroid are Rayleigh. In this thesis, we will study a subclass of transversal matroids $\mathcal{L}$, called lattice path matroids, which was first introduced by Bonin, de Mier and Noy in [3]. I will give a brief review of this class of matroids in the first half chapter 3. In the rest of chapter 3, I will present the main result as the following theorem. *For all lattice path matroids M, $\Delta M(e,f) \geq 0$ where $x_i \geq 0$ for all i of the ground set of M.* That is $\mathcal{L}$ is a class of Rayleigh matroids. The proof that I will give is a combinatorial proof by constructing an injective function from $\mathcal{B}_{ef} \times \mathcal{B}^{ef}$ to $\mathcal{B}_e^f \times \mathcal{B}_f^e$ which preserves elements in pairs of the corresponding bases, which implies that in the expression

$$M_f^e M_e^f - M_{ef} M^{ef}$$

all terms in $M_{ef} M^{ef}$ are included in $M_f^e M_e^f$. Hence, the Rayleigh difference of a lattice path matroid is just a polynomial where each term has a non-negative coefficient. Therefore, if we give any non-negative real assignment, then the value of Rayleigh difference is greater that or equal to 0. In the last chapter, I introduce generalized Catalan matroids which is a subclass of $\mathcal{L}$, we wonder does this class admits strong Rayleigh property. In addition, I have also introduce positroid which is a larger class of matroid that has $\mathcal{L}$ as its subclass.

Chapter 2

Preliminaries

2.1 Matroids

In this section, I will give a brief review of matroid theory. Most of the fact I have reviewed can be found in Oxley's book [11].

2.1.1 Definitions

A *matroid* M is a pair of sets $(E, \mathcal{I})$, where set E is a finite set and $\mathcal{I}$ is a collection of subsets of E that satisfy the following conditions:

- (**I0**) $\emptyset \in \mathcal{I}$.
- (**I1**) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
- (**I2**) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element e of $I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The set E is called the *ground set* of M (denoted as $E(M)$) and sets in $\mathcal{I}$ are called *independent* sets of M (denoted as $\mathcal{I}(M)$).

Lemma 2.1.1. *Let $M = (E, \mathcal{I})$ and $X \subseteq E$. Let I_1 and I_2 be two maximal independent set in X, then $|I_1| = |I_2|$.*

Proof. Suppose not, then there exist I_1, I_2 such that $|I_1| < |I_2|$. By ***(I2)***, there exists e in $I_2 - I_1$ such that $I_1 \cup \{e\}$ is in $\mathcal{I}$. This contradicts to the maximality of ***(I1)***. Hence, $|I_1| = |I_2|$ if they are both maximal. □

Let $M = (E, \mathcal{I})$ and $X \in E$, define $rank_M(X) = max\{|I| : I \in X \text{ and } I \in \mathcal{I}\}$. A maximum independent set is called a *basis* of M.

Lemma 2.1.2. *Let B_1 and B_2 be two bases of a matroid M, then $|B_1| = |B_2|$.*

Proof. Suppose for the sake of contradiction, there are two bases such that $|B_1| < |B_2|$. Then by (**I2**), there exists $e \in B_2 - B_1$ such that $B_1 \cup \{e\}$ is also an independent set. But this contradicts to the maximality of a basis. Hence, the desired result is obtained. □

Lemma 2.1.3. *Let $\mathcal{B}$ denote the set of bases of a matroid M. Then*

- *(**B1**) $\mathcal{B} \neq \emptyset$.*
- *(**B2**) For each $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 - B_2$, there exists $f \in B_2 - B_1$ such that $(B_1 - \{e\}) \cup \{f\} \in \mathcal{B}$.*

Proof. (***B1***) follows from (***I0***). Then consider two bases $B_1, B_2 \in \mathcal{B}$, by (***I2***) the result follows. □

Theorem 2.1.4. *Let $\mathcal{B}$ be a collection of subset of E, then $\mathcal{B}$ is the collection of bases of a matroid if and only if it satisfies (**B1**) and (**B2**).*

Proof. Clearly by lemma 2.1.3, if $\mathcal{B}$ is the collection of bases of M, then (***B1***) and (***B2***) hold.
For the other direction, let $\mathcal{I}$ be the collection of subsets of E that are contained in some B in $\mathcal{B}$. (***I0***) is satisfied since (***B1***). (***I1***) is satisfied since the choices of $\mathcal{I}$. For (***I1***), let $I_1 \subseteq B_1$ and $I_2 \subseteq B_2$ where $|I_1| < |I_2|$. If $B_1 = B_2$, then we are done. So assume B_1 is not equal to B_2. □

We can define a matroid by **Theorem 2.1.4**. This is the bases definition of a matroid. One of the amazing parts of matroid theory is that there are plenty of equivalent definitions of a matroid for different purposes of study.

There are also a circuit definition, and a rank functions definition for matroids. However, in this work the bases definition is what we need.

2.1.2 Examples

In this section, we will introduce several classes of matroids, including *graphic matroids, uniform matroids and representable matroids.* They are all important classes of matroids in matroid theory.

For a graph $G = (V, E)$, define $M(G) = (E, \mathcal{I})$ where $\mathcal{I} = \{F \subseteq E : G = (V, F) \text{ is a forest } \}$. This is called the *cycle matroid* of G. Clearly, the set of all spanning forests is $\mathcal{B}$ of $M(G)$.

Example 2.1.5. Consider the following graph. The independent set of $M(G)$ will be the subsets of [7] where the indicated edges do not contain a cycle. For example, $\{1, 3, 6\} \in \mathcal{I}$.

The class of matroids that is related to graphs is an important subject in matroid theory. A matroid is *graphic* if it is isomorphic to the cycle matroid of a graph. However, a graphic matroid is not enough to determine the graph. If G is a graph, then adding any single vertex to G will not change its cycle matroid. So to determine a graph from a graphic matroid, we may assume the desired graph G has no single vertex. Even though this is not enough. In Oxley's book [11] section 5.3, he introduced three operation on

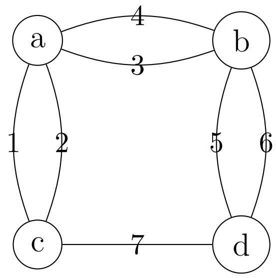

Figure 2.1. G

graph, which are *vertex identification*, *vertex cleaving* and *twisting* (or "Whitney flips"). These operations all result in cycle matroids of two different graphs being the same. Since none of those operations can be applied on a 3-connected loopless graph, such a graph can be determined by its cycle matroid uniquely. Proof of this fact can also be found in [11] Section 5.3.

Example 2.1.6. Let $r \leq n$ be non-negative integers. Let E be a set with n elements. Define $U_{r,n} = (E, \mathcal{I})$ where $\mathcal{I} = \{I \subseteq E : \ |I| \leq r\}$. This is called a *uniform matroid.* We can easily check, $\mathcal{B}(U_{r,n}) = \{B \subseteq E : |B| = r\}$ is the set of bases of this matroid.

Example 2.1.7. Let E be $[n]$ and A be an r by n matrix over a field $\mathbb{F}$. Define $M(A) = (E, \mathcal{I})$, where $\mathcal{I} = \{I \subseteq E : \text{if } I \text{ gives linearly independent columns in } A\}$. Then $M(A)$ is a matroid. The matroid obtained from A is called the *column matroid* or *vector matroid* of A.
Consider the graph G from example 2.1.5, and give an orientation to it,

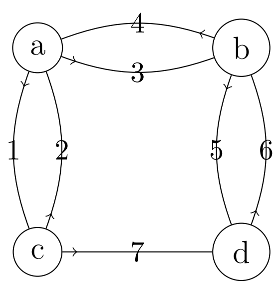

Figure 2.2. Directed G

Then consider the signed incidence matrix of the directed graph over rational field,

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

It is easy to verify that $M(A)$ and $M(G)$ have the same independent sets and bases.

A matroid M is $\mathbb{F}$-*representable* if M is the column matroid of a matrix A over $\mathbb{F}$, denoted by $M_{\mathbb{F}}(A)$. Clearly, in the above example, A is a representation of $M(G)$. And we have the following lemma for graphic matroids.

Lemma 2.1.8. *Let $D(G)$ be an arbitrary orientation of a graph G, let $\mathbb{F}$ be a field. Then the incidence matrix of $D(G)$ is a matrix representation of $M(G)$ over $\mathbb{F}$.*

Proof. In [11], Lemma 5.1.3. □

A $GF(2)$-representable matroid is called *binary*.

It is not hard to check that Example 2.1.7 is a $GF(2)$-representable matroid if we replace every -1 by 1 and let the ground field be $GF(2)$.

A matroid is *regular* if it is representable over all fields.

Theorem 2.1.9. *Graphic matroids are regular.*

Proof. People can find the proof in [11], Proposition 5.1.5. □

Therefore, Example 2.1.7 is a regular matroid as well.

Of course, there are matroids which are not representable, one of the smallest examples is the V*á*mos matroid.

2.1.3 Operations on Matroids

Dual of a matroid

Let $M = (E, \mathcal{B})$ be a matroid, and define $\mathcal{B}^* = \{B \subseteq E : E - B \in \mathcal{B}\}$.

Theorem 2.1.10. *$M^* = (E, \mathcal{B}^*)$ is a matroid.*

Proof. It can be found in [11], Theorem 2.1.1. □

This M^* is called the *dual matroid* of M. Of course there is a rank function definition, and a circuits definition (we did not provide rank and circuits definition in this work because we do not use it) for dual of a matroid, and so on. However, the basis definition for dual is a better choice than others in this paper. And it is very easy to verify (***B1***) and (***B2***) to prove the above theorem.

Example 2.1.11. Dual graph G^* of example 2.1.5 shown in figure 2.3. Consider $M(G^*)$, it is the matroid dual of $M(G)$ in example 2.1.5. Moreover, we have the following theorem.

Theorem 2.1.12. *G is planar if and only if $M(G^*)$ is graphic.*

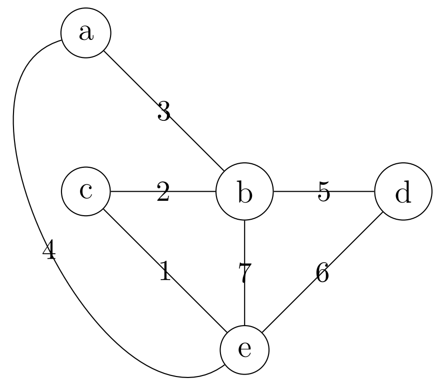

Figure 2.3. G^* dual of G

Proof. In [11], Theorem 5.2.2. □

Let $\mathbb{F}$ be any field. Given a finite set E, let V be a subspace of $\mathbb{F}^E$, define $V^\perp = \{x \in \mathbb{F}^E : x^T y = 0, \text{ for each } y \in V\}$. $V^\perp$ is called the *orthogonal space* of V.

Theorem 2.1.13. *Let $A_1 \in \mathbb{F}^{r_1 \times E}$ and $A_2 \in \mathbb{F}^{r_2 \times E}$. If $Rowspace(A_1)^\perp = Rowspace(A_2)$, then $M(A_1)^* = M(A_2)$.*

Proof. In [11], Theorem 2.2.8. □

Theorem 2.1.13 tells us that the class of representable matroid is dual-closed.

Deletion and contraction

Let $M = (E, \mathcal{I})$ be a matroid, S be a subset of E. Define $\mathcal{I}' = \{I \subseteq (E - S) : I \in \mathcal{I}\}$.

Lemma 2.1.14. *Let $M \backslash S = (E - S, \mathcal{I}')$, then $M \backslash S$ is a matroid, called deletion of M with respect to S.*

Proof. The proof can be found in [11] Section 3.1. It is easy to verify ***I0,I1,I2***. □

After deletion a representable matroid is still representable, since given a representation of the primal matroid and deleting the corresponding column will give the representation of the resulting matroid.

Define $M/S = (M^* \backslash S)^*$ to be the *contraction* of M with respect to S.

A matroid M' obtained by applying deletions and contractions to M is called a *minor* of M.

Example 2.1.15. Consider the graph G from Example 2.1.5, $G/\{7\}$ is shown in figure [2.4] It is easy to verify that $M(G/\{7\}) = M(G)/\{7\}$.

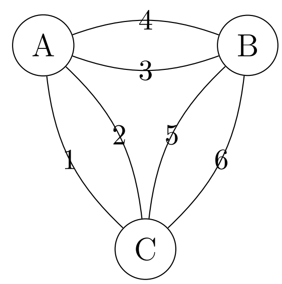

Figure 2.4. $G/\{7\}$

Theorem 2.1.16. *If N is a minor of M, then N can be written as $M/I\backslash I^*$, where $I \in \mathcal{I}(M)$ and $I^* \in \mathcal{I}(M^*)$.*

Proof. This proof can be found in [11] Theorem 3.3.2 $\square$

Clearly, the class of representable matroids is closed under minors since they are closed under duals and closed under deletions. However, class of representable matroid is a big class of matroids. Later we will introduce a class of matroids, *transversal matroids*. This class of matroids is not closed under minors.

Direct Sums

Let $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ be matroids with $E_1 \cap E_2 = \emptyset$, define $M_1 \oplus M_2 = (E_1 \cup E_2, \mathcal{I})$ where $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$. $M_1 \oplus M_2$ is a matroid, and it is called the *direct sum* of M_1 and M_2.
For example, if we have two edge disjoint graphs G_1 and G_2, then $M(G_1) \oplus M(G_2) = M(G_1 \cup G_2)$. Later, we will show a example of direct sum of two lattice path matroids.

2.1.4 Binary Matroids and Regular Matroids

Characterizing a class of matroids by its excluded minors is one of the main streams in matroid theory. In this section, we will introduce some characterizations for the class of binary matroids and regular matroids.

Recall, a matroid M is binary if it is representable over $GF(2)$. There are plenty of characterization of binary matroids, and the following theorem show that uniform matroid $U_{2,4}$ can not be a minor of a binary matroid. In fact, if a matroid has no minor that is isomorphic to $U_{2,4}$ then it is binary matroid.

Theorem 2.1.17. *Let M be a matroid, then M is binary if and only if M has no $U_{2,4}$-minor.*

Proof. In [11], Theorem 6.5.4. $\square$

Recall, a matroid M is regular if it is representable over all field.

Theorem 2.1.18. *Let M be a matroid, then the following statements are equivalent*

1. *M is regular.*
2. *M is $GF(2)$-representable and $GF(q)$-representable for some odd q.*
3. *M is $GF(2)$-representable and $\mathbb{Q}$-representable.*
4. *There exists a matrix representation A of M, where A is totally unimodular.*

Proof. This is Theorem 6.6.3 in [11] $\square$

Recall Theorem 2.1.9. To prove it, we need to prove the sign incidence matrix of a graph is totally unimodular.

2.2 Rayleigh Matroids

In this section, I will give the definition of Rayleigh matroid and give a literature review of the work done in this area.

2.2.1 Definitions

Bases Enumerator

Let $M = (E, \mathcal{B})$, let x_e for all $e \in E\}$ be indeterminants that are indexed by the ground set E. For a subset S of E, let $x^S = \prod_{e \in S} x_e$. Define the *basis enumerator* of M is as the following polynomial

$$M(x) = \sum_{B \in \mathcal{B}} x^B$$

where $\mathcal{B}$ is the set of bases of a matroid M.

Rayleigh Difference

Recall in the introduction, we can partition $\mathcal{B}$ into 4 sets according to a pair of elements $e, f \in E(M)$ which are not loops as follows

- $\mathcal{B}_e^f = \{B \in \mathcal{B} : e \in B, f \in E - B\}$.
- $\mathcal{B}_f^e = \{B \in \mathcal{B} : f \in B, e \in E - B\}$.
- $\mathcal{B}_{ef} = \{B \in \mathcal{B} : e, f \in B\}$.
- $\mathcal{B}^{ef} = \{B \in \mathcal{B} : e, f \in E - B\}$.

Define M^e and M_f as follows:

$$M^e = M(x)\big|_{x_e=0}, \quad \text{and} \quad M_f = \frac{\partial M(x)}{\partial x_f}.$$

Then we can write

$$M^{ef} = \sum_{B \in B^{ef}} x^B \tag{2.1}$$

$$M_{ef} = \frac{\sum_{B \in B_{ef}} x^B}{x_e x_f} \tag{2.2}$$

$$M_e^f = \frac{\sum_{B \in B_e^f}}{x_e} \tag{2.3}$$

$$M_f^e = \frac{\sum_{B \in B_f^e}}{x_f} \tag{2.4}$$

So we have the following expression

$$M(x) = M^{ef} + x_e M_e^f + x_f M_f^e + x_e x_f M_{ef} \tag{2.5}$$

Define *Rayleigh difference* of M with respect to e, f is defined as

$$\Delta M(e, f) = M_f^e M_e^f - M_{ef} M^{ef}$$

Then the question of determining whether a pair elements has negative correlation

$$\frac{M_e}{M(x)} \geq \frac{M_{ef}}{M_f}$$

will be translated to determine whether the following Rayleigh difference is non-negative.

If the Rayleigh difference is non-negative for every pair of elements with all non-negative real assignment, then the corresponding matroid is *Rayleigh*. Rayleigh matroid was first introduce by Chow and Wagner in [7]

If the Rayleigh difference is non-negative for every pair of elements with all real assignment, then the corresponding matroid is *strongly Rayleigh*.

2.2.2 Rayleigh Matroids

Some examples

In other word, if a matroid is Rayleigh then $\Delta M(e, f) \geq 0$ for all $\vec{x}$ in $\mathbb{R}_{\geq 0}^{E-\{e,f\}}$. Following theorem indicate known classes of Rayleigh matroids and some connection between some other classes of matroids.

Theorem 2.2.1. *The following classes of matroids are Rayleigh:*

1. *Regular matroids.*

2. *Uniform matroids.*

3. *Matroids with rank or corank no more than* 3.

4. *Vámos matroid is Rayleigh.*

1 and *2* are proved in [7] Section3. *3* is proved in [13] Theorem 1.1. *4* can be verified through Theorem 3(c) in [16].

The following theorem indicates that the class of Rayleigh matroids is closed under those operations. It can be found in [7].

Theorem 2.2.2. *Let $\mathcal{R}$ be class of Rayleigh matroids*

1. *$\mathcal{R}$ is closed under duals and minors.*

2. *$\mathcal{R}$ is closed under* 2*-sums.*

Theorem 2.2.3.
- *Every Rayleigh matroid is balanced.*
- *A binary matroid is Rayleigh if and only if it does not contain S_8 as a minor.*
- *A binary matroid is balanced if and only if it is Rayleigh.*

Reader can find the proof of 2.2.2 and 2.2.3 in [7] if interested. The proof is technical and not important to this thesis, hence it is omitted here. And it is important to noticed that not all balanced matroid are Rayleigh, in [**??**] Choe and Wagner give an example of this in Theorem 5.11.

Half Plane Property and Strongly Rayleigh Property

A polynomial $P(\mathbf{x}) = \sum_{\alpha} c_\alpha x^\alpha$ in several complex variables x_e $e \in E\}$ has the *half plane property* if when $\mathrm{Re}(x_e) > 0$ for all $e \in E$, then $P(\mathbf{x}) \neq 0$. A matroid $M = (E, \mathcal{B})$ is a *half plane property matroid* if its bases enumerator has the half plane property. This class of polynomial was first introduced by Choe, Oxley Sokal and Wagner in [6]. Later Br$\bar{a}$nd$\acute{e}$n proved the following theorem in [1].

Theorem 2.2.4. *A matroid is a half plane matroid if and only if it is a strongly Rayleigh matroid.*

It is a surprising result that a complex conditions of multinomial is equivalent to a real non-negative conditions. Reader can find the proof of this theorem in [1] Section 5 if interested.

In this thesis, I will provide a combinatorial technique to prove the Rayleigh property of lattice path matroid which will be introduced in next chapter. Unfortunately, I have not found a good combinatorial approach for the strongly Rayleigh property.

Chapter 3

Lattice Path Matroids

In this chapter, we will first introduce a widely studied class of matroids, the *transversal matroids*. Then we will introduce a subclass that can be identified from a pair of lattice paths from $(0,0)$ to (n,m), which was first introduced by Joseph Bonin, Anna de Mier and Marc Noy, in [3] (2002). The class of lattice path matroids is a subclass of *transversal matroids* and it is minor-closed and dual-closed. Furthermore, $\mathcal{L}$ is closed under direct sum operation, and a particular case of 2-sum operation. We will demonstrate the basic structure properties of lattice path matroids. Then we will present the main theorem of this paper.

3.1 Transversal Matroids

3.1.1 Definition

A *set system* is a pair of sets $(E, \mathcal{A})$, where E is a finite set and $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$ is a set with element being subsets of E. For $i \neq j$, A_i is not necessarily distinct from A_j. We denote the indices of A as $J = [m]$

Example 3.1.1. Let $E = \{1, 2, 3, 4, 5, 6, 7\}$, and $\mathcal{A} = \{A_1, A_2, A_3\}$ where $A_1 = \{1, 2, 7\}$, $A_2 = \{3, 4, 7\}$ and $A_3 = \{5, 6, 7\}$ and $J = \{1, 2, 3\}$. Then $(E, \mathcal{A})$ is a set system as described above.

A *transversal* of the set system $(E, \mathcal{A})$ is a subset B of E, such that there exists a bijection

$$\phi : J \to B$$

where $\phi(j) \in A_j$ for all j in J.

Example 3.1.2. Consider the previous example 3.1.1, let $B = \{1, 4, 7\}$, we can have $\phi(1) = 1$, $\phi(2) = 4$ and $\phi(3) = 7$.

A *partial transversal* of a set system $(E, \mathcal{A})$ with $\mathcal{A} = \{A_j : j \in J\}$ is a transversal of some subsystem $(E, \mathcal{A}^{'})$ where $A^{'}$ is a subset of $\mathcal{A}$. Thus a partial transversal of $(E, \mathcal{A})$ is a subset I of E such that there is an injection

$$\psi : I \to J$$

where for all $i \in I$, i is in $A_{\psi(i)}$.

Example 3.1.3. Consider the set system in example 3.1.1, let $I = \{1, 7\}$, then we can have I be a transversal of $(E, \{A_1, A_2\})$. However, we can also have I be a transversal of $(E, \{A_1, A_3\})$.

From the above example, we noticed that the bijection between a subset of E and J may not be unique. That indicates a transversal or a partial transversal does not give the bijection or injection.

Theorem 3.1.4. *Let $(E, \mathcal{A})$ be a set system where $\mathcal{A} = \{A_j : j \in J\}$, let $\mathcal{I}$ be the collection of its partial transversals. Define $M = (E, \mathcal{I})$. Then M is a matroid.*

Proof. Let A be a 0-1 matrix over rational field where rows of A are indexed by J and columns of A are indexed by E, and the entry of i-th row and j-th column is 1 if and only if j is in A_i. Then linearly independent columns of A correspond to partial transversals. Hence, we realize that M is a matroid. Such a matroid is called a *transversal matroid*. □

Remark 3.1.5. For a set system and the corresponding transversal matroid M,

1. Transversals are bases of M.
2. Partial transversals are independent sets of M.
3. Every transversal matroid is representable over all sufficiently large fields, and it is representable over all infinite fields. See the proof in [2] Theorem 2.5.

This class of matroids was first introduced by Jack Edmonds and D. R. Fulkerson in [8] in the 1960s. It is easy to check all uniform matroids are transversal. Some graphic matroids are transversal and some are not. In the following example we will introduce a non-transversal graphic matroid. In [11] Chapter 10, Theorem 10.4.7 gives the precise characterization of which graphic matroids are transversal.

3.1.2 Facts

Theorem 3.1.6. *The following statements are equivalent for a matroid M:*

1. *M is graphic and transversal.*
2. *M is regular and transversal.*

3. M is binary and transversal.

Example 3.1.7. Consider example 2.1.15, we will show $M(G/\{7\})$ is not transversal.

Assume $M(G/\{7\})$ is a transversal matroid for a set system, then the set system will have ground set $E = [6]$. From the figure, we can easily check $\mathcal{A}$ must be $\{\{1,2\},\{3,4\},\{5,6\}\}$ since $\{1,2\},\{3,4\},\{5,6\}$ are not independent. However this gives $\{1,3,5\}$ is independent, and those three edge is a circuit in $G/\{7\}$. Hence, this arises a contradiction. Therefore, $M(G/\{7\})$ is not transversal.

Theorem 3.1.8. *Transversal matroids are not minor-closed and not dual-closed.*

Proof. Consider Example 2.1.5 it is easy to check $M(G)$ is transversal. Clearly, Example 3.1.7 show that transversal matroids are not minor-closed. Assume transversal matroids are dual closed, since transversal matroids are deletion closed we will arise transversal matroids are minor-closed. This contradicts the fact they are not minor-closed. Hence, the desired result is obtained. $\square$

A *gammoid* is a minor of a transversal matroid. It is stated in [11] Proposition 3.2.10 and 3.2.12 that every transversal matroid is a gammoid and the class of gammoids is closed under minors and under duality. Because of this, the class of gammoids is the smallest minor-closed class that contains all transversal matroids. It is a natural question, what is the largest minor-closed class that is contained in the class of transversal matroids. In this paper, we will study a subclass of transversal matroids that is called *lattice path matroid*, which is minor-closed.

3.2 Lattice Path Matroid

A *lattice path* is a sequence of steps $\{(1,0),(0,1)\}$ going from $(0,0)$ to (n,m). We can also consider a lattice path as a sequence of letters 'E' and 'N' with $E = (1,0)$ and $N = (0,1)$. Let $P = p_1p_2\cdots p_{m+n}$ and $Q = q_1q_2\cdots q_{m+n}$ be two lattice paths from $(0,0)$ to (n,m), then we say P is weakly above Q if for all $1 \le i \le m+n$ the total number of N steps in P is greater than or equal to the total number of N steps in Q up to step i. Every lattice path L between P and Q from $(0,0)$ to (n,m) has m north steps, Denoted the indices of north steps of L as $B(L)$. Define $M[P,Q] = (E,\mathcal{B})$, where $E = [m+n]$, $\mathcal{B}$ is collection of all $B(L)$ where L is a lattice path between P and Q.

Theorem 3.2.1. *(Theorem 3.1 in [3]) M[P,Q] is a transversal matroid, and $\mathcal{B}$ is the collection of bases.*

Proof. Let $\{p_{s(1)}, p_{s(2)}, \cdots, p_{s(m)}\}$ be the set of N-steps of P where $s(1) < s(2) < \cdots < s(m)$; for Q, let $\{q_{t(1)}, q_{t(2)}, \cdots, q_{t(m)}\}$ be the set of N-steps. Since P is weakly above Q, $s(i) \le t(i)$ for all $i \in [m]$. Let N_i be the interval $[s_i, t_i]$ of integers. Let $E = [m+n]$ and $\mathcal{A} = \{N_1, N_2, \cdots, N_m\}$. Let L be a lattice path between P and Q, and the indices of its north steps are $B(L) = \{l(1), l(2), \cdots, l(m)\}$. Since if L is a lattice path between

P and Q, $s(i) \leq l(i) \leq t(i)$ for all $i \in [m]$. Hence we can construct a bijection from $[m]$ to $B(L)$ by letting $\phi(i) = l(i)$. So $B(L)$ is a transversal of $(E, \mathcal{A})$. Therefore, $M[P, Q]$ is the transversal matroid of set system $(E, \mathcal{A})$. □

Let M be a matroid. Then M is a *lattice path matroid* if M is isomorphic to some $M[P, Q]$. We denote the class of lattice path matroids by $\mathcal{L}$.

Example 3.2.2. Let $P = NNENEEEENEE$ and $Q = EENEEENEENN$, then $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $N_1 = \{1, 2, 3\}$, $N_2 = \{2, 3, 4, 5, 6\}$, $N_3 = \{4, 5, 6, 7, 8, 9\}$,$N_4 = \{8, 9, 10\}$

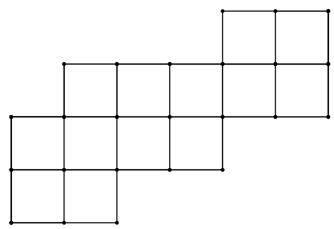

Figure 3.1. Lattice presentation

Theorem 3.2.3. *$\mathcal{L}$ is closed under duals.*

Proof. A basis for M^* is the complement of a basis for M. By definition, a basis of M is the set of N-steps of a lattice path, and its complement is the set of E-steps. Hence, simply by reflecting the representation of M along the line of $y = x$, then we will obtain a the representation of M^*. □

Example 3.2.4. For example, the dual representation of Example 3.2.2 is transversal matroid of $(E, \mathcal{A})$, where $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $\mathcal{A} = \{N_1, N_2, N_3, N_4, N_5, N_6\}$, $N_1 = \{1, 2, 3\}$, $N_2 = \{2, 3, 4, 5\}$, $N_3 = \{4, 5, 6\}$, $N_4 = \{5, 6, 7\}$, $N_5 = \{7, 8, 9\}$, $N_6 = \{8, 9, 10\}$. And the lattice path presentation is shown as follow:

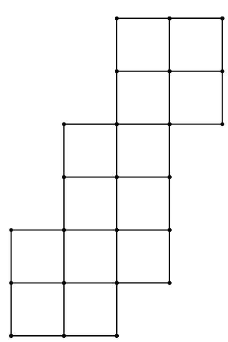

Figure 3.2. Lattice presentation of the dual matroid

Remark 3.2.5. According to the definition of $\mathcal{L}$ the following fact can be verified easily. Given $M[P, Q]$.

1. $rank(M[P, Q]) = m$.
2. Independent sets of $M[P, Q]$ are partial transversals of $([m+n], \{N_1, \cdots, N_m\})$.
3. If $s(i) = t(i)$ for some i, then $s(i)$ is an isthmus of $M[P, Q]$. An *isthmus* is an element that appears in every basis.
4. Let B is a basis of $M[P, Q]$, we can find its lattice path presentation L as

$$L = s_1 s_2 \cdots s_{m+n}$$

with

$$s_i = \begin{cases} N, & \text{if } i \in B \\ E, & \text{if } i \notin B \end{cases}$$

3.3 $\mathcal{L}$ is minor-closed, and closed under direct sum

In this section, we will show that $\mathcal{L}$ is minor-closed by the corresponding operations on lattice path presentation of a lattice path matroid.

3.3.1 Deletion and contraction

Consider single element deletion of $M[P, Q]$. Delete element e can be realized as an operation on the lattice path presentation of $M[P, Q]$. It will be in the following 3 cases:

1. For an isthmus e, we delete e from both P and Q to get a new pair of P' and Q'. $M[P, Q] \backslash \{e\} \cong M[P', Q']$.
2. For a loop e (it means e is not in any N_i for all $i \in [m]$), we delete e from both P and Q get a new pair of P' and Q'. $M[P, Q] \backslash \{e\} \cong M[P', Q']$.
3. For neither an isthmus nor a loop e, if e is north step on P, delete the first east step after e from P; if e is east step, delete e from P. If e is north step on Q, delete the last east step before e from Q; if e is east step on Q, delete e from Q. These deletions will give a new pair of P' and Q'. $M[P, Q] \backslash \{e\} \cong M[P', Q']$. For details, people can find this in [4] Theorem 3.1.

Consider single element contraction, we can just apply the deletion on $M[P, Q]^*$. For an isthmus or a loop e, we can just delete from both P and Q as before. For e is neither an isthmus nor a loop, if e is north step on P, delete e from P; if e is east step, delete the last north step before e from P. If e is north step on Q, delete e from Q; if e is east step on Q, delete the first north step after e from Q. These deletion will give a new pair of P' and Q', and $M[P, Q]/\{e\} \cong M[P', Q']$.

Example 3.3.1. Consider another pair of lattice paths,

$$P = NNEENNENEENEE$$

$$Q = EEENEEENENNNN$$

Then the lattice path presentation of $M[P,Q]$ is shown in Figure 3.3 (left), we simply omit the edge in between P and Q consider $M[P,Q]\backslash\{4\}$, then the step 4 in P is an east step; the last east step in Q before 4 is 3. Delete this two, we will have

$$P^{'} = NNENNENEENEE$$

$$Q^{'} = EENEEENENNNN$$

Then the lattice path presentation of $M[P,Q]$ is shown in Figure 3.3 (right) Consider

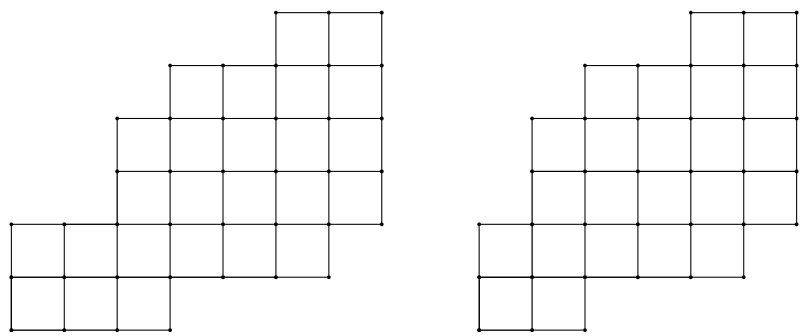

Figure 3.3. Example 3.3.1

$M[P,Q]/\{4\}$ then the last north step before 4 in P is 2; the step 4 is north step in Q. Delete this two, we will have

$$P^{'} = NEENNENEENEE$$

$$Q^{'} = EEEEEENENNNN$$

The lattice path presentation of $M[P^{'},Q^{'}]$ is shown in Figure 3.4.

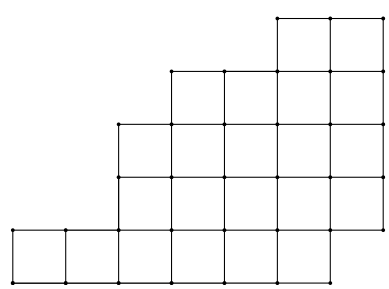

Figure 3.4. Example 3.3.1

Example 3.3.2. Consider

$$P = NNEEENNEENNENEENEE$$

$$Q = EENNEEENNNEEEEENNN$$

The lattice path presentation of $M[P,Q]$ will be Figure 3.5(left). Clearly 5 is a loop and 10 is an isthmus. Deleting or contracting 5 and 10 will give the same result as Figure 3.5(right).

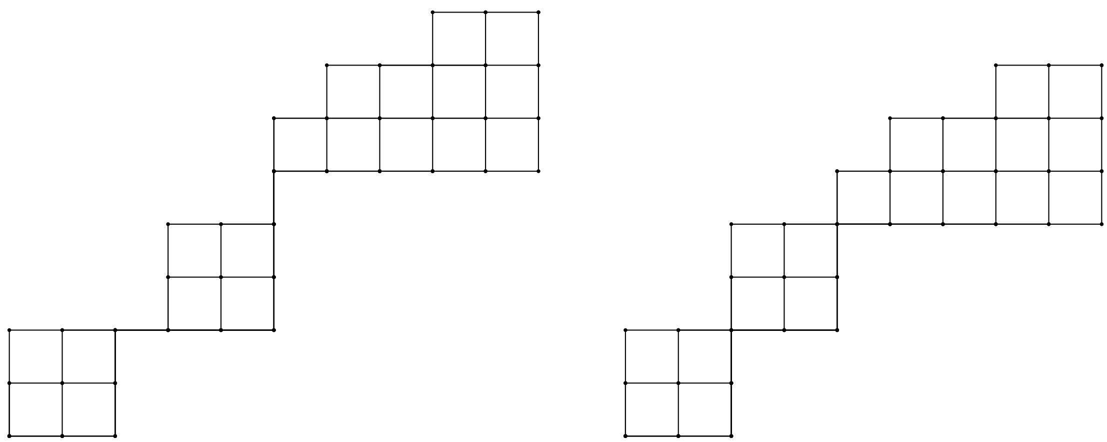

Figure 3.5. Example 3.3.2

3.3.2 Direct sum

Let $M[P_1,Q_1]$ and $M[P_2,Q_2]$ be two lattice path matroid, then $M[P,Q] = M[P_1,Q_2] \oplus M[P_2,Q_2]$ has the lattice path presentation $P = P_1P_2$ and $Q = Q_1Q_2$. Simply, just put two lattice path presentation together identify the ending point of P_1 and starting point of P_2.

Example 3.3.3. Consider

$$P_1 = NENEENEENE$$

$$Q_1 = EEENEEENNN$$

$$P_2 = NEENEENE$$

$$Q_2 = EEEENENN$$

Then the $M[P,Q] = M[P_1,Q_2] \oplus M[P_2,Q_2]$ will have the following lattice path presentation

Theorem 3.3.4. *$\mathcal{L}$ is closed under minors, duals and direction sum.*

Proof. This is really a proof by picture. According to the description of each operation on a lattice path matroid M_L, we can easily find the corresponding lattice path presentation of the minor derived from M_L by deletions and contractions. There is more detailed description in [4] Section 3. $\square$

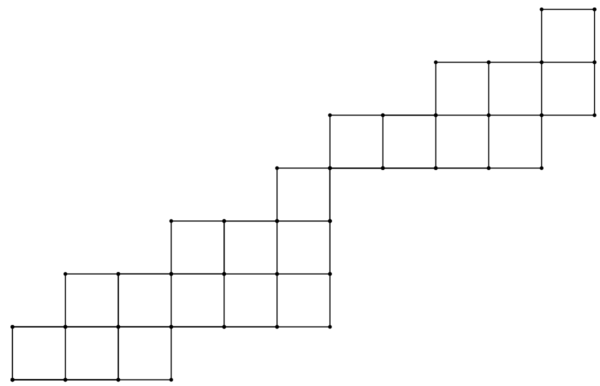

Figure 3.6. Example 3.3.3

3.3.3 Excluded Minors

As we mention in chapter 1, characterizing a minor-closed class of matroids by finding its excluded minors is a major topic in matroid theory. The following theorem is proved by J.E. Bonin is in his work [5] Theorem 3.1. The notation for these excluded minors is explained there.

Theorem 3.3.5. *A matroid is a lattice path matroid if and only if it has none of the following matroids as minors:*

1. $A_n = P_n^{'} + x$, *for* $n \geq 3$.
2. $B_{n,k} = T_n(U_{n-1,n} \oplus U_{n-1,n} \oplus U_{k-1,k})$ *and its dual* $C_{n+k,k}$, *for* $n \leq k \leq 2$.
3. $D_n = (P_{n-1} \oplus U_{1,1}) + x$ *and its dual* E_n, *for* $n \geq 4$.
4. *the rank-*3 *wheel,* $\mathcal{W}_3$, *the rank-*3 *whirl,* $\mathcal{W}^3$.
5. *the matroid* R_3 *and its dual* R_4.

3.4 Rayleigh Property

Recall that the Rayleigh difference of a matroid is defined as

$$\Delta M(e,f) = M_f^e M_e^f - M_{ef} M^{ef}.$$

Let M be a lattice path matroid. Then for each term in M_e^f, the corresponding basis has lattice path presentation that uses step e as a north step and does not use f as a north step, that is a lattice path representation of element $B \in \mathcal{B}_e^f$. For the other three polynomials, they have the respective corresponding lattice paths. Hence, if we can find a injective function $F : \mathcal{B}^{ef} \times \mathcal{B}_{ef} \to \mathcal{B}_f^e \times \mathcal{B}_e^f$ such that for $F(B_1, B_2) = (B_1^{'}, B_2^{'})$ $x^{B_1} x^{B_2} = x^{B_1^{'}} x^{B_2^{'}}$, then we will conclude that for a lattice path matroid M, $\Delta M(e,f)$ has only positive terms. Hence, the lattice path matroid is Rayleigh. It turns out we can find such an injective function for any lattice path matroid. Therefore, the class $\mathcal{L}$ is a subclass of class of Rayleigh matroids.

Theorem 3.4.1. *For all lattice path matroids M, $\Delta M(e,f) \geq 0$ where $x_i \geq 0$ for all i of the ground set of M.*

Unfortunately, we are not able to find a natural injective function from $\mathcal{B}^{ef} \times \mathcal{B}_{ef}$ to $\mathcal{B}^{e}_{f} \times \mathcal{B}^{f}_{e}$. However, we are still able to prove this theorem through a careful construction of an injective-like correspondence. First, we pick a pair of bases (B_1, B_2) in $\mathcal{B}^{ef} \times \mathcal{B}_{ef}$, then find the corresponding lattice path presentation of these two bases (P_1, P_2). Record the steps that they agree on and ignore them. We will have a pair of lattice paths (G_1, G_2) symmetric across the line $y = x$. After this, we have five cases to discuss. For each of them we find a corresponding pair $(G_1^{'}, G_2^{'})$. Then we put back the information that we ignored from the first step. This will give a pair of lattice paths which are presentation of a pair of bases $(B_1^{'}, B_2^{'})$ in $\mathcal{B}^{e}_{f} \times \mathcal{B}^{f}_{e}$. According the operations that we do, the condition $F(B_1, B_2) = (B_1^{'}, B_2^{'})$ $x^{B_1}x^{B_2} = x^{B_1^{'}}x^{B_2^{'}}$ has been satisfied, which finishes the proof. Let us look into the details.

The constructions in of the proof are complicated. So, I provide the following example to be a reference for how it works step by step.

Example 3.4.2. Let $M[P,Q]$, where

$$P = NNEENNNEEEEENNEEEEENEE$$
$$Q = EEEEENNEEEEENEEENNNNN$$

then according the definition, we have $M = ([20], \mathcal{A})$, where

$$\begin{aligned}
A_1 &= \{1,2,3,4,5,6\}\\
A_2 &= \{2,3,4,5,6,7\}\\
A_3 &= \{5,6,7,8,9,10,11,12\}\\
A_4 &= \{6,7,8,9,10,11,12,13,14,15,16\}\\
A_5 &= \{7,8,9,10,11,12,13,14,15,16,17\}\\
A_6 &= \{12,13,14,15,16,17,18\}\\
A_7 &= \{13,14,15,16,17,18,19\}\\
A_8 &= \{18,19,20\}
\end{aligned}$$

Then the lattice path presentation of M is shown in Figure 3.7

3.4.1 A forgetting function

Let $M[P,Q]$ be a lattice path matroid, and let $(B_1, B_2) \in \mathcal{B}^{ef} \times \mathcal{B}_{ef}$. Then take the lattice path presentation of (B_1, B_2), denoted as (P_1, P_2). So P_1 and P_2 are two lattice paths from $(0,0)$ to (m,n) between P and Q. Consider P_1 and P_2 as words formed by Ns and

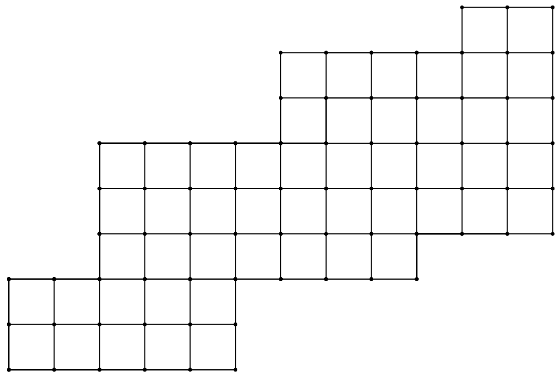

Figure 3.7. Lattice Path presentation of $M[P,Q]$

Es. Then $P_1 = s_1s_2\cdots s_{m+n}$ where $s_e = E$ and $s_f = E$, $P_2 = t_1t_2\cdots t_{m+n}$ where $t_e = N$ and $t_f = N$, and each $s_i, t_i \in \{N, E\}$. Let $C(P_1,P_2) = \{i : s_i = t_i\}$ and also record $C_r = \{(i, s_i) : s_i = t_i\}$ (for remembering them later). Let $D(P_1,P_2) = [m+n]\backslash C(P_1,P_2)$. Let $k = |D(P_1,P_2)|$ so that $D(P_1,P_2) = \{i(1), i(2), \cdots, i(k)\}$ where $i(1) < i(2) < \cdots < i(k)$. Define the *forgetting function* f by:

$$f(B_1,B_2) = (G_1,G_2,C_r)$$

where $G_1 = s_{i(1)}\cdots s_{i(k)}$ and $G_2 = t_{i(1)}\cdots t_{i(k)}$.

Lemma 3.4.3. *G_1 and G_2 are two lattice paths from $(0,0)$ to $(\frac{k}{2},\frac{k}{2})$, for $k = |D(P_1,P_2)|$, which are symmetric across the line $x = y$.*

Proof. By the construction of forgetting function G_1 and G_2 are different at every step and end at the same point. So they are symmetric along $y = x$ and end at $y = x$. Clearly, k is an even number. Hence the total number of N-step is $k/2$. □

Lemma 3.4.4. *f is an injective function.*

Proof. Since we recorded C_r, which indicates the position of steps that we have forgotten, and also indicates whether the step is N or E, we can retrieve (P_1,P_2) by adding back those steps to the corresponding position. Hence, for each (G_1,G_2,C_r), it can only be derived from an unique pair of (P_1,P_2). Therefore, there is an unique pair (B_1,B_2) such that $f(B_1,B_2) = (P_1,P_2,C_r)$. □

Remark 3.4.5. Write $G_1 = s_{i(1)}s_{i(2)}\cdots s_{i(k)}$, then $s_e = E$ and $s_f = E$ must be left in G_1. Similarly, write $G_2 = t_{i(1)}t_{i(2)}\cdots t_{i(k)}$, then $t_e = N$ and $t_f = N$ must be left in G_2. And we let $i(c) = e$ and $i(d) = f$ for later convenience.

Example 3.4.6. In example 3.4.2, pick $e = 7$ and $f = 16$, then

$$P_1 = NENENN\underbrace{E}_{s_7}ENEEENEE\underbrace{E}_{s_{16}}NENE$$

$$P_2 = EENEEN\underbrace{N}_{t_7}EEEEEEEN\underbrace{N}_{t_{16}}ENNN$$

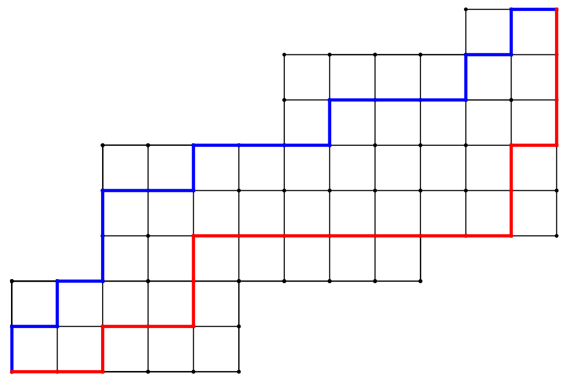

Figure 3.8. Lattice Path representation of P_1 and P_2

Then $C(P_1, P_2) = \{2, 3, 4, 6, 8, 10, 11, 12, 14, 19\}$, so $c = 3$ and $d = 7$ for later reference.

$$G_1 = NN\underbrace{E}_{s_7}NNE\underbrace{E}_{s_{16}}NEE$$
$$G_2 = EE\underbrace{N}_{t_7}EEN\underbrace{N}_{t_{16}}ENN$$

The figure of G_1 and G_2 is given as follow:

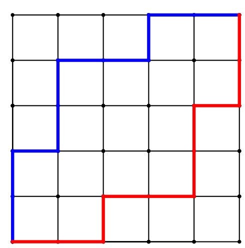

Figure 3.9. G_1 (in blue) and G_2 (in red)

3.4.2 A 1-to-1 correspondence, and a 2-to-2 correspondence

From the last section, we get (G_1, G_2) by applying the forgetting function to a pair of lattice paths that is a lattice path presentation of $(B_1, B_2) \in \mathcal{B}^{ef} \times \mathcal{B}_{ef}$. In this section, we will find the corresponding pair (G_1', G_2') of applying *forgetting function* to a pair of bases $(B_1', B_2') \in \mathcal{B}_f^e \times \mathcal{B}_e^f$. Write G_1 and G_2 as

$$G_1 = s_{i(1)}s_{i(2)} \cdots s_{i(k)}$$
$$G_2 = t_{i(1)}t_{i(2)} \cdots t_{i(k)}$$

From the forgetting function we observe that G_1 and G_2 differ at each step. Define *altitude* by

$$alt(l) = y - x$$

where (x, y) are the coordinates of the ending point of step $s_{i(l)}$ and $1 \leq l \leq k$.

And to find the correspondence that we want for (G_1, G_2), we will need to consider the following 5 cases:

Before I introduce the correspondence, I will draw the following diagram to help readers to follow the complicated case analysis for cases 2-5. According to the following Figure 3.10, consider node A is (G_1, G_2).

1. If case 2 or case 3 happens, then we have node C as $(G_1^{'}, G_2^{'})$ in the correspondence. Hence, a 1-to-1 correspondence holds for these cases

2. If case 4 or case 5 happens, then we discover that we have another two nodes B corresponding to (G_1^{*}, G_2^{*}) and D corresponding to $(G_1^{''}, G_2^{''})$. Hence, a 2-to-2 correspondence holds for theses cases

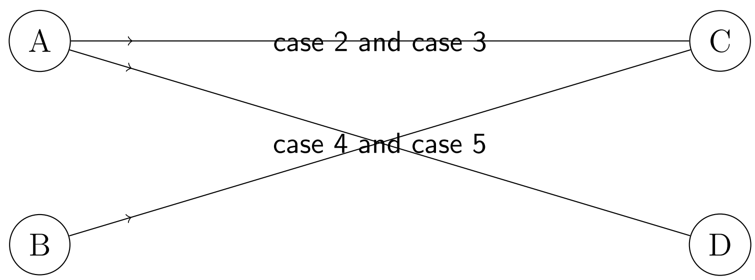

Figure 3.10. Reference for the correspondence

Now we are ready for the case analysis,

- **Case 1**: If there exist a l such that $e \leq i(l) < f$ and $alt(l) = 0$, that is G_1 and G_2 touch the diagonal $y = x$ between step s_e and s_f at step $s_{i(l)}$, then we say the corresponding $G_1^{'}$ and $G_2^{'}$ is the pair of lattice paths obtained by exchanging the steps in G_1 and G_2 after $s_{i(l)}$ as follows:

$$G_1^{'} = s_{i(1)} \cdots s_{i(l)} t_{i(l+1)} \cdots t_{i(k)}$$
$$G_2^{'} = t_{i(1)} \cdots t_{i(l)} s_{i(l+1)} \cdots s_{i(k)}$$

To avoid ambiguity, we choose $i(l) = min\{i(l) : e \leq i(l) < f,\ alt(s_{i(l)}) = 0\}$.

Example 3.4.7. Suppose that after the forgetting function, we have the following

$$G_1 = NE \underbrace{E}_{s_7} ENN \underbrace{E}_{s_{16}} ENN$$
$$G_2 = EN \underbrace{N}_{t_7} NEE \underbrace{N}_{t_{16}} NEE$$

Notice that $alt(6) = 0$ is the first index after s_7 with altitude 0, so we will switch every step after $s_{i(6)}$, the result is

$$G_1^{'} = NE\underbrace{E}_{s_7^{'}}ENN\underbrace{N}_{s_{16}^{'}}NEE$$

$$G_2^{'} = EN\underbrace{N}_{t_7^{'}}NEE\underbrace{E}_{t_{16}^{'}}ENN$$

This operation is shown in figure 3.11

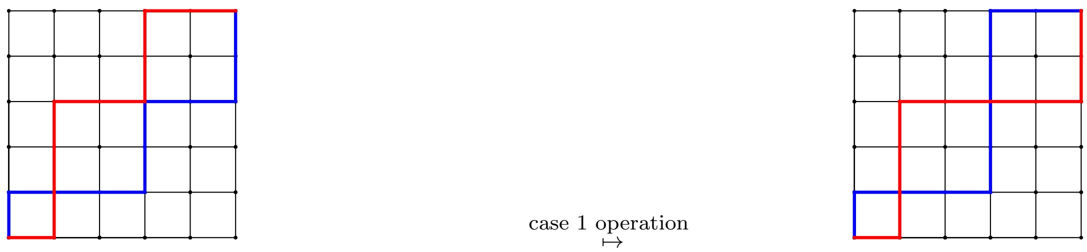

Figure 3.11. G_1 (blue) and G_2 (red) $G_1^{'}$ (blue) and $G_2^{'}$ (red)

- **Case 2**: For all $e \leq i(l) < f$ we have $alt(l) < 0$, that is between e and f, G_1 is totally bellow G_2. We know $alt(d-1) < 0$ (recall $i(d) = f$), so there must exist steps $f < i(b)$ such that $alt(b) = alt(d-1)$. To avoid ambiguity, choose the one with smallest index $i(b)$. Then switch the steps from $i(d)$ including $i(d)$ to $i(b)$ both on G_1 and G_2. After this, let $G_1^{'}$ be the result path of G_1 and $G_2^{'}$ be the result path of G_2, then the result can be realized as follows:

$$G_1^{'} = s_{i(1)} \cdots s_{i(d-1)} t_f t_{i(d+1)} \cdots t_{i(b)} s_{i(b+1)} \cdots s_{i(k)}$$
$$G_2^{'} = t_{i(1)} \cdots t_{i(d-1)} s_f s_{i(d+1)} \cdots s_{i(b)} t_{i(b+1)} \cdots t_{i(k)}$$

Example 3.4.8. Suppose that after the forgetting function, we have the following

$$G_1 = NE\underbrace{E}_{s_7}ENE\underbrace{E}_{s_{16}}NNN$$

$$G_2 = EN\underbrace{N}_{t_7}NEN\underbrace{N}_{t_{16}}EEE$$

Notice that $alt(6) = -2$ and $s_{i(8)}$ is the first position after f that has altitude equaling to -2. Then we will switch steps $s_{i(7)}$ and $s_{i(8)}$, the result is

$$G_1^{'} = NE\underbrace{E}_{s_7^{'}}ENN\underbrace{N}_{s_{16}^{'}}NEE$$

$$G_2^{'} = EN\underbrace{N}_{t_7^{'}}NEE\underbrace{E}_{t_{16}^{'}}ENN$$

This operation is shown in Figure 3.12

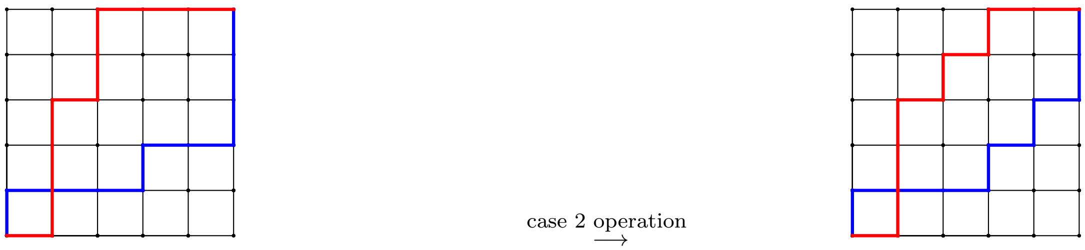

Figure 3.12. G_1 (blue) and G_2 (red) $\quad G_1'$ (blue) and G_2' (red)

- **Case 3**: For all $e \leq i(l) < f$ we have $alt(l) > 0$, that is between e and f, G_1 is totally above G_2. We know $alt(c) > 0$ (recall $i(c) = e$), so there must exists steps $i(1) \leq i(a) < e$ such that $alt(a) = alt(c)$. To avoid ambiguity, choose the one with largest $i(a)$. Then switch the steps from $i(a+1)$ to $i(c)$ both on G_1 and G_2. After this, let G_1' be the resulting path of G_2 and G_2' be the resulting path of G_1. Since we want G_1' to be the path using f but not e and G_2' be the path use e but not f. Then the result can be realized as follows:

$$G_1' = t_{i(1)} \cdots t_{i(a)} s_{i(a+1)} \cdots s_{i(c)} t_{i(c+1)} \cdots t_{i(k)}$$
$$G_2' = s_{i(1)} \cdots s_{i(a)} t_{i(a+1)} \cdots t_{i(c)} s_{i(c+1)} \cdots s_{i(k)}$$

Example 3.4.9. Suppose that after the forgetting function, we have the following

$$G_1 = NN \underbrace{E}_{s_7} NEN \underbrace{E}_{s_{16}} NEE$$
$$G_2 = EE \underbrace{N}_{t_7} ENE \underbrace{N}_{t_{16}} ENN$$

Notice that $altitude(s_{i(3)}) = 1$ and $s_{i(1)}$ is the last position that has altitude equaling to 1, then we will switch steps $s_{i(2)}$ and $s_{i(3)}$, the result is

$$G_1' = EN \underbrace{E}_{s_7'} ENE \underbrace{N}_{s_{16}'} ENN$$
$$G_2' = NE \underbrace{N}_{t_7'} NEN \underbrace{E}_{t_{16}'} NEE$$

This operation is shown in Figure 3.13

Remark 3.4.10. We call operation in case 2 and case 3 *flip in* since the action moves closer to line $y = x$.

- **Case 4**: This case has the same initial condition as case 2: for all $e \leq i(l) < f$ $alt(l) < 0$. We know $alt(d-1) < 0$, so there must exist steps $f < i(b)$ such that $alt(b) = alt(d-1)$. However, we may reach ambiguity here. In case 2, we switch

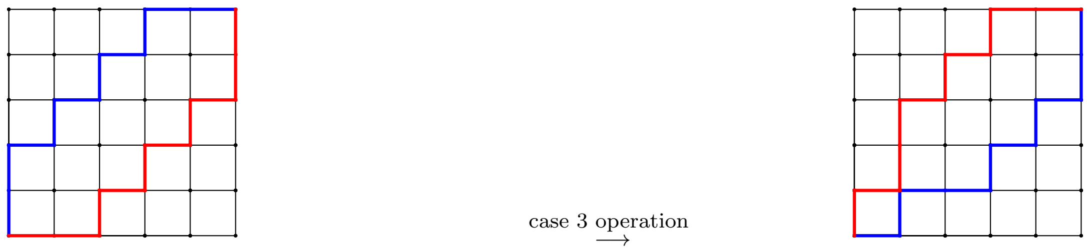

Figure 3.13. G_1 (blue) and G_2 (red) $\quad G_1^{'}$ (blue) and $G_2^{'}$ (red)

the steps from $i(d)$ to $i(b)$ (recall $i(d) = f$) to get $(G_1^{'}, G_2^{'})$, but operation in case 2 on (G_1, G_2) may end up with the same $(G_1^{'}, G_2^{'})$ as operation in case 3 on another pair of (G_1^{*}, G_2^{*}). Under this circumstance, we realize that it is legal to switch steps of (G_1, G_2) from $i(a)$ to $i(c)$ (recall $i(c) = e$), since $(G_1^{'}, G_2^{'})$ and (G_1, G_2) have the same steps between $i(a)$ and $i(c)$. Then for this (G_1, G_2), we are allowed to switch steps from $i(a)$ to $i(c)$ without making the lattice go over the boundary. Here, the 'boundary' is the smallest shape containing all of G_1, G_2, G_1^{*} and G_2^{*}. We require this because we want the result of the remembering function(will introduced in Section 3.4.3) is still between P and Q. Doing as stated, then $G_1^{''}$ be the result of path G_2 and $G_2^{''}$ be the result of path G_1, since we want $G_1^{''}$ be the path use f but not e and $G_2^{''}$ be the path use e but not f. Then the result can be realized as follows:

$$G_1^{''} = t_{i(1)} \cdots s_{i(a)} \cdots s_{i(c)} t_{i(c+1)} \cdots t_{i(k)}$$
$$G_2^{''} = s_{i(1)} \cdots t_{i(a)} \cdots t_e s_{i(c+1)} \cdots s_{i(k)}$$

Example 3.4.11. Suppose that after the forgetting function, we have the following

$$G_1 = EN \underbrace{E}_{s_7} EEN \underbrace{E}_{s_{16}} NNN$$
$$G_2 = NE \underbrace{N}_{t_7} NNE \underbrace{N}_{t_{16}} EEE$$

Notice that $alt(7) = -2$ and $s_{i(9)}$ is the first position after step $i(7)$ that has altitude equaling to -2, then we will switch steps $s_{i(8)}$ and $s_{i(9)}$. The result is as follows and figure 3.14

$$G_1^{'} = EN \underbrace{E}_{s_7^{'}} EEN \underbrace{N}_{s_{16}^{'}} ENN$$
$$G_2^{'} = NE \underbrace{N}_{t_7^{'}} NNE \underbrace{E}_{t_{16}^{'}} NEE$$

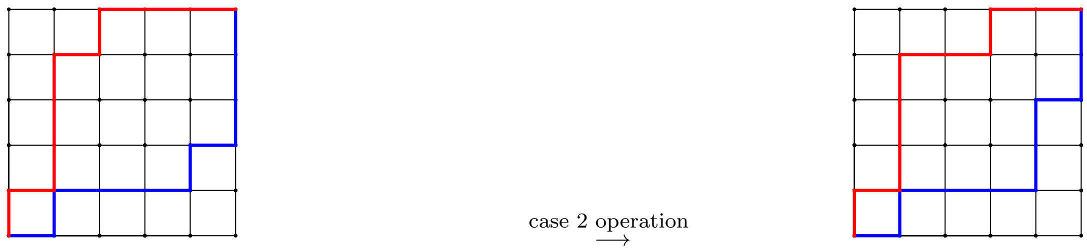

Figure 3.14. G_1 (blue) and G_2 (red) $\quad G_1^{'}$ (blue) and $G_2^{'}$ (red)

As we mentioned above, this $(G_1^{'}, G_2^{'})$ may also be obtained from a pair of (G_1^*, G_2^*) by operation in case 3. Consider the following

$$G_1^* = NN\underbrace{E}_{s_7^*}NNE\underbrace{E}_{s_{16}^*}NEE$$

$$G_2^* = EE\underbrace{N}_{t_7^*}EEN\underbrace{N}_{t_{16}^*}ENN$$

Clearly, for all step from $i(3)$ to $i(8)$, $alt(l) > 0$, and $s_{i(1)}$ is the last position that has the same alititude as $s_{i(3)}$. Then switch $s_{i(2)}$ and $s_{i(3)}$, and exchange the resulting two paths, we will have the same $(G_1^{'}, G_2^{'})$, this operation is shown in Figure 3.15

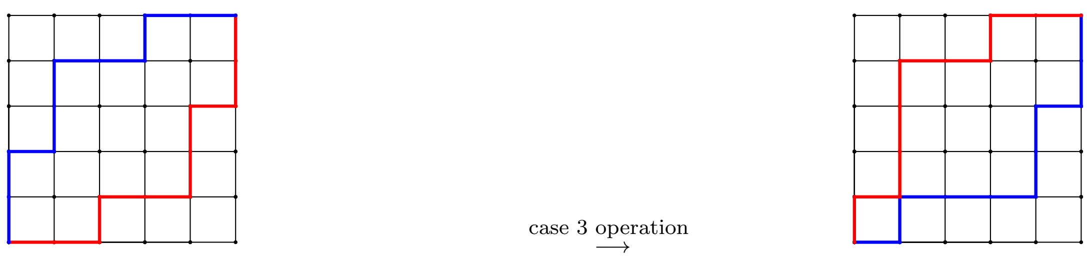

Figure 3.15. G_1^* (blue) and G_2^* (red) $\quad G_1^{'}$ (blue) and $G_2^{'}$ (red)

Therefore, we switch $s_{i(2)}$ and $s_{i(3)}$ for (G_1, G_2), we will get the following, and it slattice presentation in Figure 3.15

$$G_1^{''} = NN\underbrace{E}_{s_7^{''}}NNE\underbrace{N}_{s_{16}^{''}}EEE$$

$$G_2^{''} = EE\underbrace{N}_{t_7^{''}}EEN\underbrace{E}_{t_{16}^{''}}NNN$$

And this $(G_1^{''}, G_2^{''})$ can not be obtained from any other pair of (G_1, G_2).

- **Case 5**: This case is an analogue of case 4. (G_1, G_2) has the same initial condition as case 3. Operation in case 3 on (G_1, G_2) may end up with the same $(G_1^{'}, G_2^{'})$ as

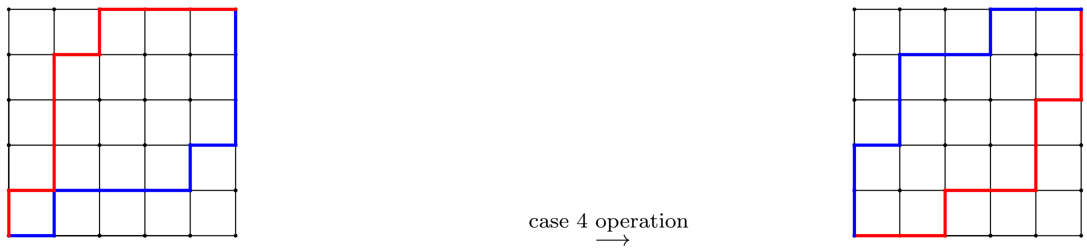

Figure 3.16. G_1 (blue) and G_2 (red) $\quad G_1''$ (blue) and G_2'' (red)

operation in case 2 on another pair of (G_1^*, G_2^*). Then switch the steps from $i(d)$ (recall $i(d) = f$) to $i(b)$ both on G_1 and G_2. The result can be realized as follows:

$$G_1' = s_{i(1)} \cdots t_{i(d)} t_{i(d+1)} \cdots t_{i(b)} s_{i(b)+1} \cdots s_{i(k)}$$
$$G_2' = t_{i(1)} \cdots s_{i(d)} s_{i(d+1)} \cdots s_{i(b)} t_{i(b)+1} \cdots t_{i(k)}$$

Example 3.4.12. According example 3.4.2 we have

$$G_1 = NN \underbrace{E}_{s_7} NNE \underbrace{E}_{s_{16}} NEE$$
$$G_2 = EE \underbrace{N}_{t_7} EEN \underbrace{N}_{t_{16}} ENN$$

We will have Figure 3.9 as (G_1, G_2). It belongs to case 3, and we can figure out that its correspondence as follows

$$G_1' = EN \underbrace{E}_{s_7'} EEN \underbrace{N}_{s_{16}'} ENN$$
$$G_2' = NE \underbrace{N}_{t_7'} NNE \underbrace{E}_{t_{16}'} NEE$$

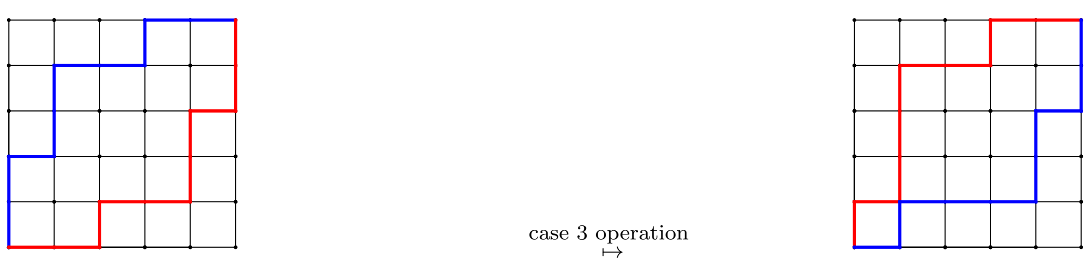

Figure 3.17. G_1 (blue) and G_2 (red) $\quad G_1'$ (blue) and G_2' (red)

However, this (G_1', G_2') can be obtained from

$$G_1^* = EN\underbrace{E}_{s_7^*}EEN\underbrace{E}_{s_{16}^*}NNN$$

$$G_2^* = NE\underbrace{N}_{t_7^*}NNE\underbrace{N}_{t_{16}^*}EEE$$

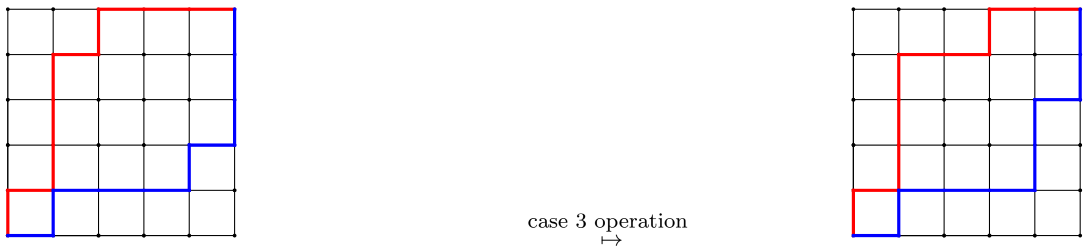

Figure 3.18. G_1^* (blue) and G_2^* (red) $G_1^{'}$ (blue) and $G_2^{'}$ (red)

From this, we figure out that we are in case 2, so we switch $s_{i(7)}$ and $s_{i(8)}$ in G_1, and then we will have the following in Figure 3.19

$$G_1^{''} = NN\underbrace{E}_{s_7^{''}}NNE\underbrace{N}_{s_{16}^{''}}EEE$$

$$G_2^{''} = EE\underbrace{N}_{t_7^{''}}EEN\underbrace{E}_{t_{16}^{''}}NNN$$

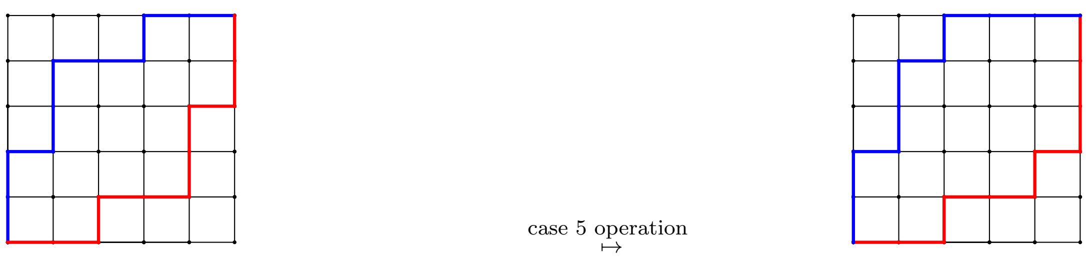

Figure 3.19. G_1 (blue) and G_2 (red) $G_1^{''}$ (blue) and $G_2^{''}$ (red)

Remark 3.4.13. We call operation in case 4 and case 5 *flip out* since they make the corresponding path further towards to line $y = x$

Remark 3.4.14.

1. What we desired to find is a one-to-one correspondence. However we may not be able to find it for every pair of (G_1, G_2), since we may reach the same image through case 2 and case 3.

2. In case 4, we will end up with a 2-to-2 correspondence. We will have (G_1, G_2) in case 2 condition, if its correspondence $(G_1^{'}, G_2^{'})$ can also be reach from a pair (G_1^{*}, G_2^{*}) in case 3, then we will have $(G_1^{''}, G_2^{''})$ that is not correspondence for any thing we have already covered. We can set $(G_1^{''}, G_2^{''})$ be the correspondence for (G_1, G_2) and set $(G_1^{'}, G_2^{'})$ be the correspondence for (G_1^{*}, G_2^{*}).

3. In case 5, it is analogous to case 4. We can consider (G_1^{*}, G_2^{*}) as our initial condition in case 3, and its correspondence is $(G_1^{'}, G_2^{'})$. Then there exists (G_1, G_2) in case 2 has this $(G_1^{'}, G_2^{'})$ as a correspondence. Then we can do the other operation on (G_1^{*}, G_2^{*}). This will give us a pair $(G_1^{*'}, G_2^{*'})$ that is never reached before. We can set this to be the correspondence pairs.

3.4.3 Remembering function

Define the *remembering function* rem by

$$rem(G_1^{'}, G_2^{'}, C_r) = (P_1^{'}, P_2^{'})$$

where $(P_1^{'}, P_2^{'})$ is the lattice path presentation of $(B_1^{'}, B_2^{'})$. The way to obtain $P_1^{'}$ and $P_2^{'}$ is according to C_r, since when we forget the common steps we record which steps and whether each step is N or E. We can just put it back into positions where we delete(or contract) them accordingly. Then we will have $P_1^{'}$ and $P_2^{'}$, which are lattice paths from $(0, 0)$ to (m, n) as well.

Lemma 3.4.15. *The remembering function is injective.*

Proof. We define it to be the inverse of *forgetting function*, by lemma 3.4.4 it is an injective function as well. □

Lemma 3.4.16. *$P_1^{'}$ is the representation of a basis in $\mathcal{B}_f^e$ and $P_2^{'}$ is the representation of a basis in $\mathcal{B}_e^f$.*

Proof. Under case 1, case 2 and case 3. $G_1^{'}$ and $G_2^{'}$ are staying between G_1 and G_2 since the parts we modified moved closer towards to $y = x$ line. Under case 4 and case 5, $G_1^{'}$ and $G_2^{'}$ are not bounded by G_1 and G_2, however the parts we flip out are legal since there are another pair G_1^{*} and G_2^{*} bounding $G_1^{'}$ and $G_2^{'}$. Then after we apply the remembering function to $G_1^{'}$ and $G_2^{'}$, $P_1^{'}$ and $P_2^{'}$ are bounded by P and Q. Hence, they are lattice paths in between P and Q. Then take their indices of north step will give a basis of $M[P, Q]$. Moreover, by construction we have for the correspondence, we know for $G_1^{'}$ the $i(c)$ step is E and the $i(d)$ step is N; similarly for $G_2^{'}$ the $i(c)$ step is N and the $i(d)$ step is E. Then after the remembering function, the eth step of $P_1^{'}$ is E and the fth step of $P_1^{'}$ is N. So the associated basis has f as an element but not e. Similarly, the associated basis of $P_2^{'}$ has e as an element but not f. This gives us the desired result.

□

Example 3.4.17. Continue the example we start from this chapter and in case 5, we have

$$G_1'' = NN\underbrace{E}_{s_7}NNE\underbrace{N}_{t_{16}}EEE$$
$$G_2'' = EE\underbrace{N}_{t_7}EEN\underbrace{E}_{s_{16}}NNN$$

from Example 3.19 and from Example 3.4.2 we have

$$C_r = \{(2,E),(3,N),(4,E),(6,N),(8,E),(10,E),(11,E),(12,E),(14,E),(19,N)\}$$

Then, applying rem to (G_1'', G_2'', C_r) gives

$$P_1' = NENENN\underbrace{E}_{s_7}ENEEENEE\underbrace{N}_{s_16}EENE$$
$$P_2' = EENEEN\underbrace{N}_{t_7}EEEEEEEN\underbrace{E}_{t_16}NNNN$$

And the correspondence bases of P_1' and P_2' are:

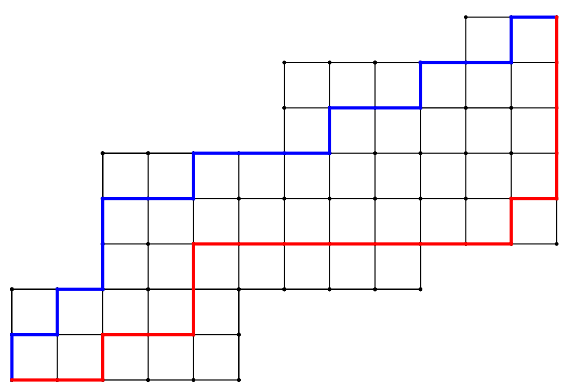

Figure 3.20. P_1' (blue) and P_2' (red)

$$B_1' = \{1,3,5,6,9,13,16,19\}$$
$$B_2' = \{3,6,7,15,17,18,19,20\}$$

And from example 3.4.2, we know the correspondence bases for P_1 and P_2 are :

$$B_1 = \{1,3,5,6,9,13,17,19\}$$
$$B_2 = \{3,6,7,15,16,18,19,20\}$$

As one may notices, this construction preserves the elements of the pair of bases. Now we are ready to prove Theorem 3.4.1.

3.5 $\mathcal{L}$ is Rayleigh

In this section, we will prove Theorem 3.4.1

Proof. Let $M[P, Q]$ be a lattice path matroid. Let (B_1, B_2) be in $\mathcal{B}^{ef} \times \mathcal{B}_{ef}$.
Claim: For each choice there is a corresponding $(B_1^{'}, B_2^{'})$ in $\mathcal{B}_f^e \times \mathcal{B}_e^f$ such that $x^{B_1}x^{B_2} = x^{B_1^{'}}x^{B_2^{'}}$ and the function $F : \mathcal{B}^{ef} \times \mathcal{B}_{ef} \to \mathcal{B}_f^e \times \mathcal{B}_e^f$ is injective.
By Lemmas 3.4.4 and 3.4.15, we can find a$(B_1^{'}, B_2^{'})$ corresponding to (B_1, B_2). Now, let us see why $(B_1^{'}, B_2^{'})$ has the same elements as (B_1, B_2). In order to find this $(B_1^{'}, B_2^{'})$, we apply forgetting function on (B_1, B_2). To accomplish this, first we derive the lattice path presentation of (B_1, B_2), that is P_1 and P_2. Indices of those north steps in P_1, P_2 are exactly the elements in (B_1, B_2). Then we ignore those steps which agree on P_1 and P_2, record the forgetting function in C_r. Element in C_r is (i, W), where i indicates the position of the step and W indicates whether this step is N or E. So if (i, N) in C_r, then both B_1 and B_2 contains i as an element. After we ignore the common information on P_1 and P_2, we have $G_1 = s_{i(1)} \cdots s_{i(k)}$ and $G_2 = t_{i(1)} \cdots t_{i(k)}$. Since G_1 and G_2 differ at each step, for every $1 \le l \le k$ either $t_{i(l)} = N$ or $s_{i(l)} = N$. That is, for every $1 \le l \le k$ $i(l)$ is either in B_1 or B_2. Then go through the construction in Section 3.2.2, we find the corresponding $(G_1^{'}, G_2^{'})$. Apply the remembering function on $(G_1^{'}, G_2^{'})$ to get $P_1^{'}$ and $P_2^{'}$. The indices of north steps of $P_1^{'}$ and $P_2^{'}$ are elements in $B_1^{'}$ and $B_2^{'}$.
Let S be the set of elements in both B_1 and B_2. Then $S = \{i : (i, W) \in C_r \text{ and } W = N\}$. Since we add $\{i : (i, W) \in C_r\}$ back in $(G_1^{'}, G_2^{'})$ to get $P_1^{'}$ and $P_2^{'}$, $S = \{i : i \in B_1^{'}, i \in B_2^{'}\}$. And for those elements appear only in one of B_1 and B_2, they are $i(l)$ for $1 \le l \le k$. They also appear only in one of $B_1^{'}$ and $B_2^{'}$, when $i(l)$ is in $B_1^{'}$ then the $i(l)$ step in $G_1^{'}$ is north step; similarly if $i(l)$ is in $B_2^{'}$ then the $i(l)$ step in $G_2^{'}$ is north step. Hence, we obtained that $B_1 \cap B_2$ is equal to $B_1^{'} \cap B_2^{'}$, and $B_1 \Delta B_2$ is equal to $B_1^{'} \Delta B_2^{'}$ as well.
Let $\Delta M(e, f)$ be the Rayleigh difference with respect to e, f,

$$\Delta M(e, f) = M_e^f M_f^e - M_{ef} M^{ef}$$

Recall that

$$M^{ef} = \sum_{B \in B^{ef}} x^B$$

$$M_{ef} = \frac{\sum_{B \in B_{ef}} x^B}{x_e x_f}$$

$$M_e^f = \frac{\sum_{B \in B_e^f}}{x_e}$$

$$M_f^e = \frac{\sum_{B \in B_f^e}}{x_f}$$

$$\Delta M(e, f) = \frac{\sum_{(B_1, B_2) \in \mathcal{B}_e^f \times \mathcal{B}_f^e} x^{B_1} x^{B_2} - \sum_{(B_1, B_2) \in \mathcal{B}_{ef} \times \mathcal{B}^{ef}} x^{B_1} x^{B_2}}{x_e x_f}$$

According to the injective F showed by the construction, every term left in $\Delta M(e,f)$ has positive coefficient. Hence, $\Delta M(e,f) \geq 0$ for all $x \geq 0$.
We can conclude that, given any lattice path matroid $M[P,Q]$ for any pair $e, f \in E(M[P,Q])$ $\Delta M(e,f) \geq 0$ for all $x \geq 0$. Therefore, the class of lattice path matroid $\mathcal{L}$ is Rayleigh. $\square$

Corollary 3.5.1. *Uniform matroids are Rayleigh.*

Proof. Consider the lattice path matroid $M[P,Q]$ with the following lattice path presentation, which is a rectangular shape,

$$P = \underbrace{N \cdots N}_{r \text{ terms}} \underbrace{E \cdots E}_{n-r \text{ terms}}$$
$$Q = \underbrace{E \cdots E}_{n-r \text{ terms}} \underbrace{N \cdots N}_{r \text{ terms}}$$

Then this is a transversal matroid with set system $(E, \mathcal{A})$ where $E = [n]$, $A_i = \{i, i+1, i+2, i+n\}$. And $U_{r,n}$ can be realized to be this transversal matroid as well. Hence, every uniform matroid is isomorphic to a rectangular shape lattice matroid. Therefore, uniform matroids are Rayleigh. $\square$

Chapter 4

Future Direction

In this chapter, we will review classes of matroids with stronger condition, which is called the Strong Rayleigh property, and how it is related to half-plane property of polynomials. We wonder whether lattice path matroids satisfies negative correlated condition for more than two elements, for example three elements. Then we will introduce a large class of matroids that includes $\mathcal{L}$, called positroids. We ask the natural question, whether positroids are Rayleigh. If not, what is the boundary between lattice path matroids and positroids.

4.1 Strong Rayleigh Property, and Generalized Catalan matroids

Recall in section 2.2.2, if a matroid is Rayleigh then $\Delta M(e,f) \geq 0$ for all $\vec{x}$ in $\mathbb{R}_{\geq 0}^{E-\{e,f\}}$. A matroid is *strongly Rayleigh* if $\Delta M(e,f) \geq 0$ for all $\vec{x}$ in $\mathbb{R}^{E-\{e,f\}}$. Matroids that are strongly Rayleigh are clearly Rayleigh, but the converse is not always true. Hence, it is a natural question to ask which Rayleigh matroids are also strongly Rayleigh matroids. Choe, Oxley, Sokal and Wagner proved that regular matroids are strongly Rayleigh in [6].

In our work, we showed $\mathcal{L}$ is Rayleigh. It is natural to ask is $\mathcal{L}$ strongly Rayleigh? Unfortunately, the answer is no. So we want to ask which subset of $\mathcal{L}$ is strongly Rayleigh. Consider the following class of lattice path matroids.

A lattice path matroid $M[P,Q]$ is a generalized Catalan matroids if the lattice path presentation of Q is

$$\underbrace{E\cdots E}_{n}\underbrace{N\cdots N}_{m}.$$

Example 4.1.1. Let $Q = EEEEEEENNNNN$ and $P = NEENNNENENNE$, then $M[P,Q]$ has the following lattice path presentation.

Generalized Catalan matroids are somehow easier to study, using Corollary 2.2 in [15]

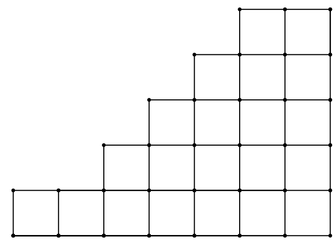

Figure 4.1. Generalize Catalan Matroid

Consider the following weight bases enumerator

$$M_{\mathcal{C}}(x) = \sum_{B \in \mathcal{B}} c_B x^B$$

where c_B is the weight on base B. By carefully choosing c_B for each B, we might artificially make the bases enumerator satisfy the half-plane property. This suggests we could have a $\mathcal{C}$-Rayleigh property for the class of generalized Catalan matroids.
Quesion 1: Can we prove for some generalized Catalan matroids M that$M_{\mathcal{C}}(\mathbf{x})$ satisfies half-plane property for all $c_B = 1$. That is, is there a subclass of Generalized Catalan matroids is strongly Rayleigh.

4.2 Positroid

In [10], Oh introduced a class of matroids, called *positroids*. A positroid is a matroid that can be represented by a $k \times n$ matrix with nonnegative maximal minors. In section 6 Lemma 21, Oh proved that all lattice path matroids are positroids. Since we know lattice path matroids are Rayleigh, we want to ask the following question:
Question 2: Is the class of positroids Rayleigh? If not, what is the largest subclass of it is Rayleigh? That is, can we characterize what property will a positroid need to be a Rayleigh matroid.

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